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Chiral Hypermaps of Small Genus

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Abstract. A hypermap $\mathcal{H}$ is a cellular embedding of a 3-valent graph $\mathcal{G}$ into a closed surface which cells are 3-coloured (adjacent cells have different colours). The vertices of $\mathcal{G}$ are called flags of $\mathcal{H}$ and let us denote by F the set of flags. An automorphism of the underlying graph which extends to a colour preserving self-homeomorphism of the surface is called an automorphism of the hypermap. If the surface is orientable the automorphisms of $\mathcal{H}$ split into two classes, orientation preserving and orientation reversing automorphisms. It is not difficult to observe that $|Aut(\mathcal{H})| \leq |\mathrm{F}|$ while for the group of orientation preserving automorphisms we have $|Aut^+(\mathcal{H})| \leq |\mathrm{F}|/2$. A hypermap satisfying $|Aut^+(\mathcal{H})| = |\mathrm{F}|/2 = |Aut(\mathcal{H})|$ will be called chiral. Hence chiral hypermaps have maximum number of orientation preserving symmetries but they are not "mirror symmetric".
The main goal of this paper is to classify all chiral hypermaps on surfaces of genus at most four. It follows that they consist of the infinite families of chiral toroidal hypermaps of types $(2,3,6)$, $(2,4,4)$, $(3,3,3)$, and their duals, and two exceptional chiral hypermaps (up to duality) of types $(3,3,7)$ and $(4,4,5)$. These exceptional chiral hypermaps are members of regular hypermaps with metacyclic oriented monodromy groups.

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1. Introduction

A map is a cellular decomposition of a closed surface. Maps on orientable surfaces can be described by means of two permutations R and L, with L being involutory, such that the group $\langle R, L\rangle$ acts transitively on the set D of darts of the map. The triple (D, R, L) determines the topological map up to isomorphism. If we relax the condition $L^2 = 1$ by considering any couple R, L of permutations of D generating a transitive group $G = \langle R, L\rangle$ of permutations we end with an (algebraic) definition of an oriented hypermap $\mathcal{H}^+ = (D, R, L)$. All the important notions such as genus, automorphism group, regularity and so on, extend naturally to hypermaps. The hypermap $\mathcal{H}$ is orientably regular if G acts regularly on D, and it is regular if it is orientably regular and the assignment $R \mapsto R^{-1}$ and $L \mapsto L^{-1}$ extends to a group automorphism. Orientably regular maps which are not regular will be called chiral. Orientably regular hypermaps are particularly nice objects since they make a bridge between geometry and algebra. In a sense, to study orientably regular hypermaps means to study two-generator groups with prescribed couples of generators.

By the well-known Hurwitz bound the size of the group $G = \langle R, L\rangle$ of an orientably regular hypermap $\mathcal{H} = (D, R, L)$ is bounded by 84(g-1), where $g > 1$ is the genus. One of the central problems in the theory of maps and hypermaps is the problem of classification of all orientably regular hypermaps on a fixed underlying surface. Using the Euler formula it is not difficult to see that spherical orientably regular hypermaps consist of the five Platonic solids, and of two infinite families of types $(1, n, n)$, $(2, 2, n)$ and their duals. Orientably regular maps on torus were classified by Coxeter and Moser [11], and the generalisation to hypermaps was done by Corn and Singerman [10]. In [4] the classification problem for double torus is settled. As concerns surfaces of higher genera only partial results are known. For instance, Conder and Dobcsányi [9] with the help of a computer program gave a list of all orientably regular maps up to genus 15.

In this paper we carry out the classification of chiral hypermaps with genus at most four. Chiral hypermaps form an interesting subset of the family of all orientably regular hypermaps, see for instance [11, 17, 6, 5]. They have maximum number of orientation preserving automorphisms but they are not isomorphic with their mirror images. It is worth to mention that by examining the list of orientably regular maps up to genus 15 [9] one can see that surfaces of genera 2, 3, 4, 5, 6, 9 and 13 support no chiral maps. The main result of the paper implies that there is no chiral hypermap on surface of genus 2, while each of the surfaces of genus 3 and 4 supports (up to duality) exactly one chiral hypermap, both with metacyclic oriented monodromy group (see [5] for a description of an infinite family of chiral hypermaps with metacyclic oriented monodromy group).

2. Preliminaries

A *topological hypermap* $\mathcal{H}$ is a cellular embedding of a connected trivalent graph $\mathcal{G}$ into a compact surface S, without boundary and not necessarily orientable, such that the cells are 3-coloured (say by black, grey and white colours) with adjacent cells having different colours. Numbering the colours 0, 1 and 2, and labelling the edges of $\mathcal{G}$ with the missing adjacent cell number, we can define 3 fixed points free involutory permutations r_i, $i = 0, 1, 2$, on the set F of vertices of $\mathcal{G}$; each r_i switches the pairs of vertices connected by i-edges (edges labelled i). The elements of F are called *flags* and the group G generated by r_0, r_1 and r_2 will be called the *monodromy group*[1] $Mon(\mathcal{H})$ of the hypermap $\mathcal{H}$. The cells of $\mathcal{H}$ coloured 0, 1 and 2 are called the *hypervertices*, *hyperedges* and *hyperfaces*, respectively. Since the graph $\mathcal{G}$ is connected, the monodromy group acts transitively on F and orbits of $\langle r_0, r_1\rangle$, $\langle r_1, r_2\rangle$ or $\langle r_0, r_2\rangle$ on F determine hypervertices, hyperedges and hyperfaces, respectively.

Given a topological hypermap $\mathcal{H}$ we can derive virtually six topological hypermaps on the same surface by permuting the three colours 0, 1, 2 of their cells; in fact, for each permutation $\sigma \in S_3 = S_{\{0,1,2\}}$ we define the *σ-dual* $D_\sigma\mathcal{H}$ to be the hypermap on the same surface, with the same underlying trivalent graph $\mathcal{G}$, whose hypervertices, hyperedges and hyperfaces are the cells coloured 0σ, 1σ and 2σ, respectively.

If the surface S is orientable, then we can, and as a rule we will, fix an orientation, for instance the counter-clockwise orientation. The subgroup G^+ generated by r_1r_2 and r_2r_0 acts on F with two orbits F^+ and F^-. Let $D = \mathrm{F}^+$ be the orbit such that r_1r_2 and r_2r_0 locally act on D like counter-clockwise rotations around hypervertices and hyperedges, respectively. It is well-known that an orientable hypermap $\mathcal{H}$ can be described via the associated *oriented hypermap* $\mathcal{H}^+ = (D, R, L)$, where D is the set of darts, and $R = r_1r_{2|_{\mathrm{F}^+}}$, $L = r_2r_{0|_{\mathrm{F}^+}}$. Generally, let D be an abstract set of darts and R and L be two permutations of D such that $\langle R, L\rangle$ acts transitively on D. Then the triple (D, R, L) defines a unique topological hypermap $\mathcal{H}$. Thus one can study properties of orientable hypermaps via their algebraic counterparts. Since all the hypermaps considered in this paper will be orientable we will freely use the term hypermap meaning either the topological hypermaps or its algebraic description by means of two permutations R and L generating the oriented monodromy group $Mon(\mathcal{H}^+) = G = \langle R, L\rangle$. We say that $\mathcal{H}$ is *orientably regular* if $Mon(\mathcal{H}^+)$ acts regularly on D, and that $\mathcal{H}$ is *regular* if moreover the assignment $R \mapsto R^{-1}$, $L \mapsto L^{-1}$ extends to a group automorphism. An orientably regular map which is not regular will be called *chiral*. An *orientation preserving automorphism* of $\mathcal{H}$ is a permutation of D commuting with both R and L. The group $\mathrm{Aut}^+(\mathcal{H})$ of orientation preserving automorphisms acts semi-regularly on D, and the action is regular if and only if $\mathcal{H}$ is orientably regular. In other words, a chiral hypermap has maximum number of orientation preserving automorphisms but it admits no orientation reversing automorphism. Since the action of G on darts in an orientably regular hypermap is regular, we can identify the darts of the hypermap with the elements of G, while the action of G on D can be interpreted as the right multiplication by elements of G. One can visualise this action by constructing the Cayley graph of G with respect to its two generators R and L.

An important and convenient way to visualise hypermaps is by bipartite maps introduced by Walsh in [22]. Topologically, a map can be seen as a cellular embedding of a graph in a compact surface and a hypermap as a cellular embedding of hypergraph in a compact surface. Since hypergraphs are in essence bipartite graphs (with one monochromatic set of vertices representing the hypervertices and the other monochromatic set of vertices representing the hyperedges) a hypermap can be viewed as a bipartite map. In fact, given any topological

[1] This group has been called the monodromy group of $\mathcal{H}$ [15, 19], the connection group of $\mathcal{H}$ [23] and the Ω-group of $\mathcal{H}$ [1].

hypermap $\mathcal{H}$ we can construct a topological bipartite map $W(\mathcal{H})$, called the *Walsh bipartite map* associated to $\mathcal{H}$ by taking first the dual of the underlying 3-valent map and then deleting the vertices (together with the edges attached to them) lying inside the hyperfaces of $\mathcal{H}$. The resulting map is bipartite with one monochromatic set of vertices lying on the faces coloured black, representing the hypervertices of $\mathcal{H}$, and the other monochromatic set lying on the faces coloured grey, representing the hyperedges. In Figure 1 we show the Walsh map of the Fano plane hypermap embedding.

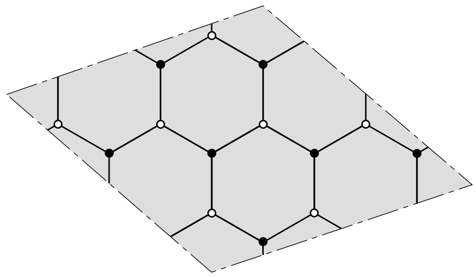


Figure 1: The Walsh map of the Fano plane embedding in Torus.

This construction can be reversed: given any topological bipartite map M, where the vertices are bipartitioned in black and grey, we construct an associated topological hypermap $W^{-1}(M)$=TD(M) by truncating the dual map D(M); the faces of the resulting 3-valent map TD(M) contains the vertices and the face-centres of the original map and are henceforth 3-colourable black, grey and white, with all these colours meeting at each vertex of TD(M).

To construct algebraically the (oriented) Walsh bipartite map $W_{\mathcal{H}} = W(\mathcal{H}) = (\bar{D}, R, L)$ from an (oriented) hypermap $\mathcal{H}^+ = (D, R_0, R_1)$ we set $\bar{D} = D \times C_2$, ($C_2 = \{0, 1\}$ denotes the additive group with two elements), $(x, i)R = (xR_i, i)$, and $(x, i)L = (x, i + 1)$, for $i = 0, 1$. Denote by $D_0 = D \times \{0\}$. The assignment $R_0 \mapsto R_{|D_0}$, $R_1 \mapsto LRL_{|D_0}$ defines a monomorphism from the oriented monodromy group $Mon(\mathcal{H}^+)$ into the oriented monodromy group $Mon({W_{\mathcal{H}}}^+)$ of the Walsh map. Having in mind that an automorphism of a hypermap is a colour preserving automorphism of the Walsh bipartite map, it is straightforward to prove the following theorem:

Theorem 1. *Let $\mathcal{H}$ be a hypermap and let its Walsh map be orientably regular. Then $\mathcal{H}$ is orientably regular, and $\mathcal{H}$ is chiral if and only if the Walsh map $W(\mathcal{H})$ is chiral.*

By the *type* of an orientably regular hypermap $\mathcal{H} = (D, R, L)$ we mean a triple $\{l, m, n\}$ of integers, where $l = ord(R)$, $m = ord(L)$ and $n = ord(RL)$. Note that the size of a cell corresponding to a hypervertex, a hyperedge, a hyperface in the associated topological hypermap is $2l$, $2m$, $2n$, respectively. A hypermap of type $\{l, 2, n\}$ for some l and n is called *map*. To each orientably regular hypermap $\mathcal{H}$ we associate the sequence of integers (called the *$\mathcal{H}$-sequence*) of the form $[N, \{l, m, n\}, \{V, E, F\}, |G|]$, where $\{l, m, n\}$ is the type (l, m, n) of $\mathcal{H}$; V, E, F are the number of hypervertices, hyperedges and hyperfaces of $\mathcal{H}$, respectively; G is the oriented monodromy group of $\mathcal{H}$ and $N = -\chi = 2g - 2 = |G| - V - E - F$ is the negative characteristic of the underlying surface of $\mathcal{H}$. Let us remark that by the definitions the following equations hold: $lV = mE = nF = |G|$. Since $|G| \leq 42N$ for $N > 0$, the number of possible $\mathcal{H}$-sequences for a fixed $N > 0$ is finite.

By the *chirality index* of an orientably regular hypermap with the oriented monodromy group $G = \langle R, L \rangle$ we mean the size of the smallest normal subgroup K of G such that the quotient $G/K = \langle KR, KL \rangle$ admits an automorphism inverting both generators KR and KL. Clearly, $\kappa(\mathcal{H})$ divides the size of G and $\kappa(\mathcal{H}) > 1$ provided $\mathcal{H}$ is chiral. The chirality index can be viewed as a measure of how much a given hypermap deviates from being mirror symmetric. Further information on the chirality index one can find in [6] or in [5]. To display the chirality index we shall write the $\mathcal{H}$-sequence in the extended form $[N, \{l, m, n\}, \{V, E, F\}, |G|, \kappa]$ as well.

For further information on maps, hypermaps and their algebraic counterparts the reader is referred to [7, 10, 12, 15, 16, 19, 21].

Next we list theorems which are the mains results of [5] and which play a vital role in this paper.

Theorem 2. *If $\mathcal{H}$ is an orientably regular hypermap with 1 or 2 hyperfaces then $\mathcal{H}$ is regular.*

Theorem 3. *If $\mathcal{H}$ is chiral with 3 hyperfaces of valency n then $n \geq 7$ and its oriented monodromy group is the metacyclic group $Mon(\mathcal{H}^+) = \langle a, b \mid a^n = 1, b^3 = a^s, bab^{-1} = a^r \rangle$, for some $s \in \{0, \dots, n-1\}$ and $r \in \{2, \dots, n-1\}$ satisfying $(r-1)s = 0\,(mod\, n)$ and $r^3 = 1\,(mod\, n)$; moreover, different solutions (r, s) correspond to different (non-isomorphic) hypermaps. Vice-versa, the group G with the above presentation defines an oriented hypermap (G, b, ab) (where b and ab acts on G by right multiplication) which is chiral and has 3 hyperfaces.*

Theorem 4. *If $\mathcal{H}$ is a chiral hypermap with 4 hyperfaces of valency n then $n \geq 5$ and its oriented monodromy group is the metacyclic group $Mon(\mathcal{H}^+) = \langle a, b \mid a^n = 1, b^4 = a^r, bab^{-1} = a^t \rangle$, for some $r \in \{0, \dots, n-1\}$ and $t \in \{1, \dots, n-1\}$ satisfying $(n, t) = 1$, $t^4 = 1\,(mod\, n)$ but $t^2 \neq 1\,(mod\, n)$, and $r(t-1) = 0\,(mod\, n)$; moreover different solutions (r, t) correspond to different hypermaps. Vice-versa, the group G with the above presentation defines an oriented hypermap (G, b, ab) (where b and ab act on G by right multiplication) which is chiral and has 4 hyperfaces.*

As we have seen two-generator groups play an important role in investigation of orientably regular hypermaps. In what follows we briefly explain a technical tool useful in study of actions of such groups.

Given a group $G = \langle a, b \rangle$, acting on some set X, by an (a, b)-*diagram* we mean a Schreier coset diagram where the right cosets are replaced by elements of X and the right multiplication by the action of G on X. The underlying undirected graph associated to any (a, b)-diagram is a 4-valent graph. If G acts transitively on X then the (a, b)-diagram is necessarily connected; in this case the elements of X corresponds to cosets of the point-stabiliser of any $x \in X$ in G and the (a, b)-diagram is no more then the Schreier coset diagram determined by the action and the stabiliser. We shall use a convention that drawing (a, b)-diagrams we glue two oppositely directed edges joining the same couple of vertices into one undirected edge. Also we will not assign the direction of edges keeping in mind that different orientations of monochromatic cycles of the same picture may give rise to non-isomorphic groups.

When the action of G on X projects onto the action of G on another set Y, that is, when the G-set X projects onto the G-set Y, then $|Y|$ divides $|X|$ and the (a,b)-diagram on X is said to be an *unfolded* (a,b)-diagram on X of the (a,b)-diagram on Y; if $\frac{|X|}{|Y|}=k$ then each element $y \in Y$ is "lifted" to k elements in X. Usually we will use the same notation to denote the permutation representation of $a, b \in G$ acting on different set of objects, bearing in mind the actions in general may be not faithful.

Let $\mathcal{F}$ denote the set of hyperfaces of a hypermap $\mathcal{H}$. If $\mathcal{H}$ is orientably regular then the orientation preserving automorphism group $G = Aut^+(\mathcal{H})$ acts transitively on $\mathcal{F}$ and on the set X of the stabilisers $Stab(f)$, $f \in \mathcal{F}$; this set forms a regular partition (that is, all sets are of the same size) of $\mathcal{F}$ and its elements are called *face-classes*. The action of G on $\mathcal{F}$ projects onto the action of G on X and consequently the number $|X|$ divides $|\mathcal{F}|$. If G is generated by a, b then the (a,b)-diagram on $\mathcal{F}$ projects onto the (a,b)-diagram on X. The induced action on faces-classes is investigated in [2].

Elements of Sylow theory will be used in the proof of the main result. The following lemma proved in [23] will be useful. For further information about Sylow theory the reader is referred to [8, 23].

Lemma 5. *Let G be a group of size $|G| = p^n(p+1)q$, where p is a prime and q is a number coprime to p, let S be a Sylow p-subgroup and $N = N(S)$ be its normaliser. If G has $p+1$ Sylow p-subgroups then the following statements hold:*

i) *There exists $s \in S$ cyclically permuting (by conjugation) the other p Sylow p-subgroups and so cyclically permuting the respective normalcies,*

ii) *If S is abelian or $N(S) = S$ then the intersection of S with one of its conjugates coincides with the p-core, and hence is normal in G.*

iii) *G acts transitively (by conjugation) on the complement C of the union of all conjugates of N provided $|C| = p$.*

3. Chiral hypermaps of genus $g \leq 4$

As it was already mentioned, there is no chiral map on the sphere, and the only toroidal chiral maps are the Coxeter maps $\{4,4\}_{a,b}$ and $\{6,3\}_{a,b}$, where $(b-c)bc \neq 0$. From the Garbe classification [14] of orientably regular maps up to genus 7 it follows that there are no chiral maps of genus $2 \leq g < 7$. Recently, Conder and Dobcsányi [9] have determined all orientably regular maps up to genus 15 and this classification says that from genus 7 up to 15, the surfaces of genus 9 and 13 support no chiral maps.

What about hypermaps?

The answer for $N \leq 6$, where $N = 2g-2$ denotes the negative characteristic follows.

$\boldsymbol{N=-2}$. *There are no chiral hypermaps on the sphere.*

Up to duality all orientably regular hypermaps on the sphere are maps. These are the five Platonic solids and two infinite families of types $(2,2,n)$, $(n,1,n)$ and their duals. All these maps are regular.

$N = 0$. *Besides the Coxeter chiral maps, the only chiral hypermaps on the Torus are the hypermaps of type (3,3,3) whose Walsh maps are the chiral Coxeter bipartite maps of type (3,2,6).*

In fact if $\mathcal{H}$ is a chiral hypermap in a torus then either it is a Coxeter chiral map or, by Euler-Poincaré formula, its type is $(3, 3, 3)$. Singerman and Corn [10] proved that a hypermap $\mathcal{H}$ on the torus is orientably regular if and only if the Walsh map $W(\mathcal{H})$ is orientably regular. Theorem 1 now implies that $\mathcal{H}$ is chiral if and only if $W(\mathcal{H})$ is chiral. Thus all the chiral hypermaps of type (3,3,3) can be obtained by the reverse of Walsh construction from the bipartite Coxeter chiral maps of type (3,2,6). The Fano plane imbedded in Torus is one element in this family.

$N = 2$. *There is no chiral hypermap on this surface, see [4].*

Let the negative Euler characteristic $N = |G| - V - E - F > 0$ be fixed. As it was already mentioned we have just finitely many $\mathcal{H}$-sequences to consider. Up to duality, in the enumeration tables (cases $N = 4$ and $N = 6$) below we only display possible $\mathcal{H}$-sequences for which $2 < l \leq m \leq n$.

$N = 4$. *Up to a duality, there are only two chiral hypermaps on a surface of* $N = 4$, *which are mirror image of each other. They have $\mathcal{H}$-sequence* $[4, \{7,3, 3\}, \{4, 7, 7\}, 21, \kappa = 7]$ *and their oriented monodromy group is metacyclic.*

Proof. By Theorem 2 any orientably regular hypermap with one or two hyperfaces is regular. Table 1 below lists the possibilities for $\mathcal{H}$-sequences of hypermaps that are not maps (that is, with $l \geq 2$) with 3 or more hyperfaces.

Table 1

#	N	l	m	n	V	E	F	darts
1	4	5	3	5	3	5	3	15
2	4	4	4	4	4	4	4	16
3	4	7	3	3	3	7	7	21
4	4	4	3	4	6	8	6	24
5	4	6	3	3	4	8	8	24
6	4	5	3	3	6	10	10	30
7	4	4	3	3	12	16	16	48

Theorem 4 eliminates items 2 and 5. Theorem 3 eliminates item 1 and says that the $\mathcal{H}$-sequence in item 3 corresponds to a chiral hypermap $\mathcal{H}$ with metacyclic oriented monodromy group and chirality index $\kappa = 7$. Below it is pictured its dual $D_{(0\,2)}\mathcal{H}$ of type (3,3,7).

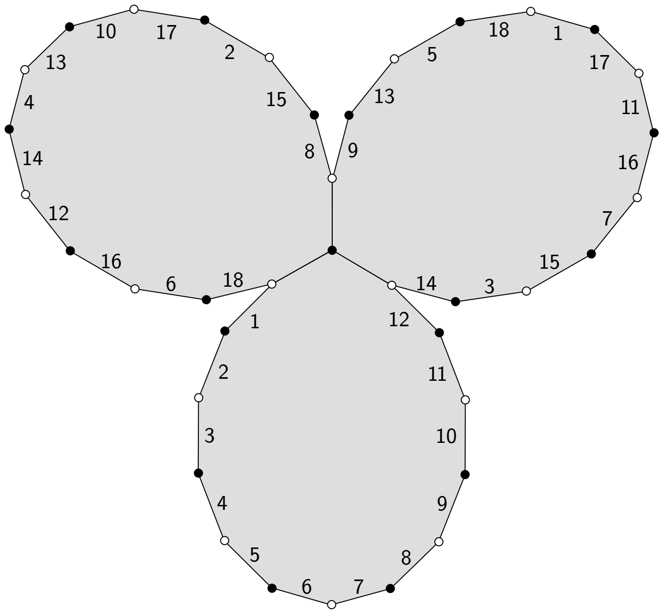


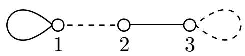
Figure 2: Walsh representation of a chiral hypermap of type (3,3,7) and genus 3.

It remains to eliminate the items 4, 6 and 7.

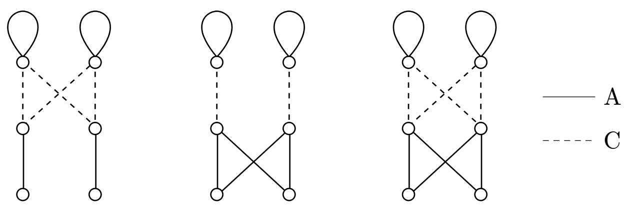
To fix notations, let a, b and c denote rotations one step about a hyperface, an adjacent hypervertex and an adjacent hyperedge, respectively; any two of these rotations generate the orientation preserving group $G = Aut^{+}(\mathcal{H})$, whose size gives the number of darts of $\mathcal{H}$.

#	N	l	m	n	V	E	F	darts
4	4	4	3	4	6	8	6	24

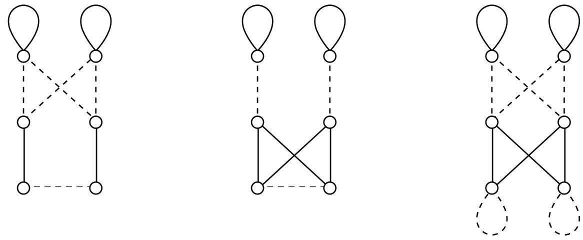
The transitive action of G on the 6 hyperfaces a must fix 2 hyperfaces and so according to Corollary 15 [23], G acts transitively on three face-classes 1, 2 and 3 with a fixing, say 1, and permuting 2 with 3; in fact the (a,b)-diagram on these 3 face-classes must be connected and as the rotation b has order at most 2 on the face-classes it must fix at least one and so none of a and b can fix more than one face-class. Then we must have $a = (2,3)$ and $b = (1,2)$ as actions on the face-classes.



We unfold this diagram to diagrams on the action on the 6 hyperfaces and possible partial unfoldings are



Taking into account that ab must have order 3 or 1 then we have 3 possible diagrams:



I II III

Diagrams I and II give permutation groups of size 24 so they give rise to orientably regular maps with 24 darts and so they cannot give item 4. Diagram III gives rise to a hypermap with 24 darts and $\mathcal{H}$-sequence [4, {4, 3, 4}, {6, 8, 6}, 24]. In this case the equations $b^2ab^2 = a^{-1}$, $a^2ba^2 = b^2$ hold and one can see that these equations define G; but then by the Substitution Test [18] the function $a \to a^{-1}$, $b \to b^{-1}$ extends to an automorphism of G and so diagram III gives rise to a regular hypermap. □

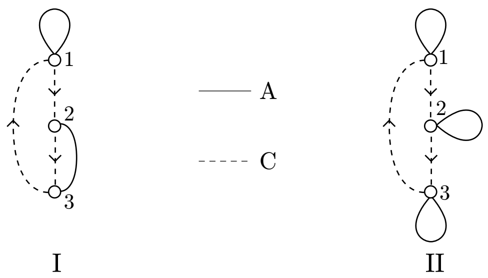
#	N	l	m	n	V	E	F	darts
6	4	5	3	3	6	10	10	30

If a Sylow 3-subgroup is normal then factoring it out would give an $\mathcal{H}$-sequence [−12, {5, 1, 1}, {2, 10, 10}, 10] which is clearly impossible. Then the number n_3 of Sylow 3-subgroups must be 10. Let ν_i denote the number of elements of order i in G. Then $\nu_1 = 1$, $\nu_2 = n_2$, $\nu_3 = 2n_3$ and $\nu_5 = 4n_5$. As $\nu_3 = 20$ then ν_5 cannot be bigger then 9, so $n_5 = 1$ and then $|G| = 1+\nu_2+20+4+\nu_6+\nu_{10}+\nu_{15}+\nu_{30}$. Now n_2 cannot be 1 because if so then factoring out the Sylow 2-subgroup would give an $\mathcal{H}$-sequence [−2, {5, 3, 3}, {3, 5, 5}, 15]; in this $\mathcal{H}$-sequence $n_3 = 1$ but the Sylow 3-subgroup cannot be factored out. Hence $\nu_2 > 1$ and then $\nu_{30} = 0$; then $\nu_2 + \nu_6 + \nu_{10} + \nu_{15} = 5$ implies that $n_2 = 3$ or 5. If $n_2 = 5$ then $\nu_6 = \nu_{10} = \nu_{15} = 0$; but if N is the normaliser of a Sylow 2-subgroup then $|N| = 6$. As N contains only one element of order 2 and two elements of order 3 then the other two non-trivial elements must have order 6 which is a contradiction. Then $n_2 = 3$ and the normaliser N of a Sylow 2-subgroup has order 10. As N has only one involution and one cyclic group of order 5, N, and consequently G, has two elements of order 10, and only two; but any element of order 10 gives rise to 4 distinct elements of order 10 which is a contradiction. Item 6 is eliminated.

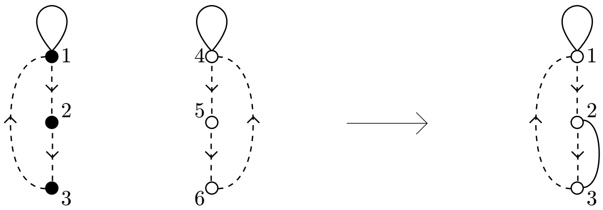
Note. This argument actually proves that there is no orientably regular hypermap with $\mathcal{H}$-sequence [4, $\{5, 3, 3\}$, $\{6, 10, 10\}$, 30].

#	N	l	m	n	V	E	F	darts
7	4	4	3	3	12	16	16	48

The cyclic group C generated by a is a 2-group hence by Sylow theory there are 2-subgroups S_1 and S_2 such that $C < S_1 < S_2$ where $|S_1| = 8$ and $|S_2| = 16$. Then the action of G by right multiplication on the 12 right cosets $G/_rC$ projects over the action of G on the 6 right cosets $G/_rS_1$ and this action projects over the action of G on the 3 right cosets $G/_rS_2$. This means that the diagram on the cosets of S_2 can be 2-unfolded to a diagram on the cosets of S_1 and in turn this diagram can be 2-unfolded to a diagram on the cosets of C. As $a \in S_2$ then we have two possibilities for a (a, b)-diagram on cosets of S_2:

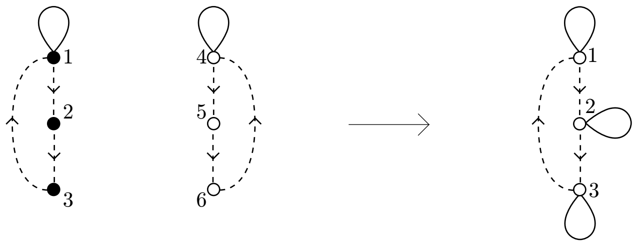


A partial unfolding of diagram I towards a diagram on cosets of S_1 is:

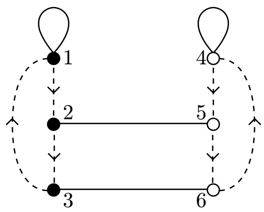


In order to get a connected diagram the coset 2 must be joined to the coset 6. In all possible connections, ab at 1 (and at 2) acts like an involution which is not possible. So diagram I cannot be unfolded to a diagram on cosets of S_1.

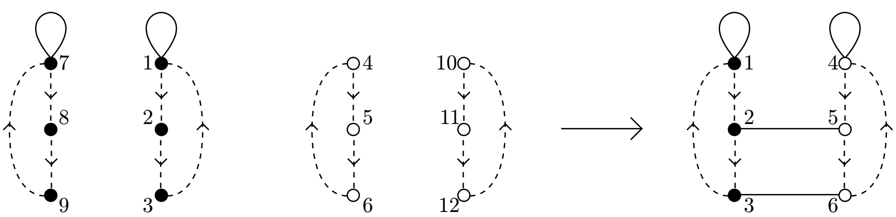
Without any loss of generality, a partial unfolding of diagram II can be as shown below:



To connect the diagram so that ab acts with order 3 we have only one possible solution:



This diagram gives rise to a dual of the tetrahedron $D_{(1\,2)}\mathcal{T}$. We unfold now this diagram to a diagram on the cosets of C. A partial unfolding is:



To complete the unfolding in order to get a connected diagram with ab acting with order 3, we have 6 possible solutions for a:

1) a=(2,11)(5,8)(3,6,9,12)(4,10)
2) a=(2,5)(8,11)(3,12,9,6)(4,10)
3) a=(2,5,8,11)(3,6)(9,12)(4,10)
4) a=(2,11,8,5)(3,12)(9,6)(4,10)
5) a=(2,11,8,5)(3,6,9,12)
6) a=(2,5,8,11)(3,12,9,6)

The first four possibilities give rise to permutation groups $G = \langle a, b\rangle$ with size 96 which is too big. The last two possibilities give rise to permutation groups G with size 48. In both these two cases the relations $a^4 = b^3 = (ab)^3 = (ab^2)^3 = 1$ hold and since these define a group of order 48 the two permutation groups $G = \langle a, b\rangle$ corresponding to items 5 and 6 are isomorphic and have presentation

$$\langle a, b \mid a^4 = b^3 = (ab)^3 = (ab^2)^3 = 1\rangle.$$

By the Substitution Test the function $a \to a^{-1}$, $b \to b^{-1}$ is an automorphism of G and so this presentation correspond to a regular hypermap. The proof is complete.

$\boldsymbol{N = 6}$**.** *Up to a duality, there are only two chiral hypermaps on a surface of* $N = 6$*, which are mirror image of each other. They have* $\mathcal{H}$*-sequence* $[6, \{5, 4, 4\}, \{4, 5, 5\}, 20, \kappa = 5]$ *and their oriented monodromy group is metacyclic.*

Proof. Table 2 below lists the possible $\mathcal{H}$-sequences of hypermaps that are not maps with 3 or more hyperfaces.

Table 2

#	N	l	m	n	V	E	F	darts
1	6	5	5	5	3	3	3	15
2	6	6	3	6	3	6	3	18
3	6	5	4	4	4	5	4	20
4	6	4	4	4	6	6	6	24
5	6	6	3	4	4	8	6	24
6	6	9	3	3	3	9	9	27
7	6	4	3	4	9	12	9	36
8	6	6	3	3	6	12	12	36
9	6	5	3	3	9	15	15	45
10	6	4	3	3	18	24	24	72

Theorem 3 eliminates the items 1, 2 and 6. Theorem 4 eliminates the item 5 and says that the $\mathcal{H}$-sequence corresponding to item 3 comes from a chiral hypermap with metacyclic oriented monodromy group and chirality index $\kappa = 5$. It remains to eliminate the items 4, 7, 8, 9 and 10.

#	N	l	m	n	V	E	F	darts
4	6	4	4	4	6	6	6	24

Let $\mathcal{H}$ be an orientably regular hypermap with this $\mathcal{H}$-sequence. The number n_3 of Sylow 3-subgroups is 1 or 4.

a) Suppose that $n_3 = 1$. Then the Sylow 3-subgroup S is normal in G. Factoring it out we get an $\mathcal{H}$-sequence [2, {4, 4, 4}, {2, 2, 2}, 8]. This orientably regular hypermap has 2 hyperfaces, so by Theorem 2 it must be regular. By [1] there is only one regular hypermap of type $\{4,4,4\}$ on a surface of characteristic $\chi = -2$ and it is $\epsilon_4^{2f}(2,1)$. Its oriented monodromy group E has presentation

$$\langle a, b \mid a^4 = b^4 = (ab)^4 = a^2b^2 = (ba)^2a^2 = 1\rangle.$$

Then $\mathcal{H}$ is a smooth 3-fold cover of $\epsilon_4^{2f}(2,1)$. Let $\psi : G \longrightarrow E$ be the 3-fold cover and let $K = C_3$ be its kernel. The elements a^2b^2 and $(ba)^2a^2$ belong to K and they cannot be trivial elements in G, otherwise by the Substitution Test, $E \cong G$, which is not possible. Then any of them generate K and so we have two possibilities:

$$1)\quad (ba)^2a^2 = a^2b^2 \iff baba^{-1} = a^2b^2$$
$$2)\quad (ba)^2a^2 = (a^2b^2)^{-1} = b^2a^2$$

If (2) holds then $bab^{-1} = a^{-1}$ and so $\langle a\rangle \lhd G$. As $G = \langle a\rangle\langle b\rangle$ and $\langle a\rangle \cap \langle b\rangle = \{1\}$ then G is a split extension of $\langle a\rangle$ by $\langle b\rangle$ which implies that $|G| = 16$, a contradiction. Hence (1) must hold. Then G has presentation

$$\langle a, b \mid a^4 = b^4 = (ab)^4 = 1, baba^{-1} = a^2b^2, (a^2b^2)^3 = 1, (a^2b^2)^a = (a^2b^2)^{\pm 1},$$
$$(a^2b^2)^b = (a^2b^2)^{\pm 1}\rangle.$$

But $(a^2b^2)^a = (a^2b^2)^{\pm 1} \iff b^2a = ab^2$ and $(a^2b^2)^b = (a^2b^2)^{\pm 1} \iff b^{-1}a^2 = a^2b^{-1}$. Then $(a^2b^2)^3 = (a^2b^2)^2a^2b^2 = a^2b^2$ and so $a^2b^2 = 1$ in G, a contradiction.

b) Hence $n_3 = 4$. Let S_i, $i = 1, 2, 3, 4$, be the four Sylow 3-subgroups and N_i their respective normalcies. Then $|N_i| = 6$ and so N_i is either a cyclic group or a dihedral group. By Lemma 5 i) there is some element $s \in S_1$ not in S_2 that cyclically permutes (by conjugation) the three Sylow 3-subgroups S_2, S_3 and S_4. Take the action of $G = \langle a, b\rangle$ by conjugation on the four Sylow 3-subgroups S_i. Then we have a homomorphism $\psi : G \longrightarrow \mathrm{S}_4$. As G acts transitively on the four Sylow 3-subgroups then either (I) one of $a\psi$, $b\psi$ is a 4-cycle or (II) one of $a\psi$, $b\psi$ is an even involution. In both cases $G\psi$ contains a 3-cycle and so $|G| \geq 12$.

(I) In this case $G\psi \cong \mathrm{S}_4$. But in [4] it is shown that all orientably regular hypermaps with G isomorphic to S_4 are regular.

(II) Without any loss of generality, suppose that a acts as an even involution. Then a^2 fixes (by conjugation) all Sylow 3-subgroups, that is, a^2 belongs to all normalcies N_i. Then $\langle a^2\rangle$ is in the core of $N = N_1$. If $N_i = D_3$, as N_i is generated by any two distinct involutions, then the core $N^\star$ must be a cyclic group of order 2, that is $N^\star = \langle a^2\rangle$ and $\langle a^2\rangle$ is normal in G. If $N_i = C_6$, as C_6 contains only one involution then $N^\star = \langle a^2\rangle$ and $\langle a^2\rangle$ is normal in G. In both cases $\langle a^2\rangle$ is normal in G. Factoring it out we must get the $\mathcal{H}$-sequence (the others possibilities give $N \leq -3$) [0, {4, 4, 2}, {3, 3, 6}, 12]. By Theorem 3 it must correspond to a regular map (up to a duality), but by [11] there is no regular map on the torus with type $\{4,4\}$ and size 12.

#	N	l	m	n	V	E	F	darts
7	6	4	3	4	9	12	9	36

We start by showing first the following lemma:

Lemma. *If $\mathcal{H}$ with the above $\mathcal{H}$-sequence is chiral then G contains no normal subgroup of order 3.*

Proof. Let H be a normal subgroup of order 3 in G and let b be a rotation one step about a hyperedge. If $b \in H$ then factoring H out we would get an $\mathcal{H}$-sequence $[-6, \{4, 1, 4\}, \{3, 12, 3\}, 12]$ which is clearly impossible. Hence $b \notin H$. Then factoring H out we get a hypermap $\mathcal{H}'$ with $\mathcal{H}$-sequence $[2, \{4, 3, 4\}, \{3, 4, 3\}, 12]$. By Theorem 3, $\mathcal{H}'$ must be regular. By [4] there is only one regular hypermap on a surface of characteristic $\chi = -2$ and this has binary dihedral group $\widetilde{D_3} = \langle A, B \mid A^4 = B^3 = 1, A^2 = (BA)^2\rangle$. But $\mathcal{H}'$ cannot be "smoothly" 3:1 fold covered: In fact, let $\psi : G \longrightarrow \widetilde{D_3}$ be a "smooth" 3:1 fold cover and $K = C_3$ its kernel. Let $A = a\psi$ and $B = b\psi$. As $A^2 = (BA)^2$ is the extra relation that defines the binary dihedral then a^2 must be different from $(ba)^2$ in G, and so $(ba)^2a^{-2} = baba^{-1}$ generate the kernel K. As K is normal in G, and G is generated by ba and a, then $(baba^{-1})^{ba} = (baba^{-1})^{\pm 1} \iff ba^2 = a^2b$ and $(baba^{-1})^{a^{-1}} = (baba^{-1})^{\pm 1} \iff (ba)^2 = (ab)^2$. Consequently, G is presented by

$$\langle a, b \mid a^4 = b^3 = (ab)^4 = (baba^{-1})^3 = 1, ba^2 = a^2b, (ba)^2 = (ab)^2\rangle.$$

But this presentation determines a group of size 12, not 36. □

Back to item 7. The number n_3 of Sylow 3-subgroups of G is 1 or 4. Let b be the element of order 3 (i.e. the rotation one step about a hyperedge).

a) Suppose that $n_3 = 1$. Then the Sylow 3-subgroup S of size 9 is normal. Then S is either a cyclic group or $C_3 \times C_3$. If $S = C_9$ then S contains only one subgroup of order 3 and it is $\langle b\rangle$. $\langle b\rangle$ being characteristic must be normal in G, which cannot be the case by the above lemma. Then $S = C_3 \times C_3$ and $S = \langle b, s\rangle$ for some $s \in S$. Then G is a split extension of S by the cyclic group C_4 generated by a and so it has presentation

$$\langle a, b, s \mid a^4 = (ab)^4 = 1, b^a = x, s^a = y, b^3 = s^3 = [b, s] = 1\rangle$$

for some $x, y \in S$ with $x \notin \{b, y, y^{-1}\}$. If $x = b$ or b^{-1} then b generates a normal subgroup of order 3 in G, and if $y = s$ or s^{-1} then s generates a normal subgroup of order 3 in G, which is not possible in the advent of $\mathcal{H}$ being chiral. The only values of $x \neq b, b^{-1}, y, y^{-1}$ and $y \neq s, s^{-1}$ that make the above presentation having size 36 are displayed in the following table:

x	y	$\lvert G\rvert$
s	a^{-1}	36
s^{-1}	b	36
bs	bs^{-1}	36
$b^{-1}s$	bs	36
bs^{-1}	$b^{-1}s^{-1}$	36
$b^{-1}s^{-1}$	$b^{-1}s$	36

For the other values of x, y the above presentation has order ≤ 12. Now the equation $b^a = x$ eliminates s as a generator and as the two relations $b^a = x$ and $s^a = y$, in these 6 cases, are equivalent to the relation $a^2ba^2 = b^{-1}$, then G has presentation

$$\langle a, b \mid a^3 = b^4 = (ab)^4 = b^2ab^2a = 1\rangle.$$

But, by the Substitution Test, the assignment $a \to a^{-1}$, $b \to b^{-1}$ extends to an automorphism of G. This leads to a regular hypermap.

b) Suppose now that $n_3 = 4$. Let $S = S_0$, S_1, S_2, S_3 be the four Sylow 3-subgroups and $N = N_0$, N_1, N_2, N_3 be their respective normalcies. As $|N| = 9$ then $N_i = S_i$. By Lemma 5 i) and ii) there is an element $s \in S \backslash S_1$ that cyclically permutes (by conjugation) the three Sylow 3-subgroups S_1, S_2 and S_3, and the intersection $S \cap S_i$ is the p-core $S^\star$. We have two possibilities $|S \cap S_i| = 1$, or 3. But $|S \cap S_i| = 3$ just says that G contains a normal subgroup $S^\star = S \cap S_i$ of order 3 and by the above Lemma this cannot be the case provided $\mathcal{H}$ is assumed to be chiral.

Then $|S \cap S_i| = 1$ and so the union $\mathcal{U}$ of all conjugates of S has $4(9-1)+1 = 33$ elements. Then the complement $\mathcal{C} = G\backslash\mathcal{U}$ has just three non-trivial elements whose order is not a power of 3. In this set lie the elements of order 4 and the elements of order 2. Thus there exists only one cyclic group of order 4 (giving rise to two elements of order 4 and one element of order 2) and it must be generated by a. Then $ba = a$ or $ba = a^{-1}$, and so, $b = 1$ or $b = a^{-2}$ which contradicts the fact that b has order 3.

#	N	l	m	n	V	E	F	darts
8	6	6	3	3	6	12	12	36

The number n_3 of Sylow 3-subgroups must be 4, since otherwise factoring out the Sylow 3-subgroup would give us $\mathcal{H}$-sequence $[-6, \{2, 1, 1\}, \{2, 4, 4\}, 4]$ which is not possible. Let $S = S_0$, S_1, S_2 and S_3 be the Sylow 3-subgroups and $N = N_0$, N_1, N_2 and N_3 their normalcies. Then $|N| = 9$ and thus $N = S$. By Lemma 5 i) there is an element $s \in S$ that cyclically permutes by conjugation N_1, N_2 and N_3. By Lemma 5 ii) the intersection $N \cap N_i$ coincides with the p-core $S^\star = N^\star$. There are two possibilities: $|N^\star| = 1$, or 3. If $|N^\star| = |N \cap N_1| = 1$ then N is distinct from its conjugates. Let $\mathcal{U}$ be the union of all conjugates of N. Then $|\mathcal{U}| = 4(9-1)+1 = 33$ and so the complement $\mathcal{C} = G\backslash\mathcal{U}$ has 3 elements. By Lemma 5 iii) as $|\mathcal{C}| = 3$, G acts transitively on $\mathcal{C}$, by conjugation, so all elements in $\mathcal{C}$ have the same order. As the element a of order 6 is not in $\mathcal{U}$ then it must be in $\mathcal{C}$ and so all elements in $\mathcal{C}$ have order 6 which implies that $|\mathcal{C}|$ is even, a contradiction.

Thus $|N^\star| = 3$. Factoring $N^\star$ out we get an orientably regular hypermap $\mathcal{M}$ with one of the $\mathcal{H}$-sequences (I) $[2, \{6, 3, 3\}, \{2, 4, 4\}, 12]$ or (II) $[-2, \{2, 3, 3\}, \{6, 4, 4\}, 12]$. By Theorem 4, both cases correspond to regular hypermaps. By [1] there is no regular hypermaps on a surface of characteristic $\chi = -2$ with type $\{6, 3, 3\}$, up to a duality, so case (I) is eliminated. In case (II) $\mathcal{M}$ must be a dual of the tetrahedron. Without any loss of generality, we turn to the dual form $[6, \{3, 6, 3\}, \{12, 6, 12\}, 36]$ so that $\mathcal{M}$ is the tetrahedron. This has oriented monodromy group isomorphic to A_4. Let H be the group of $\mathcal{M}$. Then H has presentation

$$\langle x, y \mid x^3 = y^2 = (xy)^3 = 1\rangle.$$

Let $\psi : G \longrightarrow H$, $a \to x$, $b \to y$ be the branched 3-fold covering and K its kernel. Then b^2 must be a nontrivial element of K and $K = \langle b^2 \rangle$. As K is normal in G then $(b^2)^a = b^{\pm 2} \iff b^2 a = ab^2$ or $b^2 a = ab^3$. Then G has presentation

$$\langle a, b \mid a^3 = b^6 = (ab)^3 = 1, b^2 a = ab^2 \rangle$$

or

$$\langle a, b \mid a^3 = b^6 = (ab)^3 = 1, b^2 a = ab^3 \rangle.$$

Using GAP [13] we have calculated that the first presentation gives a group of size 36 while the second determines a group of size 3. Hence

$$G = \langle a, b \mid a^3 = b^6 = (ab)^3 = 1, b^2 a = ab^2 \rangle.$$

By the Substitution Test the function $a \to a^{-1}$, $b \to b^{-1}$ extends to an automorphism of G and this item corresponds to a regular hypermap.

#	N	l	m	n	V	E	F	darts
9	6	5	3	3	9	15	15	45

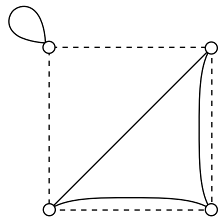
The number n_5 of Sylow 5-subgroups must be 1. Factoring out the Sylow 5-subgroup we get the $\mathcal{H}$-sequence [−6, {1, 3, 3}, {9, 3, 3}, 9] which is impossible.

#	N	l	m	n	V	E	F	darts
10	6	4	3	3	18	24	24	72

Let us consider the dual form [6, {3, 3, 4}, {24, 24, 18}, 72] instead. As in item 8 the number n_3 of Sylow 3-subgroups must be 4. Let $S = S_0$, S_1, S_2 and S_3 be the Sylow 3-subgroups and $N = N_0$, N_1, N_2 and N_3 their normalcies. Then $|N| = 18$. By Lemma 5 i) there is an element $s \in S$ that cyclically permutes by conjugation N_1, N_2 and N_3. By Lemma 5 ii) the intersection $S \cap S_i$ coincides with the p-core $S^\star$. There are two possibilities: $|S^\star| = 1$, or 3.

A) Suppose that $|S^\star| = 1$. We have two possibilities: $|N \cap N_1| = 1$ or 2. If $|N \cap N_1| = 1$ then conjugating by s, $|N \cap N_i| = 1$ for all i, that is, N is distinct from its conjugates. Let $\mathcal{U}$ be the union of all conjugates of N. Then $|\mathcal{U}| = 4(18-1)+1 = 69$ and so the complement $\mathcal{C} = G \backslash \mathcal{U}$ has 3 elements. By Lemma 5 iii) G acts transitively on $\mathcal{C}$, by conjugation, so all elements in $\mathcal{C}$ have the same order. As c of order 4 is not in $\mathcal{U}$ then it must be in $\mathcal{C}$ and so all elements in $\mathcal{C}$ have order 4 which implies that $|\mathcal{C}|$ is even, a contradiction.

Then $|N \cap N_1| = 2$. Without any loss of generality, suppose that $b \in S$. Because $|S^\star| = 1$, b of order 3 acts on the four normalcies as a 3-cycle. What about c of order 4? As 4 does not divide 18, $c \notin N = N_G(N) \iff N^c \neq N$. Then c acts on the normalcies without fixed points. There are two possibilities: c acts (I) as an even permutation or (II) as a 4-cycle. In the first case c^2 fixes all, that is, c^2 is in the core $N^\star$ of N. As $|N \cap N_1| = 2$ then $\{c^2\} = N^\star$ and so factoring $N^\star$ out gives $\mathcal{H}$-sequence [−6, {3, 3, 2}, {12, 12, 18}, 36] which is impossible. In the second case b and c act on the normalcies N_i as is depicted below.



Hence we have $a = cb^{-1}$ acting as a permutation of order 2 or 4 which is not possible.

B) $|S^\star| = 3$. Factoring $S^\star$ out gives an orientably regular hypermap $\mathcal{M}$ with the only possible $\mathcal{H}$-sequence $[2, \{3, 3, 4\}, \{8, 8, 6\}, 24]$. By [4] there is only one orientably regular hypermap on genus 2 with this type and it is $W^{-1}\{4+4, 3\}$ with automorphism group $H = \widetilde{\mathrm{A}_4} \cong SL(2, 3)$ binary tetrahedral. This group has presentation

$$\langle x, y \mid x^3 = y^3 = (xy)^4 = [(xy)^2, x] = [(xy)^2, y] = 1\rangle.$$

$$= \langle x, y \mid x^3 = y^3 = (xy)^4 = (xy)^2(yx)^2 = 1\rangle.$$

Let $\psi : G \longrightarrow H$, $a \to x$, $b \to y$, be the smooth 3-fold cover and $K = C_3$ its kernel. As $a^3 = b^3 = (ab)^4 = 1$ then $(ab)^2(ba)^2 \neq 1$ and so $K = \langle (ab)^2(ba)^2\rangle$. As G is generated by ab and b then we must have $((ab)^2(ba)^2)^{ab} = ((ab)^2(ba)^2)^{\pm 1}$ and $((ab)^2(ba)^2)^b = ((ab)^2(ba)^2)^{\pm 1}$. As

$$((ab)^2(ba)^2)^{ab} = ((ab)^2(ba)^2)^{\pm 1} \iff (ba)^2ab = ab(ba)^2$$

then

$$((ab)^2(ba)^2)^b = ((ab)^2(ba)^2)^{\pm 1} \iff (ba)^2 = (ab)^2$$

That is $(ab)^2(ba)^2 = 1$, a contradiction.

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