Projective bundles

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Dedicated to J. A. Thas on his fiftieth birthday

Abstract

A projective bundle in $\mathrm{PG}(2,q)$ is a collection of q^2+q+1 conics that mutually intersect in a single point and hence form another projective plane of order q. The purpose of this paper is to investigate the possibility of partitioning the q^5-q^2 conics of $\mathrm{PG}(2,q)$ into $q^2(q-1)$ disjoint projective bundles. As a by-product we obtain a multiplier theorem for perfect difference sets that generalizes a portion of Hall's theorem.

1 Introduction

There are $q^5-q^2 = q^2(q-1)(q^2+q+1)$ nondegenerate conics in the desarguesian projective plane $\pi_0 = \mathrm{PG}(2,q)$ of order q [6, p. 140]. Moreover, it is not hard to find (see [1, §8], [5, p. 1085], or [8]) a collection of q^2+q+1 nondenegerate conics in π_0 that mutually intersect in exactly one point, and hence serve as the "lines" of another projective plane on the points of π_0.

We will call such a collection of conics a *projective bundle*. The issue of concern for this paper is whether the q^5-q^2 conics of π_0 can be partitioned into $q^2(q-1)$ projective bundles. We exhibit a collection of $q^2(q-1)/2$ disjoint bundles for any odd prime power q, and show that a slightly larger number of disjoint bundles may be constructed for $q=3$. When q is even, a similar construction produces only $q-1$ disjoint bundles, although a computer-aided search for $q=4$ produced 30 disjoint bundles. It seems unlikely, however, that a complete partitioning of the conics of π_0 into projective bundles is possible. We also discuss the connections of this problem

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to the quadric Veronese surface of $\mathrm{PG}(5,q)$, to the construction of nondesarguesian 3-dimensional translation planes, and to perfect difference sets.

2 Disjoint projective bundles of type $\mathcal{B}(P)$

For a prime power q, $\mathrm{GF}(q)$ denotes the finite field of q elements, and $\pi_0 = \mathrm{PG}(2,q)$ the desarguesian projective plane coordinatized by $\mathrm{GF}(q)$. We embed π_0 into $\pi = \mathrm{PG}(2,q^3)$, and let σ denote the period 3 collineation of π that fixes the points of π_0. The points of π fall into three disjoint classes, depending upon the orbit structure under the group $\langle\sigma\rangle$:

- the points of π_0 (for which $P^\sigma = P$),
- the points of $\pi\backslash\pi_0$ lying on lines of π_0 (for which $P^{\sigma^2} \in PP^\sigma$),
- the points not on any line of π_0 (for which $PP^\sigma P^{\sigma^2}$ is a triangle).

The lines also fall into three disjoint classes, the duals of the point classes.

We define a *real conic* to be a conic of π that meets π_0 in a nondegenerate conic of π_0. It is a simple consequence of the classical theory (see for instance [11, Chapter X, especially exercises 9-12 on pages 296-297]) that, if P is not on any line of π_0, then there are exactly q^2+q+1 real conics through P, P^σ, and P^{σ^2} — one through each pair of points of π_0 — and these conics induce a projective bundle of π_0. We will denote this projective bundle by $\mathcal{B}(P)$ (for any point P not on a line of π_0). Of course, $\mathcal{B}(P) = \mathcal{B}(P^\sigma) = \mathcal{B}(P^{\sigma^2})$. We now address the question of finding as many disjoint projective bundles of this type as possible.

Lemma 2.1 *A disjoint pair of triangles in π that are in perspective from a point O and line ℓ have all six vertices on one conic if and only if the central collineation having center O and axis ℓ is an involution that interchanges the two triangles.*

Proof. This is an immediate consequence of Desargues' involution theorem; see [3, §9.3, especially exercise 3, p. 88]. □

Note that when q is odd, $O \notin \ell$ and the involution is a homology; when q is even, $O \in \ell$ and the involution is an elation [3, **6.32** and p. 132].

The lemma enables us to use a bundle $\mathcal{B}(P)$ to generate a family of disjoint projective bundles. For our construction we distinguish in π_0 a line ℓ and a point $Q \notin \ell$. Let $\mathcal{E}$ be the group of the q^2 elations of π_0 whose axis is ℓ, and let $\mathcal{H}$ be the group of the $q-1$ homologies of π_0 having axis ℓ and center Q.

Case 1. *q is odd.* Denote by J the involution with center Q and axis ℓ. Let $\mathcal{H}'$ be a subset of $(q-1)/2$ homologies from $\mathcal{H}$ with the property that $H \in \mathcal{H}'$ if and only if $HJ \notin \mathcal{H}'$. In particular, since $\mathcal{H}$ is a cyclic group of order $q-1$, let H be a generator of $\mathcal{H}$ and define

$$\mathcal{H}' = \{H^k \mid k = 0, 1, \cdots, (q-3)/2\}.$$

Let P be any point of π not on a line of π_0, and consider the set of images of P under the $q^2(q-1)/2$ central collineations EH' with $E \in \mathcal{E}$ and $H' \in \mathcal{H}'$. If $E_1 H_1{}'$ and $E_2 H_2{}'$ are any two such (distinct) collineations, then the images of the triangle $PP^\sigma P^{\sigma^2}$ under these two maps are clearly disjoint. The resulting two triangles, say T_1 and T_2, will be in perspective from the line ℓ and some point O, where O is the center of the central collineation $(E_1 H_1{}')^{-1}(E_2 H_2{}')$. But $(E_1 H_1{}')^{-1}(E_2 H_2{}') = E(H_1{}')^{-1} H_2{}'$ for some $E \in \mathcal{E}$ as the elations with axis ℓ form a normal subgroup of the perspectivities with axis ℓ. Since the q^2 involutions with axis ℓ are precisely $\{EJ \mid E \in \mathcal{E}\}$, the definition of $\mathcal{H}'$ and the above lemma imply that the projective bundles determined by T_1 and T_2 are disjoint.

Case 2. *q is even.* When q is even, no collineation of $\mathcal{H}$ is an involution. Arguing as above, for any point P not on a line of π_0, the image of $PP^\sigma P^{\sigma^2}$ under the $q-1$ homologies of $\mathcal{H}$ produce $q-1$ disjoint bundles.

To summarize, we have the following result.

Theorem 2.2 *The above construction produces $q^2(q-1)/2$ disjoint projective bundles of type $\mathcal{B}(P)$ when q is odd, and $q-1$ of them when q is even.*

We have been unable to find any general construction for partitioning more than half the conics of $\mathrm{PG}(2,q)$ into projective bundles using bundles of type $\mathcal{B}(P)$. Perhaps by employing different kinds of projective bundles one can do better. We will investigate this possibility in the next section. However, we first briefly describe an alternative construction for $q^2(q-1)/2$ disjoint bundles of type $\mathcal{B}(P)$ when q is odd.

Let ℓ now represent any line of π that meets π_0 in a single point, say R. By choosing left-normalized coordinates so that $\ell = < (0,1,0),(1,0,\beta) >$ and $R = (0,1,0)$, where β is a primitive element of $\mathrm{GF}(q^3)$, it is easy to see that $\{(1, r+s\beta, \beta) \mid r,s \in \mathrm{GF}(q)\}$ are the points of ℓ lying on exactly one line of π_0, while $\{(1, r+s\beta+t\beta^2, \beta) \mid r,s,t \in \mathrm{GF}(q),\ t \neq 0\}$ are the points of ℓ lying on no line of π_0. We next partition $\mathrm{GF}(q) \setminus \{0\}$ (q odd) into two subsets F_1 and F_2 so that each element of F_2 is the additive inverse of some element in F_1, and vice-versa. Then a straightforward, but somewhat cumbersome, coordinate argument shows that $\{\mathcal{B}(P) \mid P = (1, r+s\beta+t\beta^2, \beta);\ r,s \in \mathrm{GF}(q),\ t \in F_1\}$ is a collection of $q^2(q-1)/2$ mutually disjoint projective bundles. It is interesting to note that in this construction all the points P lie on a line, and hence we might call the above collection of disjoint bundles a "collinear" set.

3 Other types of projective bundles

In his Ph. D. Thesis [4], David Glynn studied three types of projective bundles. (He calls them *packings*.) Let us fix a triangle $PP^\sigma P^{\sigma^2}$ and denote it by T. The known types of projective bundles are as follows.

1. A *circumscribed bundle* consisting of all real conics containing the three vertices of T — the type $\mathcal{B}(P)$ of the previous section; these exist for all q.

2. An *inscribed bundle* consisting of all real conics that are tangent to the three sides of T; these exist for all odd q.

3. A *self-polar bundle* consisting of all real conics with respect to which T is self-polar; these exist for all odd q.

The first and third types are *bundles* in the classical sense of the word: if $f = 0, g = 0, h = 0$ are the equations of three real conics with h not a linear combination of f and g, then the bundle is the system of conics given by the equation $\lambda f + \mu g + \nu h = 0$ with $\lambda, \mu, \nu \in \mathrm{GF}(q)$. It is easy to verify that these systems are projective bundles (cf.[11, p. 297]). By contrast, the definition of an inscribed bundle (with its three conjugate imaginary common tangents) implies that any two of its conics have a unique common tangent line in π_0. The projective bundle requirement that any two of these conics meet in exactly one point of π_0 is an immediate consequence of the next result.

Theorem 3.1 *When q is odd, two real conics have one common point in π_0 and three in $\pi \setminus \pi_0$ if and only if they have one common tangent in π_0 and three in $\pi \setminus \pi_0$.*

Proof. The number of intersection points in π_0 of two real conics C and D has the same parity as the number of common tangents they have in π_0. To see this, simply count the pairs (P, t) where t is a tangent of D and $P \in t \cap C$. If P is an external point of D (on two of its tangents), then P is in exactly two pairs, while any common point (of $C \cap D$) belongs to a unique pair. Dually, any tangent t of D that is a secant of C is in two pairs, while any common tangent is in one pair. To finish the argument, observe that finding the points of $C \bigcap D$ involves solving a quartic equation; consequently, if there were three roots in $\mathrm{GF}(q)$, there would necessarily be a fourth. In other words, 1 is the only possible odd number of real roots. Dually, if the number of real tangents is odd, then that number must be 1. $\square$

Corollary

In $\mathrm{PG}(2, q)$ with q odd, the dual of a projective bundle is a collection of line conics whose envelopes constitute the conics of a projective bundle.

Glynn studied his families of conics by associating each conic of π_0 with a point of $\mathrm{PG}(5, q)$. The conics of the self-polar and circumscribed bundles correspond in this model to the points of certain planes in $\mathrm{PG}(5, q)$; the conics of an inscribed bundle correspond to the points of a quadric Veronesean. (Background information about this model can be found in [7, §25.1] or [10].)

Instead of going to five dimensions, one can study Glynn's projective bundles with the help of perfect difference sets [6, §4.2]. The points of π_0 =PG$(2, q)$ are represented by $\mathbf{Z}_N$, the integers mod N, where $N = q^2 + q + 1$; the lines are obtained from the perfect difference set $D = \{d_0, d_1, \cdots, d_q\}$ as the images of D under the *Singer cycle* S defined by $S(i) = i + 1$ (for all $i \in \mathbf{Z}_N$).

Theorem 3.2 *For $N = q^2 + q + 1$, if $r \in \mathbf{Z}_N$ is relatively prime to N and D is a perfect difference set for $PG(2, q)$, then $D/r = \{d/r \mid d \in D\}$ is the point set of a curve of degree r.*

Proof. As a collineation of π_0 =PG$(2, q)$, a Singer cycle S induces a collineation of π =PG$(2, q^3)$ that fixes the vertices $P, P^\sigma, P^{\sigma^2}$ of a triangle we denote by T. (As before, σ is the collineation that fixes the points of π_0.) Introduce new coordinates so that T is the triangle of reference $(1, 0, 0), (0, 1, 0), (0, 0, 1)$, and $(1, 1, 1)$ is a point of π_0 corresponding to some element of D. S is then represented by a diagonal matrix,

$$S = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix} \qquad \lambda, \mu, \nu \in \mathrm{GF}(q^3).$$

Note that π_0 is the orbit of $(1, 1, 1)$ under successive powers of S. Elements $d \in D$ correspond to points with coordinates $(\lambda^d, \mu^d, \nu^d)$ that satisfy the equation of some line $ax+by+cz = 0$ $(a, b, c \in \mathrm{GF}(q^3))$. For fixed $r \in \mathbf{Z}_N$ (relatively prime to N) and $d \in D$, the element d/r (mod N) corresponds to the point $(\lambda^{d/r}, \mu^{d/r}, \nu^{d/r})$, which satisfies the equation

$$ax^r + by^r + cz^r = 0,$$

an equation of degree r as desired. $\square$

A theorem of Hall [5, Theorem 4.5] states that *in any cyclic projective plane of order n, if p is a prime divisor of n, then pD is a line.* For the plane PG$(2, q)$ this follows from our theorem since $x \to x^p$ is a field automorphism for any p that divides q. Other simple consequences:

1. $\mathbf{r} = -\mathbf{1}$. Then $D/r = -D$ satisfies $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 0$ (or equivalently, $yz+zx+xy = 0$), which is a conic that contains the vertices of T. S takes $-D$ to the other conics of the circumscribed bundle. (Bruck worked out this case and the next in [1, §8]).

2. $\mathbf{r} = \frac{\mathbf{1}}{\mathbf{2}}$. Then $D/r = 2D$ satisfies $x^{1/2} + y^{1/2} + z^{1/2} = 0$ (or, when rationalized, $x^2 + y^2 + z^2 - 2xy - 2yz - 2zx = 0$), which is a conic tangent to the sides of T when q is odd. S takes $2D$ to the other conics of the inscribed bundle.

3. $\mathbf{r} = \mathbf{2}$. Then $D/r = D/2$ satisfies $x^2 + y^2 + z^2 = 0$, a conic for which T is self-polar (when q is odd). S takes $D/2$ to the other conics of the self-polar bundle. (Hall discussed this case in his famous paper [5, p. 1085].)

Note that the second example provides an independent proof that, when q is odd, an inscribed bundle is indeed a projective bundle. Moreover, as in the above examples, if $r \in \mathbf{Z}_N$ then $D/(-r)$ is a curve of degree $2r$ (just multiply $x^{-r} + y^{-r} + z^{-r} = 0$ by $x^r y^r z^r$). Similarly, rD is a curve of degree r (since $x^{1/r} + y^{1/r} + z^{1/r}$ can always be rationalized).

Using the software package CAYLEY, we conducted an exhaustive search to find the largest possible collection of disjoint projective bundles in PG$(2, 3)$, where bundles could be any one of the three known types. We found that the largest possible collection consists of 10 bundles, only one more than the set of bundles of type $\mathcal{B}(P)$ we constructed in the last section. We obtained such a collection by taking one circumscribed, three inscribed, and six self-polar bundles. The search was certainly disappointing; although conics in PG$(2, 3)$ are simply quadrangles, the outcome suggests the implausibility for any q that the conics of PG$(2, q)$ can be partitioned into projective bundles. However, it should also be mentioned that using the newly released software package MAGMA we were able to find 30 disjoint projective bundles in PG$(2, 4)$, all of type $\mathcal{B}(P)$ (the only known type for q even). Thus for $q = 4$ we were able to partition more than half the conics into projective bundles, although the pattern of these bundles remains a mystery. The only general partitioning construction we have for q even is the one previously mentioned in Section 2.

Curiously, A. Cayley [2] calculated in 1850 that when $q = 2$ our constructed "family" of disjoint bundles, with just $q - 1 = 1$ member, is best possible. Since a conic in PG$(2, 2)$ is a noncollinear triple of points, Cayley's result can be stated as follows: *there exist just two disjoint Steiner triple systems on seven points — one is formed by the lines of a* PG$(2, 2)$*, the other by the conics of a circumscribed bundle.*

4 Concluding remarks

It seems clear that new types of projective bundles must be found if one is to have any hope of partitioning all the conics of π_0 =PG$(2, q)$ into projective bundles. On the other hand, it is undoubtedly true that one can do much better than $q - 1$ disjoint projective bundles for q even, even if only bundles of type $\mathcal{B}(P)$ are used, as indicated by our previously mentioned search for $q = 4$.

As a footnote to our discussion we point out that Kirkman [9] found a complete set of seven disjoint Steiner triple systems on 9 points. Interpreting one of these systems as the 12 lines of an affine plane of order 3, the other six systems partition the parabolas of the plane into disjoint "affine bundles" Thus, the affine version of our problem has a positive solution when $q = 3$. It is easy to check that Kirkman's affine bundles do not extend to projective bundles.

In general the problem of determining the maximum number of disjoint Steiner systems on v points has attracted considerable attention, but relatively few results for block sizes larger than 4. If $S(t, k, v)$ denotes a Steiner system with parameters t, k, v and $d(t, k, v)$ denotes the maximum number of mutually disjoint $S(t, k, v)'s$ on the same point set, our results show that $d(2, q + 1, q^2 + q + 1) \geq 1 + q^2(q - 1)/2$ for any odd prime power q.

While the partitioning problem discussed in this paper is interesting in its own right, there is another motivation that should be mentioned. A partition of the conics of π_0 into circumscribed or self-polar bundles leads, via Glynn's model, to a collection of $q^3 - q^2$ disjoint planes in PG$(5, q)$ that partition the points corresponding to nondegenerate conics. If one could then find a collection of $q^2 + 1$ disjoint planes covering the remaining points, the union of these two partial spreads would give us an interesting 2-spread of $\mathrm{PG}(5, q)$, and thus a method for constructing translation planes that are 3-dimensional over their kernels.

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