Boletín de Estadística e Investigación Operativa
Vol. 27, No. 1, Febrero 2011, pp. 14-28

Investigación Operativa

Multidimensional assignment: applications and some thoughts

Federico Perea
Dept. of Applied Statistics, Operations Research and Quality
Polytechnic University of Valencia
✉ perea@eio.upv.es

Abstract

This paper reviews the multi dimensional assignment problem (MAP) and some of its applications. The MAP is a higher dimensional version of the classic two-sided assignment problem, and has applications in areas such as target tracking and robot vision. These applications are reviewed as well as the game arising from the MAP, called the m-sided assignment game (m-SAG). Some thoughts about the use of approximation algorithms in m-SAG to avoid the NP-hardness of the underlying MAP are described. The paper finishes with a brief introduction to the concept of approximation games. The characteristic function of these games cannot be calculated and is, therefore, approximated.

Keywords: Multidimensional assignment, Tracking, Games, Approximation.
AMS Subject classifications: 90B80, 91A46.

1. Introduction

The Multidimensional Assignment Problem (MAP) is an extension of the well known Linear Assignment Problem (LAP). Pierskalla in his pioneering work [13] referred to the MAP as a higher dimensional version of the standard two-dimensional linear assignment problem. Such higher dimensions can be thought as time, space, etc. Whereas the LAP consists of matching elements from two sets in an optimal way, the MAP extends the LAP by requiring the matching of t-uples from more than two sets, which need not have the same cardinality. Although the LAP can be optimally solved in polynomial time, the MAP is NP-hard for dimensions strictly larger than two, see for instance [7].

This paper gives a brief overview of the state of the art in multidimensional assignment regarding two applications that have been addressed by members of our society: the Multi-Target Multi-Sensor Tracking Problem (MTMSTP), and the m-Sided Assignment Game (m-SAG).

The MTMSTP consists of assigning plots in a radar screen to their corresponding moving objects. Its applications vary from air traffic control in airports and defense radars, to robot vision. The need for "good" solutions in a relatively short time and the computational complexity of this problem, justify that a great part of the research in multi-target multi-sensor tracking has focused on finding suitable approximation algorithms to solve it.

On the other hand, the m-SAG is a game in which the members of several pairwise disjoint sets interact in order to optimize a certain utility function. In comparison with the classic assignment game (m-SAG with $m = 2$), little has been written about the m-SAG for $m \geq 3$. One issue that, as far as the author knows, has been neglected in the literature about m-sided assignment games is the fact that the characteristic function of these games is extremely hard to obtain. At the end of this paper some ideas on how this drawback could be overcome are briefly discussed.

Some concepts on cooperative game theory will be needed in this paper. First recall that a generic finite cooperative game is a pair (N, v), where N is a finite set of players and $v : 2^N \to \mathbb{R}$ is the characteristic function, which assigns to each coalition $S \subseteq N$ a real value (it can be a benefit or a cost), with $v(\emptyset) = 0$. The main objective in cooperative game theory is the allocation of the profit obtained by the cooperation of agents. Two well-studied allocation sets are the *core* and the *kernel*. The core of (N, v) (in the case of a benefit game) is the set

$$C(N, v) = \{x \in \mathbb{R}^n : x(N) = v(N), x(S) \geq v(S), \forall\ S \subset N\}, \qquad (1.1)$$

where $x(S) = \sum_{j \in S} x_j$, for any $S \subseteq N$. Allocations in the core are stable allocations in the sense that no coalition have incentives to act separately from the rest of players since, this way, they would end up worse off (that is the meaning of the inequality $x(S) \geq v(S)$ for all $S \subset N$, also known as the *collective rationality principle*). The kernel is constituted by individually rational allocations, those $x \in \mathbb{R}^n$ such that $v(\{i\}) \leq x_i$ for all $i \in N$, for which no player i *outweighs* another player j. For more details on these concepts the reader is referred to [11].

The paper is structured as follows. In Section 2 the MAP is introduced. Section 3 is devoted to show the first application studied in this paper: the Multi-Target Multi-Sensor Tracking Problem. In Section 4, the second application of the MAP introduced here is reviewed: the Multi-Sided Assignment Game, and a brief introduction to these games and a literature review on the topic are shown. The last part of the paper is devoted to show the need of using approximation algorithms in certain cooperative games, including the concept of *approximation games* introduced here, in which the characteristic function is calculated by means of an approximation algorithm.

2. The MAP

This section reviews the MAP and formulates it using Integer Linear Programming (ILP).

An m-dimensional assignment problem consists of m pairwise disjoint sets, named $N^1, N^2, \ldots, N^m$, of the form

$$N^k = \{i^k_1, \ldots, i^k_{n_k}\} \quad k = 1, \ldots, m. \tag{2.1}$$

The assignment of agents $\{i^1, \ldots, i^m\}$, where $i^k \in N^k \ \forall\ k$, results in a benefit equal to $a_{i^1,\ldots,i^m}$ units. In such a case agents $\{i^1, \ldots, i^m\}$ are *associated.* The problem that arises when the elements of $N^1, N^2, \ldots, N^m$ must be associated so that the total benefit obtained is maximized and no agent belongs to more than one association is a MAP of dimension m, or just an m-dimensional assignment problem. Note that when $m = 2$ this problem reduces to the classic assignment problem.

This situation can be described by an ILP problem. Consider the variables $x_{i^1 \ldots i^m} \in \{0,1\} \ \forall\ i^k \in N^k,\ k = 1, \ldots, m$, satisfying that $x_{i^1 \ldots i^m} = 1$ if agents $(i^1, \ldots, i^m)$ are matched and zero otherwise. The integer linear program that models the MAP is:

$$\begin{array}{lll}
\max & \displaystyle\sum_{i^1 \in N^1} \cdots \sum_{i^m \in N^m} a_{i^1 \ldots i^m} x_{i^1 \ldots i^m} & \\
\text{s.t.} & \displaystyle\sum_{i^2 \in N^2} \cdots \sum_{i^m \in N^m} x_{i^1 \ldots i^m} \leq 1 & \forall\ i^1 \in N^1 \\
& \displaystyle\sum_{i^1 \in N^1} \cdots \sum_{i^{k-1} \in N^{k-1}} \sum_{i^{k+1} \in N^{k+1}} \cdots \sum_{i^m \in N^m} x_{i^1 \ldots i^m} \leq 1 & \begin{array}{c} \forall\ i^k \in N^k, \\ 1 < k < m \end{array} \\
& \displaystyle\sum_{i^1 \in N^1} \cdots \sum_{i^{m-1} \in N^{m-1}} x_{i^1 \ldots i^m} \leq 1 & \forall\ i^m \in N^m \\
& x_{i^1 \ldots i^m} \in \{0,1\},\ \forall\ i^1 \in N^1, \ldots, i^m \in N^m &
\end{array} \tag{2.2}$$

The constraints in the above ILP problem imply that, for a given $k \in 1, ..., m$ and a given $i_k \in N$, the variables $x_{i_1,\ldots,i_k,\ldots,i_m}$ must be equal to 1 for at most one assignment $(i_1, ..., i_{k-1}, i_{k+1}, ..., i_m)$.

Therefore, an m-dimensional assignment problem is denoted by its sets of agents and the vector of benefits

$$(N^1, \ldots, N^m; a). \tag{2.3}$$

To finish this section consider the following example with a 3-dimensional assignment problem.

Example 2.1. *Suppose that there are two factories, two warehouses and two shops. It is known that if factory i, warehouse j and shop k are associated, they together produce a benefit of a_{ijk}. Besides, both factories, warehouses and shops can be associated with only one of the others, that is, only one factory with only one warehouse with only one shop. The benefits a_{ijk} are shown in Table 3.*

a_{111}	a_{112}	a_{121}	a_{122}	a_{211}	a_{212}	a_{221}	a_{222}
3	4	2	5	2	6	5	4

Table 3: Table of benefits.

So, the formulation of this problem as a linear program is:

$$\begin{aligned} \max \quad & 3x_{111} + 4x_{112} + 2x_{121} + 5x_{122} + 2x_{211} + 6x_{212} + 5x_{221} + 4x_{222} \\ s.t. \quad & x_{111} + x_{112} + x_{121} + x_{122} \leq 1 \\ & x_{211} + x_{212} + x_{221} + x_{222} \leq 1 \\ & x_{111} + x_{112} + x_{211} + x_{212} \leq 1 \\ & x_{121} + x_{122} + x_{221} + x_{222} \leq 1 \\ & x_{111} + x_{121} + x_{211} + x_{221} \leq 1 \\ & x_{112} + x_{122} + x_{212} + x_{222} \leq 1 \\ & x_{ijk} \in \{0,1\} \ \forall \ i,j,k = 1,2. \end{aligned} \tag{2.4}$$

One optimal feasible solution to the relaxed problem of (2.4) is the vector

$$[\frac{1}{2}, 0, 0, \frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, 0] \tag{2.5}$$

which yields a value in the objective function equal to $\frac{19}{2}$.

The solution to the problem taking into account the integer constraints is

$$x_{112} = x_{221} = 1, \quad x_{ijk} = 0 \text{ otherwise} \tag{2.6}$$

which produces an objective function value equal to 9.

Thus, their optimal associations are:

- *Factory 1, warehouse 1 and shop 2.*
- *Factory 2, warehouse 2 and shop 1.*

Note that, unlike the classic assignment problem, the relaxed solution need not coincide with that obtained when taking the binary constraints into account.

3. The MTMSTP

In this section a real problem that can be modeled as a MAP is introduced: the multi-target multi-sensor tracking problem.

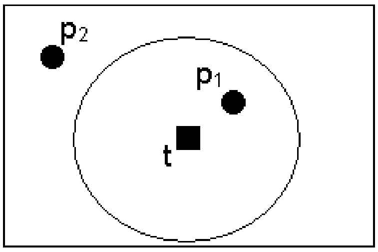

Figure 1: Association gate.

According to [2], *tracking* is the processing of measurements obtained from a *target* in order to maintain an estimate of its current state, which typically consists of:

- Kinematic components - position, velocity, acceleration, turn rate, etc.
- Feature components - radiated signal strength, spectral characteristics, radar cross-section, target classification, etc.
- Constant or slowly varying parameters - aerodynamic parameters, etc.

One of the major difficulties in the application of multi-target multi-sensor tracking involves the problem of associating measurements received by a sensor with the appropriate target, forming a so-called *track hypothesis*, or just a *track*. The term target refers to the actual object following a certain trajectory while track refers to an estimated target trajectory. It is assumed that each target in the coverage of the scanning radar can produce a maximum of one measurement during a radar scan. To determine which measurements are likely candidates to originate from a certain target, an association gate is positioned at the predicted measurement of the target in the measurement space, as proposed in [3]. A measurement could be associated with a track if the measurement falls within a defined track association gate. As an example consider a track t and two new measurements received in the last scan, p_1 and p_2.

In Figure 1 the square represents the predicted position of track t, the dots are the measurements p_1 and p_2 and the circle represents the gate that corresponds to track t. In this example p_1 is associated with track t and p_2 is not.

Now, the MTMSTP is formulated as a MAP. A more complete description of this process can be found in [18] and [14]. Let us first describe the multi-target multi-sensor tracking problem.

Suppose that a sensor starts observing the airspace periodically at a time $y_0 = 0$. During the first scan a first set of measurements is received. Let y_1 denote the point in time at which the data of the first scan, during the time interval $[y_0, y_1)$, are collected. Analogously, the second set of measurements, corresponding to the second scan, is received at instant y_2, and so on. The set of measurements collected at scan k, during the time interval $[y_{k-1}, y_k)$, and received at time instant y_k, is defined as $Z(k) = \{z_i^k\}_{i=1}^{M_k}$, where M_k denotes the total number of measurements received during scan k and z_i^k is the i^{th} measurement received within this scan. The cumulative data set for N scans is defined as $Z^N = Z(1) \cup \cdots \cup Z(N)$. A track t is defined as a set of measurements of Z^N that contains at most one measurement from each scan and consists of at least one measurement. This can be mathematically expressed as

$$t \subset Z^N \ : \ |t| \geq 1, \ |t \cap Z(k)| \leq 1 \ \forall \ k = 1, \ldots, N. \tag{3.1}$$

A feasible partition of Z^N is a set of tracks $\delta = \{t_1, \ldots, t_{|\delta|}\}$ satisfying two conditions:

1. δ must cover Z^N, that is, $\bigcup_{j=1}^{|\delta|} t_j = Z^N$.

2. Any two tracks belonging to δ may not have common measurements, that is, they must be disjoint, $t_i \cap t_j = \emptyset \ \forall \ i \neq j, \ i, j = 1, \ldots, |\delta|$.

For the sake of readability, in the rest of the paper feasible partitions will be called partitions. The set of all possible partitions of Z^N is denoted by $\Delta(Z^N)$.

The goal in multi-target multi-sensor tracking is to find a partition of Z^N that is most likely to represent the actual situation. To obtain such a partition, which need not be unique, a quality measure $Q(t)$ is assigned to each track $t \subset Z^N$, which expresses how well each measurement of t fits the assumed target's dynamic model. [18] was one of the first to introduce a likelihood function $Q(t)$ for each track t. A similar approach can be found in [14] and [22]. Since the objective is to find a partition that is most likely to be true, the problem reduces to

$$\begin{array}{rl} \max & \prod_{t \in \delta} Q(t) \\ \text{s.t.} & \delta \in \Delta(Z^N). \end{array} \tag{3.2}$$

Note that this objective function is not linear. In order to come up with a linear problem, the track score is defined as $w(t) = \log(Q(t))$ for each $t \subset Z$. Therefore, the partition that maximizes $\prod_{t \in \delta} Q(t)$ also maximizes $\sum_{t \in \delta} w(t)$, since the logarithm is a monotonic function. Besides the linearity of the linear function, an added benefit of using the logarithm is that it also reduces the round-off errors that result from multiplying small numbers (such as likelihood functions). Then, our problem reduces to

$$\begin{array}{rl} \max & \sum_{t \in \delta} w(t) \\ \text{s.t.} & \delta \in \Delta(Z^N). \end{array} \tag{3.3}$$

Since the number of tracks explosively grows with the number of scans, and therefore so does the computational complexity of the optimization problem, a sliding window approach is used. The main idea behind the sliding window technique consists of only considering the measurements of the last d scans, assuming that the assignments of the previous scans are fixed (see Figure 2). So, if a new scan is performed the window slides one scan onwards, discarding the oldest scan of the previous window. The other $d-1$ scans are maintained. After including the new scan, the number of scans within the window is again restored to d scans. So, each time a set of measurements is received, the number of considered scans remains constant and the complexity of the corresponding MAP problem does not increase. A description of the method is provided in [15]. As an example, in Figure 2 those tracks are related with the scans with a number $\leq M$.

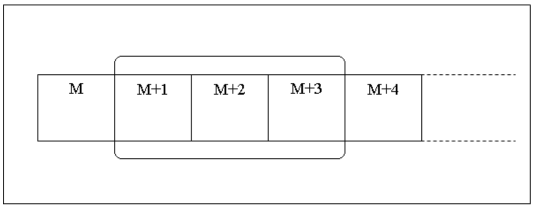

Figure 2: The sliding window contains only three scans: $d = 3$.

Now the multi-target multi-sensor tracking problem can be formulated as a multidimensional assignment problem. A track $t = \{z^0_{i_0}, z^1_{i_1}, \ldots, z^d_{i_d}\} \subset Z^N$ is either present in a partition $\delta \in \Delta(Z^N)$ or it is not, where $Z(0) = \{z^0_i\}_{i=1}^{M_0}$ is the set of previously established tracks. This corresponds to a 0-1 decision, which can be represented by the following variables:

$$x_{i_0,i_1,\ldots,i_d} = \begin{cases} 1 & \text{if } t = \{z^0_{i_0}, z^1_{i_1}, \ldots, z^d_{i_d}\} \in \delta, \\ 0 & \text{otherwhise.} \end{cases} \tag{3.4}$$

Let $c_{i_0,i_1,\ldots,i_d}$ be the score of track $t = \{z^0_{i_0}, z^1_{i_1}, \ldots, z^d_{i_d}\}$. Then, the formulation of the MTMSTP as a MAP is

$$\begin{array}{ll} \max & \displaystyle\sum_{i_0=1}^{M_0}\sum_{i_1=1}^{M_1}\cdots\sum_{i_d=1}^{M_d} c_{i_0,i_1,\ldots,i_d} x_{i_0,i_1,\ldots,i_d} \\ \text{s.t.} & \displaystyle\sum_{\substack{j=0\\ j\neq k}}^{d}\sum_{i_j=1}^{M_j} x_{i_0,\ldots,i_d} = 1, \quad \begin{array}{l} i_k = 0,\ldots,M_k, \\ k = 0,1,\ldots,d, \end{array} \\ & x_{i_0,i_1,\ldots,i_d} \in \{0,1\} \ \forall\ i_0, i_1, \ldots, i_d, \end{array} \tag{3.5}$$

which is equivalent to those found in [9] and [18]. The constraints of the problem force each measurement to be in one, and only one, track. In other words, two different objects cannot originate the same measurement.

Example 3.1. *As an example consider a toy instance with three scans, where two measurements are received in each scan as depicted in Figure 3.*

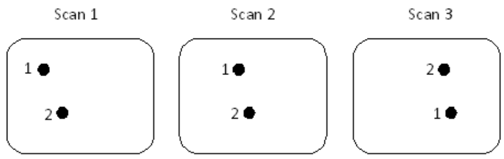

Figure 3: Three scans with two measurements each.

The first scan, Scan 0*, actually is the set of previously established tracks. At each scan, two measurements are received (in order to maintain the same notation as before, let the black dot corresponding to measurement* i *in scan* k *be* z_i^k*.) Assume that the track scores* c_{i_0,i_1,i_2} *are:*

(i_0, i_1, i_2)	(1,1,1)	(1,1,2)	(1,2,1)	(1,2,2)	(2,1,1)	(2,1,2)	(2,2,1)	(2,2,2)
$c_{i_0 i_1 i_2}$	3	4	2	5	2	6	5	4

According to this score, an object following the trajectory given by the dots 2,1,1 in scans 0,1,2, respectively, is less likely to be true than an track following the trajectory given by the dots 2,1,2 in the same scans, since the score for the first object would be 2 and the score for the second one would be 6.

The problem can be solved by substituting the parameters $c_{i_0 i_1 i_2}$ *in (3.5), which yields the solution* $x_{112} = x_{221} = 1$*, the other variables being zero. That means that our formulation would decide that there is one object following the trajectory given by measurements 1,1 and 2 in the corresponding scans 0,1 and 2, and there is a second object following the trajectory given by measurements 2,2 and 1, as depicted in Figure 4.*

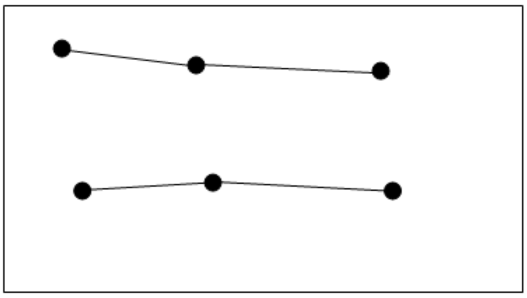

Figure 4: The trajectories followed by the two objects.

Although the problem size is reduced by considering only the last d scans the resulting MAP is still NP-hard for window sizes $d \geq 2$, since this leads to a $(d+1)-$dimensional assignment problem (remember that the m-dimensional assignment problem is NP-hard for $m \geq 3$). When tracking multiple targets, a solution to the MAP must be given before the following scan of the sensor begins, which is usually a short time. This justifies the development of efficient approximation algorithms that provide good solutions for multi-target multi-sensor tracking problems.

The GRASP (Greedy Randomized Adaptive Search Procedure) algorithm proposed in [10] proved very efficient for the MTMSTP. As for other heuristics, [26] presents a tabu search algorithm. Another research line in multi-sensor multi-target tracking leads to seeking the K-best associations, see for instance [16]. [4] used a semi-greedy algorithm to generate a set of required solutions from which the best solution was selected. More recently, [12] proposed K-greedy algorithms, which are a generalization of greedy algorithms in which, at each step, the group of K tracks that jointly yield the best improvement in the objective function are chosen. Note that when $K = 1$ these algorithms reduce to the classic greedy algorithm.

4. The m-SAG

In this section the multiple-sided extension of the classic assignment game is introduced. Borrowing from [21], "The assignment game is a model for a two-sided market in which a product that comes in large, indivisible units (e.g., houses, cars, etc.) is exchanged for money, and in which each participant either supplies or demands exactly one unit." Ever since this first appearance, the assignment game has been widely studied in the literature. For a quick good introduction to the assignment game, the reader is for instance referred to the third chapter in [5].

Let us now consider an m-dimensional assignment problem $(N^1, \ldots, N^m; a)$. Suppose that the agents interacting in the MAP, that is, the members of $N^1 \cup \cdots \cup N^m$, have conflicting objectives but, at the same time, they all want to maximize their individual benefits. Thus, a cooperative game (N, v) naturally arises, where the set of players is $N = N^1 \cup \cdots \cup N^m$. In order to obtain the characteristic function of this game, the maximum benefit that each coalition can make by themselves has to be calculated. For each $S \subset N$ define N_S^k to be the set consisting of the members of S that come from N^k, that is $N_S^k = N^k \cap S \ \forall\ k = 1, \ldots, m$. Thus, $v(S)$ is defined as the value of the linear program (2.2) with $(N_S^1, ..., N_S^m; a)$.

Definition 4.1. *Let (N, v) be such that $N = N^1 \cup \cdots \cup N^m$ with $N^i \cap N^j = \emptyset \ \forall\ i \neq j$ and $v(S)$ is obtained from (2.2) with $(N_S^1, ..., N_S^m; a)$. Then (N, v) is an m-sided assignment game.* From its definition, it can be proven that m-SAG are well defined, nonnegative, monotonic and superadditive.

So far, there has been very little written about the m-SAG. One of the main differences with respect to the classic assignment game is that, in general, m-SAG do not have core allocations. This means that, in general, it is not possible to share the general profit among the agents in such a way that all coalitions receive at least what they would make by acting on their own. Therefore cooperation among players is not guaranteed. [1] used the "three-sexes" version of the Gale and Shapley's marriage market to prove that m-SAG can have an empty core.

Later on, [17] and [23] found subclasses of m-SAG with nonempty cores. The second one presented the condition of *additivity* as a sufficient condition for the non-emptyness of the core of an m-SAG. Specifically, an additive m-SAG satisfies that the value of a matching $(i_1, ..., i_m)$ is always the result of the sum of the values of the 2-matchings $(i_k, i_{k+1}),\ k = 1, ..., m-1$.

In the last few years, a group of the University of Barcelona is actively working on the m-SAG. Examples of such efforts are [24] and [25]. The first one shows the relation between the core of an m-SAG and the competitive prices of the multilateral assignment market with buyers and different firms. The second paper studies a subset of the core: the symmetrically multilateral-bargained (SMB) allocation set. SMB allocations are core allocations that remain invariant after a negotiation process between agents, in which each agent's strength depends on what they could receive in their preferred alternative matching, provided that the other players' payoffs do not change. These allocations extend the *pairwise-bargained allocations* ([19]). They prove that in m-sided assignment games with nonempty core, the SMB allocation set is always nonempty and that, unlike the two-sided case, it does not coincide in general with the kernel.

4.1. Some thoughts about approximation algorithms and m-SAG

All the research done in the m-SAG assumes that the characteristic function of the game is known, which, as it will be discussed later, is crucial. Due to the fact that in general m-SAG may have empty core, the search for other allocations with good properties such as the Shapley value, see [20], becomes even more important than in games where core allocations can be efficiently found. But in games arising from problems that cannot be optimally solved in real time, just like MAP, the calculation of allocations like the Shapley value is, in general, extremely hard. Note that to compute such a value the characteristic function must be known, that is, it is necessary to solve Problem (2.2) for every coalition. In other words, $O(2^n)$ MAP problems (each of them is NP-hard) must be solved. This argument is enough to justify the search for techniques to efficiently allocate benefits (or costs) in m-SAG. This section proposes approximation algorithms to avoid the NP-hardness of the underlying LP problems. Approximation algorithms are procedures that find "good" (not necessarily optimal) solutions to complex problems in a "short" time. The quality of the solution is guaranteed because the approximation is optimal up to a constant factor. For example, a feasible solution x given by an $\alpha-$approximation algorithm for a maximization problem P would yield a value $f(x)$ satisfying $\alpha OPT \leq f(x) \leq OPT$, where OPT is the value of an optimal solution to P, with $0 < \alpha < 1$. Note that, if $\alpha = 0.95$, our algorithm would guarantee a solution whose value deviates less than 5% from the optimal solution. On the other hand, an $\alpha-$approximation algorithm for a minimization problem would yield solutions that satisfy $OPT \leq f(x) \leq \alpha OPT$, with $\alpha > 1$. For a detailed description on approximation algorithms see [27].

Examples of the use of approximation algorithms to efficiently allocate the benefits of a cooperative situation are [6] and [8]. The first one studies the Traveling Salesman Problem (TSP) and, since computing the length of an optimal TSP tour is NP-hard, it proposes approximately fair cost allocations (allocations where the customers can be overcharged by a certain percentage, or the supplier is allowed to run a certain deficit.) To do so they introduce a modified game with a smaller cost function. The second paper considers the problem of sharing the cost of a jointly utilized facility in a "fair" way, specifically the problem of multicast routing. They avoid the complexity of the underlying problem by a factor 2 approximation algorithm for the Steiner tree problem (a minimum spanning tree on the required vertices).

In the rest of the section a greedy mechanism to allocate benefits in an m-SAG is proposed. That is, this procedure will yield a vector $(x_1, \ldots, x_n)$ such that x_i is the payoff of player i, $\forall\ i \in N$. Note that, since the calculation of $v(N)$ may not be possible due to the complexity of the underlying LP problem, the allocation proposed may not allocate the optimal benefit, as expressed in Theorem 4.1.

Given a MAP, its set of associations is the set

$$AS = \{s : s = \{i^1, \ldots, i^m\}, i^k \in N^k\ \forall\ k = 1, \ldots, m\}. \tag{4.1}$$

Let us define the benefit of the association $s \in AS$, $a(s)$, to be the benefit generated after the association of the players in s, that is, $a(s) = a_{i^1,\ldots,i^m}$. So, by introducing the binary variable y_s, which takes value one if association S is to form and zero otherwise, the MAP (2.2) can be formulated as:

$$\begin{array}{lll} \max & \sum\limits_{s\in AS} y_s a(s) & \\ \text{s.t.} & \sum\limits_{s\,:\,i\in s} y_s \leq 1 & \forall\ i \in N \\ & y_s \in \{0,1\} & \forall\ s \in AS. \end{array} \tag{4.2}$$

Note that, just like problem (2.2), problem (4.2) maximizes the benefits obtained by the associations formed, imposing that the same player cannot belong to more than one of the associations to form.

As a first attempt for using approximation algorithms in m-sided assignment games, the *greedy allocation* is here introduced. This procedure runs as follows. Consider (N, v) an m-SAG. Iteratively select the most profitable association and remove the members of such association. Repeat this process until all players have been assigned or there are no more possible associations. When an association $s \in S$ is chosen, its benefit is equally distributed among the players that constitute s. Thus, set $x_i = a(s)/m \ \forall\ i \in s$ for every association s selected in the greedy algorithm. A pseudocode of this allocation procedure is:

Greedy allocation.

1. $x = 0, C = N$
2. Repeat.
   (a) Find the most profitable association in C, say s.
   (b) Let $a(s)$ be the benefit of association s.
   (c) For each $i \in s$ set $x_i = a(s)/m$.
   (d) $C \leftarrow C - \{s\}$.

   Until no more associations are found.
3. Output x.

For a fixed m-dimensional assignment problem, the value of the solution computed by the process above approximates the optimal solution within a factor of m, that is, V(OPT) $\leq m$V(GREEDY), where $V(OPT)$ denotes the value of the optimal solution and $V(GREEDY)$ denotes the value of the solution returned by the greedy algorithm. The proof of this result can be found in [4]. Besides, [12] provides an example to show that this approximation factor is tight.

After those arguments, the next theorem follows.

Theorem 4.1. *Let $x = (x_1, \ldots, x_n)$ be the greedy allocation for an m-SAG game (N, v). The following assertions hold:*

1. *x is a preimputation of (N, v), i.e., $x(N) \leq v(N)$.*
2. *x satisfies the individual rationality principle, i.e., $x_i \geq v(\{i\})$.*
3. $\frac{v(N)}{m} \leq x(N) \leq v(N)$.
4. *x can be computed in polynomial time.*

Proof.

1. Since $x(N)$ is the value obtained by the greedy algorithm and $v(N)$ is optimal, the result follows.

2. Since $v(\{i\}) = 0 \ \forall \ i \in N$, and $x_i \geq 0$, the result follows.

3. This comes from the result $v(OPT) \leq mv(GREEDY)$ proved in [4].

4. The complexity of the greedy algorithm for MAP has been proven to be $O(n \log n)$ in [4].

■

This allocation, despite its obvious limitations, can be used as a starting point for the use of approximation algorithms in m-sided assignment games.

Remark 4.1. ***(Thoughts on approximation games)***
To finish this paper, some thoughts about further use of approximation algorithms for cooperative games arising from complex combinatorial problems are exposed. To do so, consider (N, v) a (benefit) cooperative game whose characteristic function cannot be efficiently calculated (just like m-SAG). Let us now build the game (N, v'), where $v'(S)$ is calculated by solving the corresponding combinatorial problem by means of an approximation algorithm. Therefore, $v'(S) \leq v(S) \ \forall \ S$. (N, v') will be called the approximation game of (N, v). *A first advantage of this new game is that its characteristic function can be calculated efficiently, unlike (N, v). Besides, if $v(N)$ can be calculated, and therefore we set $v'(N) = v(N)$, then the following property directly follows:*

$$C(N, v) \subset C(N, v'). \tag{4.3}$$

The proof of this result is trivial. Let $x \in C(N, v)$. Then, for all $S \subset N$, $x(S) \geq v(S) \geq v'(S)$. Since $x(N) = v(N) = v'(N)$, then $x \in C(N, v')$.

So, although the ideal situation is to have the complete characteristic function, sometimes this is not possible. This remark intends to underline the need to go further in games arising from complex problems in order to "fairly" share the general benefit.

Acknowledgments

The author wants to thank the Spanish Ministry of Science and Innovation under project MTM2010-19576-C02-01 and the Junta de Andalucía (Spain) under grant P09-TEP-5022, for their financial support.

References

[1] Alkan, A., 1988. Nonexistence of stable threesome matchings. Math. Soc. Sci. 19, 207–209.

[2] Bar-Shalom, Y., Li, X.-R., 1995. Multitarget-Multisensor Tracking: Principles and Techniques. Storrs, CT: YBS Publishing.

[3] Blackman, S., 1986. Multiple-target tracking with radar applications. Artech House.

[4] Capponi, A., 2004. Polynomial time algorithm for data association problem in multitarget tracking. IEEE T. Aero. Elec. Sys. 40 (4), 1398–1410.

[5] Curiel, I., 1997. Cooperative Game Theory and Applications. Kluwer Academic Publisher.

[6] Faigle, U., Fekete, S. P., Hochstfittler, W., Kern, W., 1998. On approximately fair cost allocation in Euclidean TSP games. OR Spektrum 20, 29–37.

[7] Garey, M., Johnson, D., 1979. Computers and Intractability. A Guide to the Theory of NP-completness. Freeman, San Francisco.

[8] Jain, K., Vazirani, V. V., 2001. Applications of approximation algorithms to cooperative games. In: STOC'01. Hersonissos, Crete, Greece.

[9] Morefield, L., 1977. Application of 0-1 integer programming to multitarget tracking problems. IEEE T. Automat. Contr. 22 (3), 302–312.

[10] Murphey, R., Pardalos, P., Pitsoulis, L., 1998. A GRASP for the multitarget multisensor tracking problem. In: Networks, Discrete Mathematics and Theoretical Computer Science Series. Vol. 40. pp. 277–302, American Mathematical Society.

[11] Owen, G., 1982. Game Theory. Academic Press.

[12] Perea, F., de Waard, H. W., 2011. Greedy and k-greedy algorithms for multidimensional data association. IEEE T. Aero. Elec. Sys. Scheduled for 47 (3).

[13] Pierskalla, W., 1968. The multidimensional assignment problem. Oper. Res. 16 (3), 422–431.

[14] Poore, A., 1994. Multidimensional assignment formulation of data association problems arising from multitarget and multisensor tracking. Comput. Optim. Appl. 3 (1), 27–57.

[15] Poore, A. B., Rijavec, N., Barker, T., 1992. Data association for track initiation and extension using multiscan windows. In: Drummond, O. (Ed.), Proceedings of Signal and Data Processing of Small Targets. pp. 432–441.

[16] Popp, R. L., Pattipati, K. R., Bar-Shalom, Y., 2001. M-Best S-D Assignment Algorithm with Application to Multiple Target Tracking. IEEE T. Aero. Elec. Sys. 37 (1), 22–39.

[17] Quint, T., 1991. The core of an m-sided assignment game. Game. Econ. Behav. 3, 487–503.

[18] Reid, D., 1979. An algorithm for tracking multiple targets. IEEE T. Automat. Contr. 24 (6), 843–854.

[19] Rochford, S., 1984. Symmetrically pairwise-bargained allocations in an assignment market. J Econ Theory 34 (2), 262–281.

[20] Shapley, L. S., 1953. A value for n-person games. In: Kuhn, H., Tucker, A. (Eds.), Contributions to the Theory of Games, volume II. Vol. 28 of Annals of Mathematical Studies. Princeton University Press, pp. 307–317.

[21] Shapley, L. S., Shubik, M., 1971. The assignment game I: The core. Int. J. Game Theory (1), 111–130.

[22] Storms, P., Spieksma, F., 2003. Geometric three-dimensional assignment problems. Comput. Oper. Res. 7 (30), 1067–1085.

[23] Stuart, H. W., 1997. The supplier-firm-buyer game and its m-sided generalization. Math. Soc. Sci. 34, 21–27.

[24] Tejada, O., 2010. A note on competitive prices in multilateral assignment markets. Economics Bulletin 30 (1).

[25] Tejada, O., Rafels, C., 2010. Symmetrically multilateral-bargained allocations in multi-sided assignment markets. Int. J. Game Theory 39 (1), 249–258.

[26] Turkmen, I., Guney, K., Karaboga, D., 2004. Tabu search tracker for multiple target tracking. J. Electromagnet. Wave. 18 (12), 1573–1589.

[27] Vazirani, V., 2001. Approximation Algorithms. Springer.

About the author

Federico Perea Rojas-Marcos is Associate Professor at the Polytechnic University of Valencia since October 2010. He obtained his PhD in Mathematics at the University of Sevilla in 2007. During his academic career he has previously worked for the same university in the department of Statistics and Operations Research, and in the department of Applied Mathematics II. Later on he joined the department of Statistics at the University of Zaragoza. His research lines focus on game theory and transportation. He has published several papers in international journals within the areas of operations research, transportation and applied mathematics.