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# Well Centered Spherical Quadrangles

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**Abstract.** We introduce the notion of a well centered spherical quadrangle or WCSQ for short, describing a geometrical method to construct any WCSQ. We shall show that any spherical quadrangle with congruent opposite internal angles is congruent to a WCSQ. We may classify them taking in account the relative position of the spherical moons containing its sides. Proposition 2 describes the relations between well centered spherical moons and WCSQ which allow the refereed classification.

Let $L$ be a spherical moon. We shall say that $L$ is *well centered* if its vertices belong to the great circle $S^2 \cap \{(x,y,z) \in \mathbb{R}^3 : x = 0\}$ and the semi-great circle bisecting $L$ contains the point $(1,0,0)$.

If $L_1$ and $L_2$ are two spherical moons with orthogonal vertices then $L_1$ and $L_2$ are said to be *orthogonal*.

Let us consider the class $\Omega$ of all spherical quadrangles with all congruent internal angles or with congruent opposite internal angles.

**Proposition 1.** *$Q \in \Omega$ if and only if $Q$ has congruent opposite sides.*

*Proof.* It is obvious that any spherical quadrangle, $Q$, with congruent opposite sides is an element of $\Omega$.

Suppose now, that $Q$ is an arbitrary element of $\Omega$. Then $Q$ has congruent opposite internal angles say, in cyclic order, $(\alpha_1, \alpha_2, \alpha_1, \alpha_2)$, with $\alpha_i \in (0, \pi)$, $i = 1, 2$, $\alpha_1 + \alpha_2 > \pi$. Lengthening two opposite sides of $Q$ we get a spherical moon, $L$, as illustrated in Figure 1.

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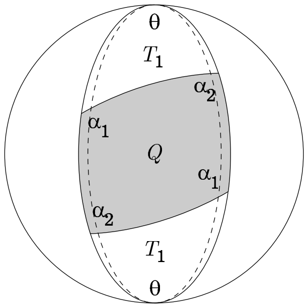


Figure 1

The moon $L$ includes $Q$ and two spherical triangles, $T_1$ and $T_2$. As $T_1$ and $T_2$ have congruent internal angles, they are congruent, and so the sides of $Q$ common to respectively $T_1$ and $T_2$ are congruent. The result now follows applying the same reasoning to the other pair of opposite sides of $Q$. □

**Proposition 2.** *Let $L_1$ and $L_2$ be two well centered spherical moons with distinct vertices of angle measure $\theta_1$ and $\theta_2$, respectively and let $Q$ be the spherical quadrangle $Q = L_1 \cap L_2$. Then $Q$ has internal angles and sides in cyclic order of the form, $(\alpha_1, \alpha_2, \alpha_1, \alpha_2)$ and $(a, b, a, b)$, respectively. Moreover, $L_1$ and $L_2$ are orthogonal if and only if $\alpha_1 = \alpha_2$, and $\theta_1 = \theta_2$ if and only if $a = b$.*

*Proof.* Let $L_1$ and $L_2$ be two well centered spherical moons with distinct vertices of angle measure $\theta_1$ and $\theta_2$, respectively. $L_1$ and $L_2$ divide the semi-sphere into 8 spherical triangles, labelled as indicated in Figure 2, $T_i$, $i = 1, \ldots, 8$ and a spherical quadrangle $Q = L_1 \cap L_2$.

Let $E$ and $N$ be vertices of $L_1$ and $L_2$, respectively, $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and $a, b, c, d$ be, respectively, the angles and sides of $Q$ in cyclic order (see Figure 2).

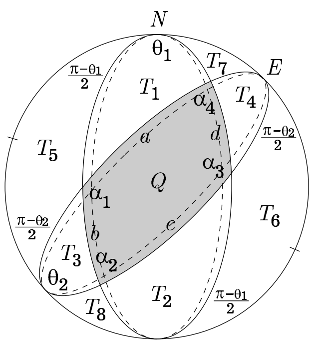


Figure 2

The triangles $T_5$ and $T_6$ are congruent (it is enough to verify that they have one congruent side and two congruent angles) and so $\alpha_1 = \alpha_3$. Also $T_7$ and $T_8$ are congruent and so $\alpha_2 = \alpha_4$. Since $T_5$ and $T_6$ are congruent and $T_7$ and $T_8$ are congruent we may conclude that $T_1$ and $T_2$ are congruent as well as $T_3$ and $T_4$ and so $a = c$ and $b = d$.

Now, $\theta_1 = \theta_2$ if and only if $T_1$ and $T_4$ are congruent, that is, if and only if $a = b$.

Besides, $L_1$ and $L_2$ are orthogonal iff $E \cdot N = 0$, where $\cdot$ denotes the usual inner product in $\mathbb{R}^3$, iff $T_6$ and $T_7$ are congruent iff $\alpha_1 = \alpha_2$. □

**Corollary 1.** *Using the same terminology as before one has*

i) *If $\theta_1 = \theta_2$ and $E \cdot N = 0$ then $Q = L_1 \cap L_2$ has congruent internal angles and all congruent sides;*

ii) *If $\theta_1 = \theta_2$ and $E \cdot N \neq 0$ then $Q = L_1 \cap L_2$ has all congruent sides and distinct congruent opposite pairs of angles;*

iii) *If $\theta_1 \neq \theta_2$ and $E \cdot N = 0$ then $Q = L_1 \cap L_2$ has congruent internal angles and distinct congruent opposite pairs of sides;*

iv) *If $\theta_1 \neq \theta_2$ and $E \cdot N \neq 0$ then $Q = L_1 \cap L_2$ has distinct congruent opposite pairs of angles and distinct congruent opposite pairs of sides.*

By a *well centered spherical quadrangle* (WCSQ) we mean a spherical quadrangle which is the intersection of two well centered spherical moons with distinct vertices.

**Proposition 3.** *Let $Q$ be a spherical quadrangle with congruent internal angles, say $\alpha \in (\frac{\pi}{2}, \pi)$, and with congruent sides, $a$. Then $a$ is uniquely determined by $\alpha$.*

*Proof.* The diagonal of $Q$ divides $Q$ in two congruent isosceles triangles of angles $(\alpha, \frac{\alpha}{2}, \frac{\alpha}{2})$. Thus, if $a$ is the side of $Q$ one has

$$\cos a = \frac{\cos\frac{\alpha}{2}(1+\cos\alpha)}{\sin\frac{\alpha}{2}\sin\alpha} = \frac{1+\cos\alpha}{1-\cos\alpha}$$

We can observe that this relation defines an increasing continuous bijection between $\alpha \in (\frac{\pi}{2}, \pi)$ and $a \in (0, \frac{\pi}{2})$. □

**Proposition 4.** *Let $Q$ be a spherical quadrangle with congruent internal angles, say $\alpha \in (\frac{\pi}{2}, \pi)$, and with congruent sides. Then $Q$ is congruent to a WCSQ.*

*Proof.* Let $Q$ be a spherical quadrangle with congruent internal angles, $\alpha \in (\frac{\pi}{2}, \pi)$, and with all congruent sides.

Consider two spherical moons well centered and orthogonal, $L_1$ and $L_2$ with the same angle measure $\theta \in (0, \pi)$ such that $\cos\theta = 2\cos\alpha + 1$ and $Q^\star = L_1 \cap L_2$, see Figure 3. Let us show that $Q$ is congruent to $Q^\star$. By Corollary 1, $Q^\star$ has congruent internal angles and congruent sides.

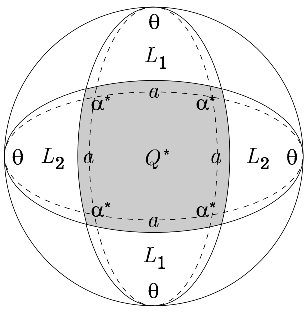


Figure 3

Denoting by $\alpha^\star \in (\frac{\pi}{2}, \pi)$ the internal angle of $Q^\star$ one has,

$$\cos\alpha^\star = -\cos^2\frac{(\pi-\theta)}{2} + \sin^2\frac{(\pi-\theta)}{2}\cos\frac{\pi}{2} = -\sin^2\frac{\theta}{2} = \frac{\cos\theta - 1}{2} = \cos\alpha$$

Thus $\alpha = \alpha^\star$ and consequently $Q$ and $Q^\star$ are congruent, since they have internal congruent angles and by the previous proposition they also have congruent sides. It should be pointed out that the relation $\cos\theta = 2\cos\alpha + 1$ defines an increasing continuous bijection between $\alpha \in (\frac{\pi}{2}, \pi)$ and $\theta \in (0, \pi)$. □

**Proposition 5.** *Let $Q$ be a spherical quadrangle with congruent internal angles, say $\alpha \in (\frac{\pi}{2}, \pi)$, and with distinct congruent opposite pairs of sides, say $a$ and $b$. Then anyone of the parameters $\alpha$, $a$ or $b$ is completely determined by the other two.*

*Proof.* Let $Q$ be a spherical quadrangle in the above conditions. For $\alpha \in (\frac{\pi}{2}, \pi)$ and $a \in (0, \pi)$, $b$ is determined by the system of equations:

$$\begin{cases} \cos b &= \cos^2\frac{(\pi-a)}{2} + \sin^2\frac{(\pi-a)}{2}\cos\theta \\ \cos\theta &= -\cos^2(\pi-\alpha) + \sin^2(\pi-\alpha)\cos b \end{cases}$$

where $\theta$ is the angle indicated in Figure 4.

Therefore,

$$\cos b = -1 + \frac{2}{1 + \cot^2\frac{a}{2}\cos^2\alpha}$$

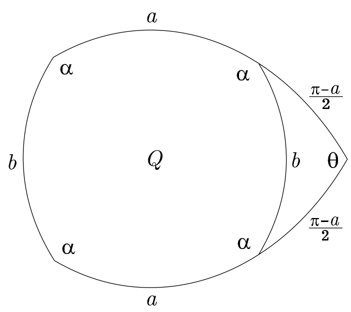


Figure 4

In a similar way, $a$ can be expressed as a function of $b$ and $\alpha$.

We shall show in next lemma that $\alpha$ can also be expressed as a function of $a$ and $b$. □

**Lemma 1.** *Let $Q$ be a spherical quadrangle with distinct congruent opposite pairs of sides, say $a$ and $b$ and with congruent internal angles, say $\alpha$. Then*

$$\cos\alpha = -\tan\frac{a}{2}\,\tan\frac{b}{2}$$

*Proof.* Let $Q$ be a spherical quadrangle in the above conditions. Lengthening the vertices of two adjacent edges, one gets two isosceles triangles with sides $a$, $\frac{\pi-b}{2}$, and $b$, $\frac{\pi-a}{2}$ respectively, see Figure 5.

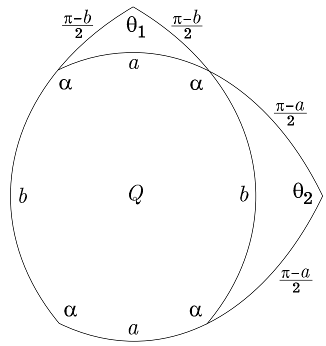

Figure 5

Let $\theta_1$ and $\theta_2$ be the internal angle measure of these triangles, see Figure 5. Then

$$\cos\alpha = -\cos\frac{\pi-\theta_1}{2}\cos\frac{\pi-\theta_2}{2} + \sin\frac{\pi-\theta_1}{2}\sin\frac{\pi-\theta_2}{2}\cos\frac{\pi}{2} = -\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}.$$

On the other hand,

$$\cos\theta_1 = \frac{\cos a - \cos^2\frac{\pi-b}{2}}{\sin^2\frac{\pi-b}{2}} = \frac{\cos a - \sin^2\frac{b}{2}}{\cos^2\frac{b}{2}}$$

and

$$\cos\theta_2 = \frac{\cos b - \cos^2\frac{\pi-a}{2}}{\sin^2\frac{\pi-a}{2}} = \frac{\cos b - \sin^2\frac{a}{2}}{\cos^2\frac{a}{2}}.$$

Thus

$$\cos\alpha = -\sqrt{\frac{1-\cos\theta_1}{2}}\sqrt{\frac{1-\cos\theta_2}{2}} = -\tan\frac{a}{2}\tan\frac{b}{2}.$$

□

**Proposition 6.** *Let $Q$ be a spherical quadrangle with congruent internal angles, say $\alpha \in (\frac{\pi}{2}, \pi)$ and with distinct congruent opposite sides, say $a \in (0, \pi)$ and $b = b(a, \alpha)$. Then $Q$ is congruent to a WCSQ.*

*Proof.* Suppose that $Q$ is a spherical quadrangle in the above conditions. We shall show that for two orthogonal well centered moons of angle measure, respectively, $\theta_1$ and $\theta_2$, the unique solution of the system of equations,

$$\begin{cases} \cos\alpha &= -\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} \\ \cos a &= \frac{\cos\theta_1+\cos^2\alpha}{\sin^2\alpha} \end{cases} \tag{1}$$

defines a well centered quadrangle (the moon's intersection) congruent to $Q$. In fact if a such well centered quadrangle exists then by Corollary 1 it has to be the intersection of two orthogonal moons $L_1$ and $L_2$ of angles $\theta_1 \in (0, \pi)$ and $\theta_2 \in (0, \pi)$, respectively.

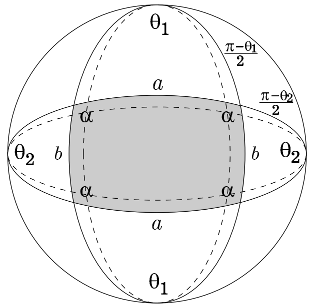


Figure 6

With the Figure 6 annotation, one has

$$\cos\alpha = -\sin\frac{\theta_1}{2}\,\sin\frac{\theta_2}{2},$$

as we have seen before. And

$$\cos a = \frac{\cos\theta_1 + \cos^2(\pi-\alpha)}{\sin^2(\pi-\alpha)} = \frac{\cos\theta_1 + \cos^2\alpha}{\sin^2\alpha}.$$

It is a straightforward exercise to show that the system of equations (1) has a unique solution and that $L1 \cap L2$ is congruent to $Q$. Observe that $\theta_1 \in (0,\,\pi)$, where the cosine function is injective and $\frac{\theta_2}{2} \in (0,\,\frac{\pi}{2})$, where the sine function is also injective. □

**Remark 1.** Let $\alpha : (0,\,\pi)\times(0,\,\pi) \to (\frac{\pi}{2},\,\pi)$ and $a : (0,\,\pi)\times(0,\,\pi) \to (0,\,\pi)$ be such that

$$\alpha(\theta_1,\theta_2) = \arccos(-\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}) \text{ and } a(\theta_1,\theta_2) = \arccos\frac{\cos\theta_1 + \sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}}{1-\sin^2\frac{\theta_1}{2}\sin^2\frac{\theta_2}{2}}.$$

The contour levels of $\alpha$ and $a$ are illustrated in Figure 7 (done by Mathematica).

We may observe that the intersection of any two contour levels of $\alpha$, and $a$ determine a unique pair of angles $(\theta_1,\,\theta_2) \in (0,\,\pi)\times(0,\,\pi)$, which means that a spherical quadrangle in the conditions of the last proposition is congruent to a well centered spherical quadrangle (the intersection of two orthogonal well centered spherical moons of angles $\theta_1$ and $\theta_2$).

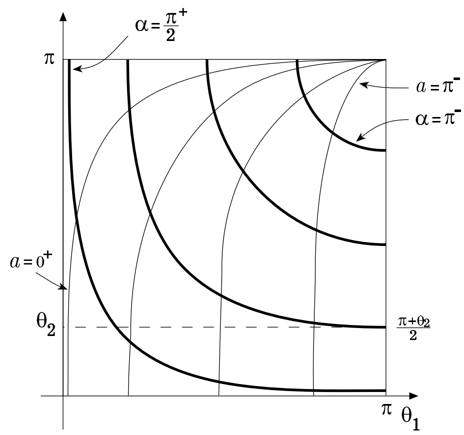


Figure 7

**Proposition 7.** *Let $Q$ be a spherical quadrangle with all congruent sides, say $a \in (0, \frac{\pi}{2})$ and with congruent opposite angles, say $\alpha_1, \alpha_2$, $\alpha_1 \geq \alpha_2$. Then $\alpha_1 \geq \arccos(1 - \frac{2}{1+\cos a})$ and anyone of the parameters $a$, $\alpha_1$ and $\alpha_2$ is completely determined by the other two.*

*Proof.* Suppose that $Q$ is in the above conditions.

1. If $\alpha_1 = \alpha_2 = \alpha$ then as seen in proposition 3, $\cos a = \frac{1+\cos\alpha}{1-\cos\alpha}$, that is, $\cos\alpha = 1 - \frac{2}{1+\cos a}$;
2. If $\alpha_1 > \alpha_2$ then a continuity argument allows us to conclude that $\alpha_1 > \arccos(1 - \frac{2}{1+\cos a}) > \alpha_2$. This can be seen dragging two opposite vertices of $Q$ along the diagonal of $Q$ containing them, see Figure 8.

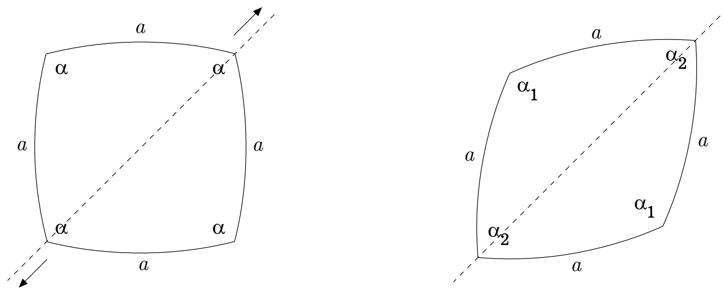


Figure 8

Now, given $a$ and $\alpha_1$, $\alpha_2$ is completely determined by the system of equations,

$$\begin{cases} \cos\alpha_2 & = -\cos^2\frac{\alpha_1}{2} + \sin^2\frac{\alpha_1}{2}\cos l \\ \cos l & = \cos^2 a + \sin^2 a \, \cos\alpha_2 \end{cases}$$

where $l$ denotes the diagonal of $Q$ bisecting $\alpha_1$, see Figure 9.

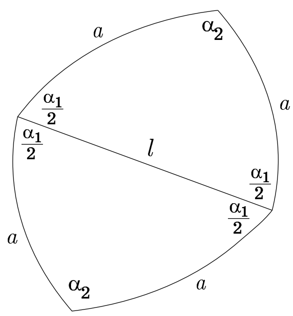


Figure 9

Thus,

$$\cos\alpha_2 = 1 - \frac{2}{1 + \tan^2\frac{\alpha_1}{2}\cos^2 a} \quad \text{and} \quad \cos a = \cot\frac{\alpha_1}{2}\cot\frac{\alpha_2}{2}.$$

□

**Proposition 8.** *Let $Q$ be a spherical quadrangle with all congruent sides, say $a \in (0, \frac{\pi}{2})$ and with congruent opposite pairs of angles, $\alpha_1, \alpha_2$, $\alpha_1 > \alpha_2$, with $\alpha_2 = \alpha_2(\alpha_1, a)$ and $\alpha_1 > \arccos(1 - \frac{2}{1+\cos a})$. Then $Q$ is congruent to a WCSQ.*

*Proof.* Let $Q$ be a spherical quadrangle as indicated above. Let us show first that when two well centered spherical moons with congruent angles, say $\theta$, and $\frac{\pi}{2} - x$, $x \in (0, \frac{\pi}{2})$ as the angle measure between them, then the following system of equations

$$\begin{cases} \cos\alpha_1 &= -\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\sin x \\ \cos a &= \frac{\cos\theta + \cos\alpha_1\cos\alpha_2}{\sin\alpha_1\sin\alpha_2} \end{cases}$$

has a unique solution which defines a WCSQ congruent to $Q$.

As seen in Corollary 1, if such well centered spherical quadrangle exists then it has to be the intersection of two well centered spherical moons with congruent angles $\theta \in (0, \pi)$, and such that the angle measure between them is $\frac{\pi}{2} - x$, $x \in (0, \frac{\pi}{2})$, see Figure 10.

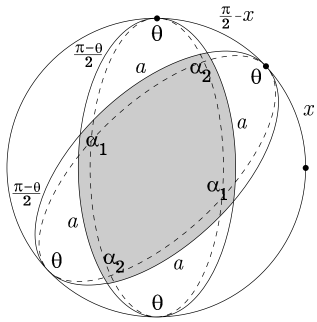


Figure 10

With the labelling of Figure 10 one has,

$$\cos\alpha_1 = -\cos^2\frac{\pi-\theta}{2} + \sin^2\frac{\pi-\theta}{2}\cos(\frac{\pi}{2}+x) = -\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\sin x$$

and on the other hand,

$$\cos a = \frac{\cos\theta + \cos(\pi-\alpha_2)\cos(\pi-\alpha_1)}{\sin(\pi-\alpha_2)\sin(\pi-\alpha_1)} = \frac{\cos\theta + \cos\alpha_1\cos\alpha_2}{\sin\alpha_1\sin\alpha_2}.$$

Using a similar argument to the one used in proposition 6 it can be seen that the solution is unique and that $Q$ is congruent to a WCSQ. □

**Remark 2.** Let $\alpha_1 : (0, \pi) \times (0, \frac{\pi}{2}) \to (\frac{\pi}{2}, \pi)$ and $a : (0, \pi) \times (0, \frac{\pi}{2}) \to (0, \frac{\pi}{2})$ be such that

$$\alpha_1(\theta, x) = \arccos(-\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\sin x)$$

and

$$a(\theta, x) = \arccos\frac{\cos\theta + (-\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\sin x)(-\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\sin x)}{\sqrt{1-(-\sin^2\frac{\theta}{2} - \cos^2\frac{\theta}{2}\sin x)^2}\sqrt{1-(-\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}\sin x)^2}}.$$

The contour levels of $\alpha_1$ and $a$ are represented in Figure 11 (done by Mathematica).

Observe that if $\alpha_1 \in (\frac{\pi}{2}, \pi)$ and $a \in (0, \frac{\pi}{2})$ such that $\alpha_1 \geq \arccos(1 - \frac{2}{1+\cos a})$ then the intersection of any two contour levels of $\alpha_1$ and $a$ is a unique point $(\theta, x) \in (0, \pi) \times (0, \frac{\pi}{2})$. In other words any spherical quadrangle in the conditions of the previous proposition is congruent to the intersection of two well centered spherical moons with the same angle measure, $\theta$, and being $\frac{\pi}{2} - x$ the angle measure between them.

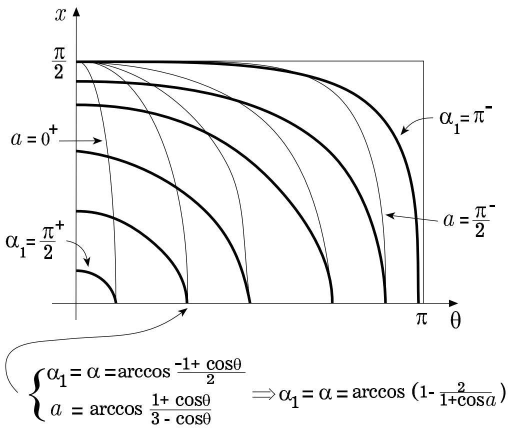


Figure 11

**Proposition 9.** *Let $Q$ be a spherical quadrangle with congruent opposite sides, say $a$ and $b$ and with congruent opposite angles, say $\alpha_1$ and $\alpha_2$ with $\alpha_1 \geq \alpha_2$. Then,*

i) $a + b < \pi$;

ii) $\alpha_1 \geq \arccos(-\tan\frac{a}{2}\tan\frac{b}{2})$;

iii) *anyone of the parameters $\alpha_1$, $\alpha_2$, $a$ or $b$ is completely determined by the other three.*

*Proof.* If $Q$ is quadrangle as described above then it follows that $0 < 2a + 2b < 2\pi$ and also $2\alpha_1 + 2\alpha_2 - 2\pi > 0$, with $\alpha_1 \in (0, \pi)$ and $\alpha_2 \in (0, \pi)$. That is, $0 < a + b < \pi$, $\alpha_1 + \alpha_2 > \pi$, $\alpha_2 \in (0, \pi)$ and $\alpha_1 \in (\frac{\pi}{2}, \pi)$, since $\alpha_1 \geq \alpha_2$.

Assume, in first place, that $\alpha_1 = \alpha_2 = \alpha$. Then, by lemma 1 we have $\cos\alpha = -\tan\frac{a}{2}\tan\frac{b}{2}$ and so $\alpha = \alpha_1 = \alpha_2 = \arccos(-\tan\frac{a}{2}\tan\frac{b}{2})$.

As before a continuity argument allows us to conclude that if $\alpha_1 > \alpha_2$, then $\alpha_1 > \arccos(-\tan\frac{a}{2}\tan\frac{b}{2}) > \alpha_2$.

Now, we show how to determine $\alpha_2$ as a function of $a$, $b$ and $\alpha_1$. The diagonal $l$ of $Q$ through $\alpha_2$ gives rise to two angles, $x$ and $y$, $(\alpha_2 = x + y)$ as illustrated in Figure 12.

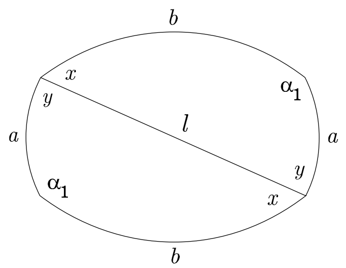


Figure 12

One has,

$$\cos l = \cos a \cos b + \sin a \sin b \cos \alpha_1.$$

Besides,

$$\cos x = \frac{\cos a - \cos b \cos l}{\sin b \sin l} \text{ and } \cos y = \frac{\cos b - \cos a \cos l}{\sin a \sin l}.$$

Since $\alpha_2 = x + y$, then $\alpha_2$ is function of $a$, $b$ and $\alpha_1$.

We can also determine $b$ as a function of $a$, $\alpha_1$ and $\alpha_2$ as follows. Let $b_1$, $b_2$ and $\theta$ be, respectively, the sides and the internal angle measure (to be determined) of the triangle obtained by lengthening the $b$ sides of $Q$, see Figure 13.

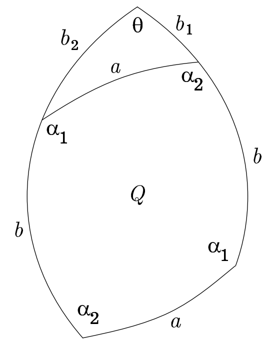


Figure 13

One has,

$$\cos\theta = -\cos\alpha_1\cos\alpha_2 + \sin\alpha_1\sin\alpha_2\cos a$$

and

$$\cos b_1 = -\frac{\cos\alpha_1 + \cos\alpha_2\cos\theta}{\sin\alpha_2\sin\theta} \quad \wedge \quad \cos b_2 = -\frac{\cos\alpha_2 + \cos\alpha_1\cos\theta}{\sin\alpha_1\sin\theta}$$

Finally, $b = \pi - (b_1 + b_2)$ is function of $a$, $\alpha_1$ and $\alpha_2$. □

**Proposition 10.** *Let $Q$ be a spherical quadrangle with congruent opposite sides, say $a$ and $b$ such that $a + b < \pi$ and with congruent internal angles $\alpha_1$, $\alpha_2$, $\alpha_1 > \alpha_2$. Let us suppose also that $\alpha_1 > \arccos(-\tan\frac{a}{2}\tan\frac{b}{2})$ and $\alpha_2 = \alpha_2(a, b, \alpha_1)$. Then, $Q$ is congruent to a WCSQ.*

*Proof.* Let $Q$ be a spherical quadrangle in the above conditions. We shall show that when we have two well centered spherical moons with angle measure $\theta_1$ and $\theta_2$ and such that $\frac{\pi}{2} - x$, $x \in (0, \frac{\pi}{2})$ is the angle measure between them, see Figure 14, then the unique solution of the system of equations

$$\begin{cases} \cos\alpha_1 &= -\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} - \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin x \\ \cos a &= \frac{\cos\theta_1 + \cos\alpha_1\cos\alpha_2}{\sin\alpha_1\sin\alpha_2} \\ \cos b &= \frac{\cos\theta_2 + \cos\alpha_1\cos\alpha_2}{\sin\alpha_1\sin\alpha_2} \end{cases}$$

defines a well centered spherical quadrangle congruent to $Q$.

As seen in Corollary 1, if a such WCSQ exists it should be the intersection of two well centered spherical moons (not orthogonal) with angles measure $\theta_1$ and $\theta_2$, $0 < \theta_i < \pi$, $i = 1, 2$ and with $\frac{\pi}{2} - x$, $0 < x < \frac{\pi}{2}$ as the angle measure between them, see Figure 14.

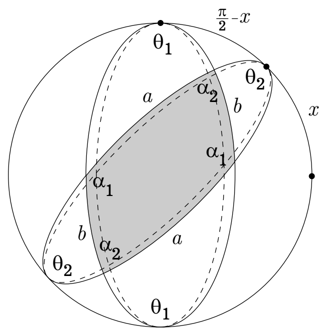


Figure 14

With the notation used in Figure 14 one has,

$$\begin{aligned}\cos\alpha_1 &= -\cos\frac{\pi-\theta_1}{2}\cos\frac{\pi-\theta_2}{2} + \sin\frac{\pi-\theta_1}{2}\sin\frac{\pi-\theta_2}{2}\cos(\frac{\pi}{2}+x)\\ &= -\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2} - \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\sin x\end{aligned}$$

On the other hand,

$$\cos a = \frac{\cos\theta_1 + \cos(\pi-\alpha_1)\cos(\pi-\alpha_2)}{\sin(\pi-\alpha_1)\sin(\pi-\alpha_2)} = \frac{\cos\theta_1 + \cos\alpha_1\cos\alpha_2}{\sin\alpha_1\sin\alpha_2}$$

and

$$\cos b = \frac{\cos\theta_2 + \cos(\pi-\alpha_1)\cos(\pi-\alpha_2)}{\sin(\pi-\alpha_1)\sin(\pi-\alpha_2)} = \frac{\cos\theta_2 + \cos\alpha_1\cos\alpha_2}{\sin\alpha_1\sin\alpha_2}.$$

As before, it is a straightforward exercise to state the uniqueness of the solution. □

## References

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