A note on the torsion elements in the centralizer of a finite index subgroup

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1 Introduction

This paper is inspired by the following theorem from algebraic topology: Let Π be the fundamental group of a closed aspherical manifold and let $1 \to \Pi \to E \to F \to 1$ be any extension of Π by a finite group F, then the set of torsion elements of $C_E\Pi$ is a characteristic subgroup of E (cf. [4], [2], [3], [1],...). In the topological setting, this group of torsion elements occurs as the kernel of a properly discontinuous action.

This theorem was proved in several steps, where the first step was the case where Π was free abelian of rank k. However, although the problem could be stated in a pure algebraic way, the author could not locate any algebraic proof. E.g. to prove the (key) case where $\Pi = \mathbb{Z}^k$, one uses the specialised theory of injective toral actions, which is not easily accessible for pure algebrists.
Nevertheless, as we will show, it is not the theorem itself which depends upon the topological properties of the group Π, rather the proofs themselves. The statement of the theorem is valid in a much more general situation and is of interest from the algebraic point of view too. Therefore we would like to present a completely algebraic approach here.

2 The results

For groups G and H, we will use the following notations: C_GH is the centralizer of H in G, $Z(G)$ denotes the center of G and $\tau(G)$ equals the set of torsion elements

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in G.

Lemma 2.1
Let $1 \to \mathbb{Z}^k \to E \to F \to 1$ *be any central extension, where* F *is a finite group. Then* $\tau(E)$ *is a characteristic subgroup of* E.

Proof : Once we know that $\tau(E)$ is a group, it follows automatically that it is characteristic. As E is a central extension of $\mathbb{Z}^k$ by F, we may view E as being the set $\mathbb{Z}^k \times F$, where the multiplication $*$ is given by

$$(a, \alpha) * (b, \beta) = (a + b + c(\alpha, \beta), \alpha\beta), \ \ \forall a, b \in \mathbb{Z}, \forall \alpha, \beta \in F, \tag{1}$$

for some 2–cocycle $c : F \times F \to \mathbb{Z}^k$. Since $\mathbb{Q}^k$ is a vector space, the inclusion map $i : \mathbb{Z}^k \to \mathbb{Q}^k$ induces a trivial map $i_* : H^2(F, \mathbb{Z}^k) \to H^2(F, \mathbb{Q}^k) = 0$ on the cohomology level. This means that there is a split short exact sequence $1 \to \mathbb{Q}^k \to E' \to F \to 1$, where E' denotes the group determined by the cocycle $i(c)$. (I.e. $E' = \mathbb{Q}^k \times F$ and multiplication is given by (1), where a and b may now belong to $\mathbb{Q}^k$.)
So there is a splitting morphism $s : F \to E'$. But it is now easy to see that s is unique, since for all $f \in F$, $s(f)$ has to be a torsion element and there is only one torsion element in E' mapping to f. This shows that $s(F) = \tau(E')$ and so $\tau(E')$ is a group. The proof finishes, by realizing that $E \subset E'$ and so $\tau(E)$ ($\subseteq \tau(E')$) has to be a group also.

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Remark 2.2

1. The proof of the lemma might suggest that the group E can be decomposed into a direct sum $E = \mathbb{Z}^k \oplus F'$ for some finite group F'. However, this is not true: consider the group $E = (\mathbb{Z} \oplus \mathbb{Z}_2) \rtimes \mathbb{Z}_2$ where $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$ and the action of $\mathbb{Z}_2$ on $\mathbb{Z} \oplus \mathbb{Z}_2$ is given by

$$^{\bar{1}}(1, \bar{0}) = (1, \bar{1}) \text{ and } ^{\bar{1}}(0, \bar{1}) = (0, \bar{1}).$$

The group $A = 2\mathbb{Z} \subseteq E$ is indeed a free abelian, central subgroup of finite index, so the conditions of the lemma are satisfied, but E cannot be seen as the direct sum of a free abelian group and a finite one.

2. If we examine the conditions of the lemma we quickly see that the lemma is false if we are not looking at central extensions (E.g. $k = 1$, $F = \mathbb{Z}_2$ and $E = \mathbb{Z} \rtimes \mathbb{Z}_2$, where $\mathbb{Z}_2$ acts non trivially on $\mathbb{Z}$) or at extensions of infinite index (E.g. $k = 0$ and $F = E = \mathbb{Z} \rtimes \mathbb{Z}_2$). However, as the following lemma shows, there is no need for a *free* abelian kernel.

Lemma 2.3
Let A *be any abelian group. If* $1 \to A \to E \to F$ *is any central extension, where* F *is finite, then* $\tau(E)$ *forms a chracteristic subgroup of* E.

Proof : $\tau(A)$ is a characteristic subgroup of A, and so there is an induced short exact sequence

$$1 \to A/\tau(A) \to E/\tau(A) \to F \to 1.$$

From this it follows that it will suffice to prove the theorem in case A is torsion free abelian.
Suppose A is torsion free and $x, y \in \tau(E)$. We have to show that $xy \in \tau(E)$. Let E' denote the group generated by x and y, and $A' = A \cap E'$. There is an induced extension

$$1 \to A' \to E' \to F' \to 1$$

where F' is some finite group. Since E' is finitely generated and F' is finite, we may conclude that A' is finitely generated too ([5, p. 117]). Therefore, we are in the situation of lemma 2.1, which implies that $\tau(E')$ is a group, and so $xy \in \tau(E') \subseteq \tau(E)$.

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Theorem 2.4
Let E be any subgroup of finite index in a given group E', then $\tau(C_{E'}E)$ is a subgroup of E'. Moreover, if E is torsion free and normal then $\tau(C_{E'}E)$ is the unique maximal normal torsion subgroup of E'.

Proof : First, let us consider the case where E is normal in E'. There is an exact sequence of subgroups of E':

$$1 \to Z(E) \to C_{E'}E \to F \to 1$$

for some finite group F. It follows immediately from lemma 2.3, that $\tau(C_{E'}E)$ is a subgroup of E. It is normal in E, since it is characteristic in another normal subgroup $(C_{E'}E)$. To proof the last statement it is enough to realize that, in case E is torsion free, any normal subgroup T, containing only torsion, commutes with E. Consider elements $e \in E$ and $t \in T$, then we see that the commutator

$$[e,t] = e^{-1}t^{-1}et = \underbrace{(e^{-1})}_{\in E}\underbrace{(t^{-1}et)}_{\in E} = \underbrace{(e^{-1}t^{-1}e)}_{\in T}\underbrace{(t)}_{\in T}$$

belongs to $E \cap T = \{1\}$ which finishes the proof for $E \triangleleft E'$.

If E is not normal in E', we replace E' by the normalizer $N_{E'}E$ of E in E'. Since $C_{E'}E \subseteq N_{E'}E$ we may apply the theorem for normal E to conclude the correctness of the theorem in the general case too.

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