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On Three-Dimensional Space Groups

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Abstract. A entirely new and independent enumeration of the crystallographic space groups is given, based on obtaining the groups as fibrations over the plane crystallographic groups, when this is possible. For the 35 "irreducible" groups for which it is not, an independent method is used that has the advantage of elucidating their subgroup relationships. Each space group is given a short "fibrifold name" which, much like the orbifold names for two-dimensional groups, while being only specified up to isotopy, contains enough information to allow the construction of the group from the name.

1. Introduction

There are 219 three-dimensional crystallographic space groups (or 230 if we distinguish between mirror images). They were independently enumerated in the 1890's by W. Barlow in

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England, E.S. Federov in Russia and A. Schönfliess in Germany. The groups are comprehensively described in the International Tables for Crystallography [10]. For a brief definition, see Appendix I.

Traditionally the enumeration depends on classifying lattices into 14 Bravais types, distinguished by the symmetries that can be added, and then adjoining such symmetries in all possible ways. The details are complicated, because there are many cases to consider.

In the spirit of Bieberbach's classical papers [2, 3], Zassenhaus described a general, purely algebraic approach to the enumeration of crystallographic groups in arbitrary dimensions, which is now widely known as the *Zassenhaus algorithm* [15]. These ideas were employed to enumerate the four-dimensional crystallographic groups [1] by computer. For a recent account of the theory and computational treatment of crystallographic groups see [11] and also [5]. For recent software, see [8, 9].

Here we present an entirely new and independent enumeration of the three dimensional space groups that can be done by hand (almost) using geometry and some very elementary algebra. It is based on obtaining the groups as fibrations over the plane crystallographic groups, when this is possible. For the 35 "irreducible" groups for which it is not, we use an independent method that has the advantage of elucidating their subgroup relationships. We describe this first.

2. The 35 irreducible groups

A group is *reducible* or *irreducible* according as there is or is not a direction that it preserves up to sign.

2.1. Irreducible groups

We shall use the fact that any irreducible group Γ has elements of order 3, which generate what we call its *odd subgroup* (3 being the only possible odd order greater than 1). The odd subgroup T of Γ is obviously normal and so Γ lies between T and its normalizer $N(T)$.

This is an extremely powerful remark, since it turns out that there are only two possibilities T_1 and T_2 for the odd subgroup, and $N(T_1)/T_1$ and $N(T_2)/T_2$ are finite groups of order 16 and 8. This reduces the enumeration of irreducible space groups to the trivial enumeration of subgroups of these two finite groups (up to conjugacy).

The facts we have assumed will be proved in Appendix II.

2.2. The 27 "full groups"

These are the groups between T_1 and $N(T_1)$. $N(T_1)$ is the automorphism group of the body centered cubic (bcc) lattice indicated by the spheres of Figure 1 and T_1 is its odd subgroup.

The spheres of any one color 0-3 correspond to a copy of the face-centered cubic lattice

$$D_3 = \{(x, y, z) \mid x, y, z \in \mathbb{Z}, x + y + z \text{ is even}\}$$

and those of the other colors to its cosets in the dual body-centered cubic lattice

$$D_3^* = \{(x, y, z) \mid x, y, z \in \mathbb{Z} \text{ or } x - \tfrac{1}{2}, y - \tfrac{1}{2}, z - \tfrac{1}{2} \in \mathbb{Z}\}$$

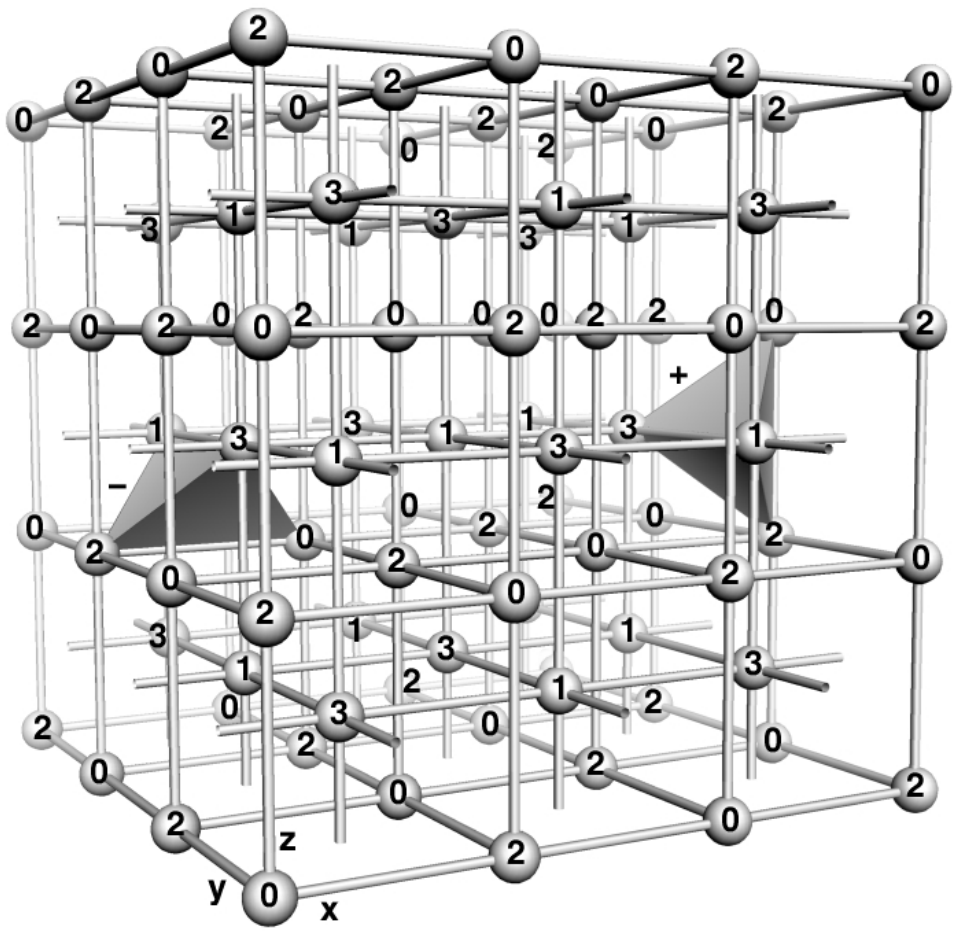


Figure 1. The body-centered cubic lattice D_3^*. The spheres colored 0 to 3 correspond to the four cosets of the sublattice D_3. The Delaunay cells fall into two classes, according as they can be moved to coincide with the tetrahedron on the left or right, by a motion that preserves the labeling of the spheres.

(the spheres of all colors). We call these cosets [0], [1], [2] and [3], [k] consisting of the points for which $x + y + z \equiv \frac{k}{2}(\text{mod}2)$.

Cells of the Delaunay complex are tetrahedra with one vertex of every color. The tetrahedron whose center of gravity is (x, y, z) is called *positive* or *negative* according as $x + y + z$ is an integer $+\frac{1}{4}$ or an integer $-\frac{1}{4}$. Tetrahedra of the two signs are mirror images of each other.

Any symmetry of the lattice permutes the 4 cosets and so yields a permutation π of the four numbers $\{0, 1, 2, 3\}$. We shall say that its image is $+\pi$ or $-\pi$ according as it fixes or changes the signs of the Delaunay tetrahedra. The group Γ_{16} of signed permutations so induced by all symmetries of the lattice is $N(T_1)/T_1$.

We obtain the 27 "full" space groups by selecting just those symmetries that yield elements of some subgroup of Γ_{16}, which is the direct product of the group $\{\pm 1\}$ of order 2, with the dihedral group of order 8 generated by the positive permutations. Figure 2 shows the subgroups of an abstract dihedral group of order 8, up to conjugacy. Our name for a subgroup of order n is n^-, n° or n^+, the superscript for a proper subgroup being $\circ$ for subgroups of the cyclic group of order 4, and otherwise $-$ or $+$ according as elements outside of this are odd or even permutations. The positive permutations form a dihedral group of order 8, which has 8 subgroups up to conjugacy, from which we obtain the following 8 space groups:

8° from $\{1,(02)(13),(0123),(3210),(13),(02),(01)(23),(03)(12)\}$
4^- from $\{1,(02)(13),(13),(02)\}$
4° from $\{1,(02)(13),(0123),(0321)\}$
4^+ from $\{1,(02)(13),(01)(23),(03)(12)\}$
2^- from $\{1,(13)\}$ or $\{1,(02)\}$
2° from $\{1,(02)(13)\}$
2^+ from $\{1,(01)(23)\}$ or $\{1,(03)(12)\}$
1° from $\{1\}$.

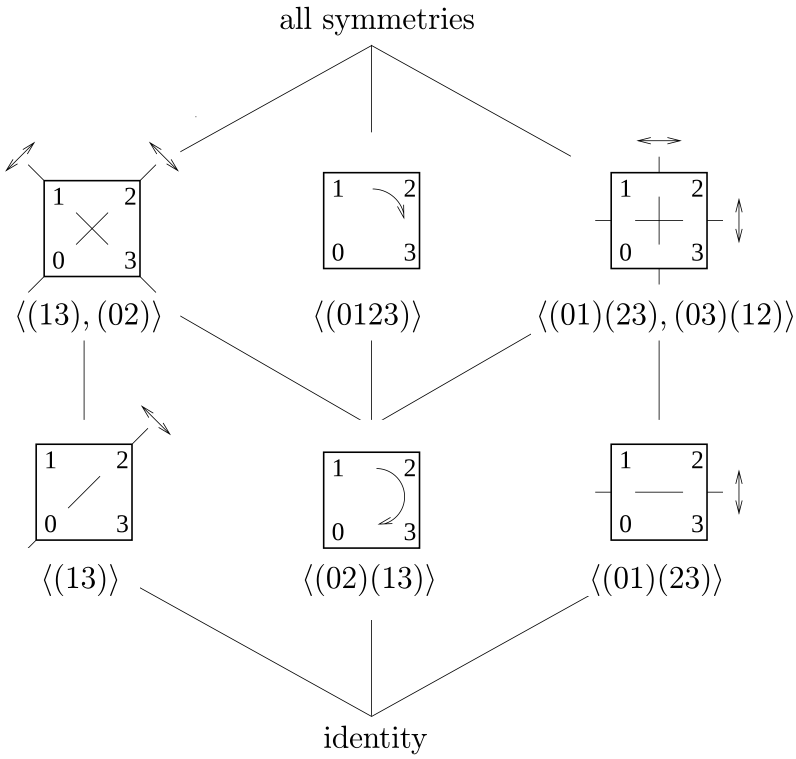


Figure 2. Subgroups of a dihedral group of order 8. The groups of order 2 and 4 on the left are generated by 1 or 2 diagonal reflections; those on the right by 1 or 2 horizontal or vertical reflections, and those in the center by a rotation of order 2 or 4.

The elements of a subgroup of Γ_{16} that contains -1 come in pairs $\pm g$, where g ranges over one of the above groups, so we obtain 8 more space groups:

$8^\circ:2$ from $\{\pm1, \pm(02)(13), \pm(0123), \pm(3210), \pm(13), \pm(02), \pm(01)(23), \pm(03)(12)\}$
$4^-:2$ from $\{\pm1, \pm(02)(13), \pm(13), \pm(02)\}$
$4^\circ:2$ from $\{\pm1, \pm(02)(13), \pm(0123), \pm(0321)\}$
$4^+:2$ from $\{\pm1, \pm(02)(13), \pm(01)(23), \pm(03)(12)\}$
$2^-:2$ from $\{\pm1, \pm(13)\}$ or $\{\pm1, \pm(02)\}$
$2^\circ:2$ from $\{\pm1, \pm(02)(13)\}$
$2^+:2$ from $\{\pm1, \pm(01)(23)\}$ or $\{\pm1, \pm(03)(12)\}$
$1^\circ:2$ from $\{\pm1\}$.

Each remaining subgroup of Γ_{16} is obtained by affixing signs to the elements of a certain group Γ, the sign being $+$ just for elements in some index 2 subgroup H of Γ. If $H = N^i$, $\Gamma = 2N^j$, we use the notation $2N^{ij}$ for this. In this way we obtain 11 further space groups:

$8^{-\circ}$ from $\{+1, +(02)(13), +(13), +(02), -(0123), -(3210), -(01)(23), -(03)(12)\}$
$8^{\circ\circ}$ from $\{+1, +(02)(13), +(0123), +(3210), -(13), -(02), -(01)(23), -(03)(12)\}$
$8^{+\circ}$ from $\{+1, +(02)(13), +(01)(23), +(03)(12), -(13), -(02), -(0123), -(3210)\}$
4^{--} from $\{+1, +(13), -(02)(13), -(02)\}$
$4^{\circ-}$ from $\{+1, +(02)(13), -(13), -(02)\}$
$4^{\circ\circ}$ from $\{+1, +(02)(13), -(0123), -(0321)\}$
$4^{\circ+}$ from $\{+1, +(02)(13), -(01)(23), -(03)(12)\}$
4^{++} from $\{+1, +(01)(23), -(02)(13), -(03)(12)\}$
$2^{\circ-}$ from $\{+1, -(13)\}$
$2^{\circ\circ}$ from $\{+1, -(02)(13)\}$
$2^{\circ+}$ from $\{+1, -(01)(23)\}$.

(For these we have only indicated one representative of the conjugacy class.)

Summary: A typical "full group" consists of all the symmetries of Figure 1 that induce a given group of signed permutations.

2.3. The 8 "quarter groups"

We obtain Figure 3 from Figure 1 by inserting certain diagonal lines joining the spheres.

[The rules are that the sphere (x, y, z)

lies on a line in direction:
$(1,1,1)$ $(-1,1,1)$ $(1,-1,1)$ $(1,1,-1)$
according as:
$x \equiv y \equiv z$ $y \not\equiv z \equiv x$ $z \not\equiv x \equiv y$ $x \not\equiv y \equiv z$ (mod2)
if x, y, z are integers, or according as:
$x \equiv y \equiv z$ $z \not\equiv x \equiv y$ $x \not\equiv y \equiv z$ $y \not\equiv z \equiv x$ (mod2)
if they are not.]

The automorphisms that preserve this set of diagonal lines form the group $N(T_2)$ whose odd subgroup is T_2. It turns out that these automorphisms preserve each of the two classes

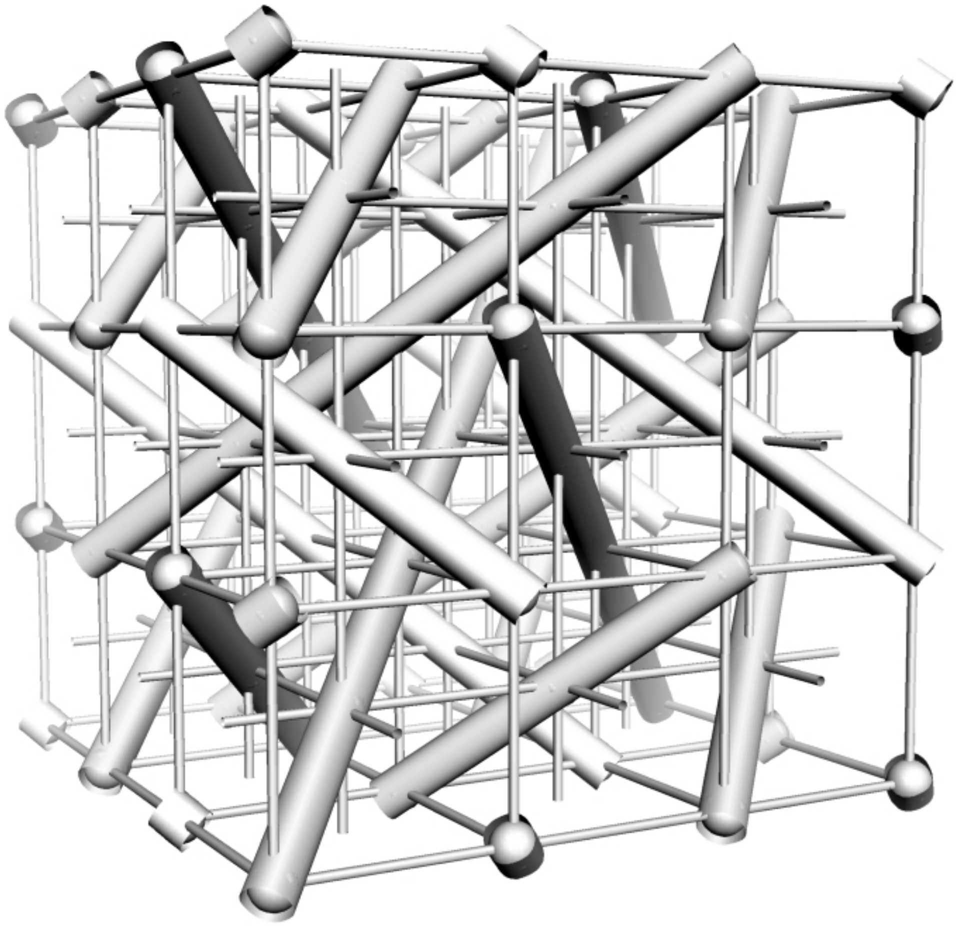


Figure 3. The cyclinders represent the set of diagonal lines fixed by the eight quarter groups.

of tetrahedra in Figure 1 so that $N(T_2)/T_2$ is the order 8 dihedral group Γ_8 of positive permutations.

So the eight "quarter groups" $N^i/4$ (of index 4 in the corresponding group N^i) are obtained from the subgroups of Γ_8 according to the following scheme:

$8^\circ/4$	from	$\{1, (02)(13), (0123), (3210), (13), (02), (01)(23), (03)(12)\}$
$4^-/4$	from	$\{1, (02)(13), (13), (02)\}$
$4^\circ/4$	from	$\{1, (02)(13), (0123), (0321)\}$
$4^+/4$	from	$\{1, (02)(13), (01)(23), (03)(12)\}$
$2^-/4$	from	$\{1, (13)\}$ or $\{1, (02)\}$
$2^\circ/4$	from	$\{1, (02)(13)\}$
$2^+/4$	from	$\{1, (01)(23)\}$ or $\{1, (03)(12)\}$
$1^\circ/4$	from	$\{1\}$.

Summary: A typical "quarter group" consists of all the symmetries of Figure 3 that induce a given group of permutations.

2.4. Inclusions between the 35 irreducible groups

Our notation makes most of the inclusions obvious: in addition to the containments N^i in $2N^{ij}$ in $2N^j$:2, each $\Gamma/4$ is index 4 in Γ, which is index 2 in Γ:2, and these three groups are index 2 in another such triple just if the same holds for the corresponding subgroups of the dihedral group of order 8. All other minimal containments have the form N^{ij} in $2N^{kl}$ and are explicitly shown in Figure 4.

To be precise, the figure does not show all minimal inclusions between irreducible space groups but only between the particular representatives that we considered here. For example, the group $4^- : 2$ contains a subgroup of index 4 that is isomorphic to $8^\circ : 2$. A complete list of subgroup relationships between space groups can be found in [10].

2.5. Correspondence with the fibered groups

Each irreducible group contains a fibered group of index 3, and when we started this work we hoped to obtain a nice notation for the irreducible groups using this idea. Eventually we decided that the odd subgroup method was more illuminating. However, we indicate these relations in Table 2a. We give a group Γ secondary descriptors such as

$$\text{[Jack]:3} \quad \text{or} \quad \text{(Jill):6}, \quad \text{say,}$$

to mean that [Jack] is the group obtained by fixing the z direction (up to sign) and (Jill) that obtained by fixing all three directions (up to sign).

These secondary descriptors are not unique names, as some space groups have identical first or second descriptors. Both descriptors together, however, happen to determine the space group uniquely. This includes those space groups with point groups $3{*}2$ or 332, for which both descriptors are identical and thus only the first one is shown.

3. The 184 reducible space groups

We now consider the space groups that preserve some direction up to sign, and accordingly can be given an invariant fibration.

3.1. On orbifolds and fibration

The rest of the enumeration is based on the concept of fibered orbifolds. The *orbifold* of such a group is "the space divided by the group": that is to say, the quotient topological space whose points are the orbits under the group [13, 12]. (So we can regard "orbifold" as an abbreviation of "orbit-manifold".)

For our purposes, a *fibration* is a division of the space into a system of parallel lines. A *fibered space group* is a space group together with a fibration that is invariant under that group. On division by the group, the fibration of the space becomes a fibration of the orbifold, each fiber becoming either a circle or an interval. We call this a *fibered orbifold.*

The concept of a fibered space group differs slightly from that of a reducible space group. The latter are those for which there exists at least one invariant direction and to obtain a

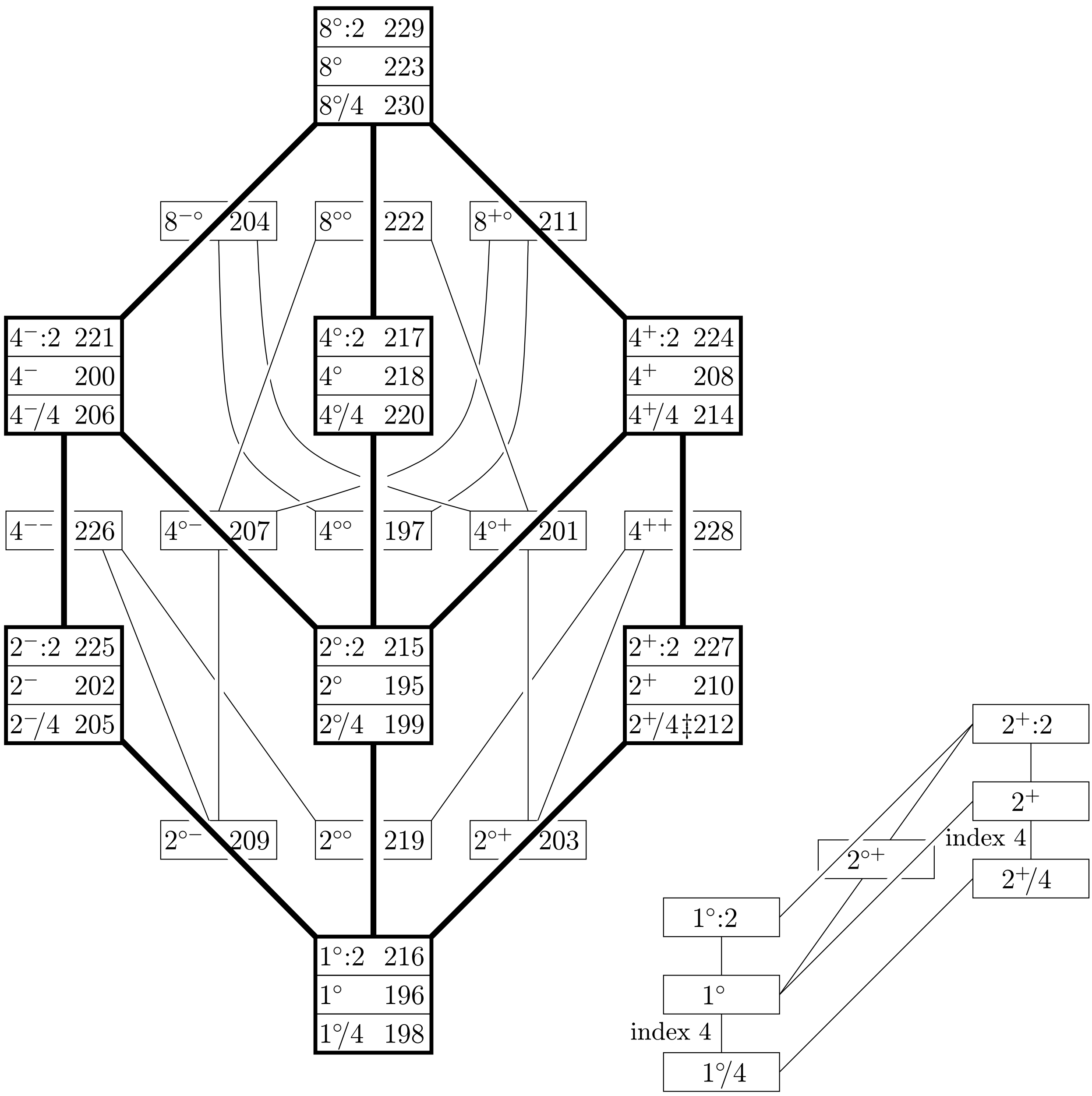


Figure 4. The 35 irreducible groups. Each heavy edge represents several inclusions as in the inset. With these conventions, the figure illustrates all 83 minimal inclusions between particular representatives of these groups: 2 for each of the 8 heavy boxes, 5 for each of the 11 heavy edges and 1 for each of the 12 thin edges. ((‡): The group $2^{+}/4$ has two enantiomorphous forms, with IT numbers 212 and 213.)

fibered space group is to make a fixed choice of such an invariant direction. This distinction leads to what we call the "alias problem" discussed in Section 5.2.

Although this paper was inspired by the orbifold concept, we did not need to consider the 219 orbifolds of space groups individually. We hope to discuss their topology in a later paper.

3.2. Fibered space groups and euclidean plane groups

Look along the invariant direction of a fibered space group and you will see one of the 17 Euclidean plane groups!

We explain this in more detail and introduce some notation. Taking the invariant direction to be z, the action of any element of the space group Γ has the form:

$$g : (x, y, z) \mapsto (a(x, y), b(x, y), c \pm z)$$

for some functions $a(x, y), b(x, y)$, some constant c, and some sign $\pm$.

Ignoring the z-component gives us the action

$$g_H : (x, y) \mapsto (a(x, y), b(x, y))$$

of the corresponding element of the plane group.

We will call g_H the *horizontal part* of g and say that it is *coupled* with the *vertical part*

$$g_V : (z) \mapsto (c \pm z).$$

3.3. Describing the coupling

The fibered space group Γ is completely specified by describing the coupling between the "horizontal" operations g_H and the "vertical" ones, g_V, for which we use the notations $c+$ and $c-$, where $c+$ and $c-$ are the maps $z \mapsto c + z$ and $z \mapsto c - z$. We say that h_H is plus-coupled or minus-coupled according as it couples to an element $c+$ or $c-$.

Geometrically, $c+$ is a translation through distance c, while $c-$ is the reflection in the horizontal plane at height $\frac{1}{2}c$. It is often useful to note that by raising the origin through a distance $\frac{c}{2}$, we can augment by a fixed amount, or "reset", the constants in all $c-$ operations while fixing all $c+$ ones.

In fact, any given horizontal operation is coupled with infinitely many different vertical operations, since the identity is. We study this by letting K denote the *kernel*, consisting of all the vertical operations that are coupled with the identity horizontal operation I. Let k be the smallest positive number for which $k+ \in K$. Then the elements $nk+$ $(n \in \mathbb{Z})$ are also in K. If K consists precisely of these elements, then the generic fiber is a circle and we speak of a *circular fibration* and indicate this by using ( )'s in our name for the group. If there exists some $c- \in K$ then we can suppose that $0- \in K$, and then K consists precisely of all $nk+$ and $nk-$ $(n \in \mathbb{Z})$. In this case, the generic fiber is a closed interval and we have an *interval fibration*, indicated by using [ ]'s in the group name.

In our tables, we rescale to make $k = 1$, so that K consists *either* of all elements $n+$ (for a circular fibration) *or* all elements $n+$ and $n-$ (for an interval one) for all integers n.

3.4. Enumerating the fibrations

To specify the typical fibration over a given plane group $H = \langle P, Q, R, \ldots \rangle$, we merely have to assign vertical operations $p\pm, q\pm, r\pm \ldots$, one for each of the generators $P, Q, R, \ldots$.

The condition for an assignment to work is that it be a homomorphism, modulo K. This means in particular that the vertical elements $c\pm$ need only be specified modulo K, which allows us to suppose $0 \le c < 1$, which for a circular fibration is enough to make these elements unique.

For interval fibrations the situation is simpler, since we need only use the vertical elements $0+$ and $\frac{1}{2}+$. (For since $0-$ is in K we can make the sign be $+$; but also $c+ \cdot K \cdot (c+)^{-1} = K$, and since $(c+)0-(c+)^{-1}$ is the map taking z to $2c-z$ this shows that $2c$ must be an integer.)

Example: the plane group $H = 632 \cong \langle \gamma, \delta, \epsilon \mid 1 = \gamma^6 = \delta^3 = \epsilon^2 = \gamma\delta\epsilon \rangle$**.**

Here we take the corresponding vertical elements to be $c\pm, d\pm, e\pm$. Then in view of $\gamma\delta\epsilon = 1$, it suffices to compute $d\pm$ and $e\pm$. The condition $\delta^3 = 1$ entails that the element $d\pm$ is either $0+$, $\frac{1}{3}+$, or $\frac{2}{3}+$. (If the sign were $-$, then the order of this element would have to be even.) The condition $\epsilon^2 = 1$ shows that $e\pm$ can only be one of $0+$, $\frac{1}{2}+$, or $0-$ and, (since the general case $e-$ can be reset to $0-$), we get at most 9 circular fibrations, which reduce to six by symmetries, namely γ, δ, ϵ couple to one of:

$$\begin{array}{llcl} 0+0+0+ & \frac{2}{3}+\frac{1}{3}+0+ & \cong & \frac{1}{3}+\frac{2}{3}+0+ \\ \frac{1}{2}+0+\frac{1}{2}+ & \frac{1}{6}+\frac{1}{3}+\frac{1}{2}+ & \cong & \frac{5}{6}+\frac{2}{3}+\frac{1}{2}+ \\ 0-0+0- & \frac{1}{3}-\frac{1}{3}+0- & \cong & \frac{2}{3}-\frac{2}{3}+0-. \end{array}$$

For interval fibrations, where we can only use $0+$ and $\frac{1}{2}+$, δ can only couple with $0+$ (because $\delta^3 = 1$), so we get at most 2 possibilities, namely that $\gamma, \delta, \epsilon, I$ couple with:

$$0+0+0+0- \quad \text{or} \quad \tfrac{1}{2}+0+\tfrac{1}{2}+0-.$$

Studying these fibrations involves calculations with products of the maps $k\pm$. Our notation makes this easy: for example, the product $(a-)(b+)(c-)$ is the map that takes z to $a-(b+(c-z)) = (a-b-c)+z$; so $(a-)(b+)(c-) = (a-b-c)+$; also, the inverse of $c+$ is $(-c)+$, while $c-$ is its own inverse.

For example for the circular fibrations of 632 we had $d\pm = 0+, \frac{1}{3}+, \frac{2}{3}+$ and $e\pm = 0+, \frac{1}{2}+, 0+$ and these define $c\pm$ via the relation $c\pm\, d\pm\, e\pm = 0+$. So, if $d\pm = \frac{1}{3}+$ and $e\pm = \frac{1}{2}+$, then $c\pm$ must be $c+$, and since

$$c+\tfrac{1}{3}+\tfrac{1}{2}+ = (c+\tfrac{1}{3}+\tfrac{1}{2})+,$$

c must be $\frac{-5}{6}$, which we can replace by $\frac{1}{6}$.

The indicated isomorphisms are consequences of the isomorphism that changes the sign of z, which allows us to replace c, d, e by their negatives (modulo 1). So we see that we have at most $6 + 2 = 8$ fibrations over the plane group 632. The ideas of the following section show that they are all distinct.

4. The fibrifold notation (for simple embellishments)

Plainly, we need some kind of invariant to tell us when fibrations really are distinct. A notation that corresponds exactly to the maps $k\pm$ will not be adequate because they are far from being invariant: for example, $0-0+0-$ is equivalent to $k-0+k-$ for every k. We shall use what we call the *fibrifold notation*, because it is an invariant of the *fibered orbifold* rather than the orbifold itself. It is an extension of the orbifold notation that solved this problem in the two-dimensional case [6, 4], see Appendix III.

The exact values of these maps are not, and should not be, specified by the notation. Rather, everything in the notation is an invariant of them up to continuous variation (isotopy) of the group. We usually prove this by showing how it corresponds to some feature of the fibered orbifold.

We obtain the fibrifold notation for a space group Γ by "embellishing", or adding information to the orbifold notation for the two-dimensional group H that is its horizontal part. (Often the embellishment consists of doing nothing. In this section we handle only the simple embellishments that can be detected by local inspection of the fibration.)

4.1. Embellishing a ring symbol ∘

A ring symbol corresponds to the relations ${}^{\alpha}\circ^{X\,Y}$: $\alpha = [X, Y]$. We embellish it to $\circ$ or $\bar{\circ}$ to mean that X, Y are both plus-coupled or both minus-coupled respectively. [We can suppose that X and Y couple to the same sign in view of the three-fold symmetry revealed by adding a new generator Z satisfying the relations $XYZ = 1$ and $X^{-1}Y^{-1}Z^{-1} = \alpha$.]

This embellishment is a feature of the fibration since it tells us whether or not the fibers have a consistent orientation over the corresponding handle.

4.2. Embellishing a gyration symbol G

Such a symbol corresponds to relations ${}^{\gamma}G$: $\gamma^G = 1$. We embellish it to G or G_g according as γ is minus-coupled or coupled to $\frac{g}{G}+$. The relation $\gamma^G = 1$ implies that G must be even if γ is minus-coupled and that c must be a multiple of $\frac{1}{G}$, if γ couples to $c+$.

The embellishment is a feature of the fibration because the behavior of the latter at the corresponding cone point determines an action of the cyclic group of order G on the circle.

4.3. Embellishing a kaleidoscope symbol $*AB\ldots C$

Here the relations are ${}^{\lambda}*^{P}A^{Q}B\ldots{}^{R}C^{S}$:

$$1 = P^2 = (PQ)^A = Q^2 = \ldots = R^2 = (RS)^C = S^2 \text{ and } \lambda^{-1}P\lambda = S.$$

We embellish $*$ to $*$ or $\bar{*}$ according as λ is plus-coupled or minus-coupled. This embellishment is a fibration feature because it tells us whether the fibers have or do not have a consistent orientation when we restrict to a deleted neighborhood of a string of mirrors.

The coupling of Latin generators is indicated by inserting 0, 1 or 2 dots into the corresponding spaces of the orbifold name. Thus AB will be embellished to AB, $A{\cdot}B$ or $A{:}B$ according as Q is minus-coupled or coupled to $0+$ or $\frac{1}{2}+$. These embellishments tell us how the fibration behaves above the corresponding line segment. The generic fiber is identified with itself by a homomorphism whose square is the identity, namely

·	(e.g. $A{\cdot}B$)	corresponds to a mirror	$\phi(x) = x$
:	(e.g. $A{:}B$)	is a Möbius map	$\phi(x) = x + \frac{1}{2}$, and
a blank	(e.g. $A\,B$)	corresponds to a link	$\phi(x) = -x$.

The numbers $A, B, \ldots, C$ are the orders of the products like PQ for adjacent generators. So, as in the gyration case, we embellish A to A or A_a according as PQ is minus-coupled or coupled to $\frac{a}{A}+$. There are two cases: If P and Q are both minus-coupled, say to $p-$ and $q-$, then PQ is coupled to $(p-q)+$, and so that $\frac{a}{A}$ is $p-q$. If they are both plus-coupled, then the subscript a is determined as 0 or $\frac{A}{2}$ by whether the punctuation marks $\cdot$ or : on each side of A are the same or different, and so we often omit it.

In pictures we use corresponding embellishments

$$\Big|\cdot \qquad \Big|: \qquad \Big| \qquad n_d \qquad n$$

to show that the appropriate reflections or rotations are coupled to:

$$0+ \qquad \tfrac{1}{2}+ \qquad k- \qquad \tfrac{d}{n}+ \qquad k-\,.$$

4.4. Embellishing a cross symbol $\times$

Here the relations are ${}^{\omega}\times^{Z}$: $Z^2 = \omega$, and we embellish $\times$ to $\times$ or $\bar{\times}$ according as Z is plus-coupled or minus-coupled (indicating whether the fibers do or do not have a consistent orientation over the corresponding crosscap).

4.5. Other embellishments exist!

These simple embellishments should suffice for a first reading, since they suffice for many groups. In a Section 6 we shall describe more subtle ones that are sometimes required to complete the notation.

5. The enumeration: detecting equivalences

The enumeration proceeds by assigning coupling maps $k\pm$ to the operations in all possible ways that yield homomorphisms modulo K, and describing these in the (completed) fibrifold notation. What this notation captures is exactly the isotopy class of a given fibration over a given plane crystallographic group H, in other words the assignment of the maps $k+$ and $k-$ to all elements of H, up to continuous variation.

5.1. The symmetry problem

The two fibrations of $*^{P}4^{Q}4^{R}2$ notated $(*4_1 4{\cdot}2)$ and $(*{\cdot}44_1 2)$ are distinct in this sense - in the first R maps to the identity, while in the second P does. However, they still give a single three-dimensional group because the plane group $H = *442$ has a symmetry taking P, Q, R to R, Q, P, see Figure 5.

So our problem is really to enumerate isotopy classes of fibrations up to symmetries: we discuss the different types of symmetry in Appendix IV. Some of them are quite subtle - how can we be sure that we have accounted for them all?

To be quite safe, we made use of the programs of Olaf Delgado and Daniel Huson [7] which compute various invariants that distinguish three-dimensional space groups. Delgado and Huson used these to find the “IT number” that locates the group in the International

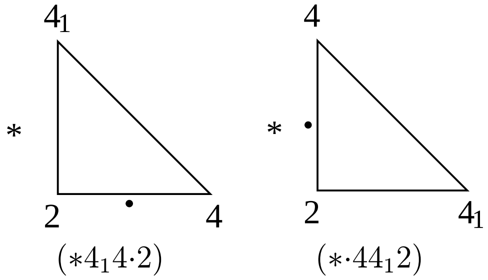


Figure 5. Two fibrations related by a symmetry.

Crystallographic Tables [10]; but our enumeration uses the invariants only, and so is logically independent of the international tabulation. The conclusion is that the reduced names we introduce in Section 5.2 account for all the equivalences induced by symmetries between fibrations over the same plane group.

5.2. The alias problem

However, a three-dimensional space group Γ may have several invariant directions, which may correspond to different fibrations over distinct plane groups H. So we must ask: which sets of names - we call them *aliases* - correspond to fibrations of the same group? This can only happen when the programs of Delgado and Huson yield the same IT number, a remark that does not in fact depend on the international tabulation, since it just means that they are the only groups with certain values of the invariants.

The alias problem only arises for the reducible point groups, namely 1, $\times$, $*$, 22, $2*$, $*22$, 222 and $*222$. In fact there is no problem for 1 and $\times$, since for these cases there is just one fibration. For $*$, 22, $2*$ and $*22$ the point group determines a unique canonical direction (see Figure 6) and so a unique *primary* name. Any other, *secondary* name, must be an alias for some primary name, solving the alias problem in these cases.

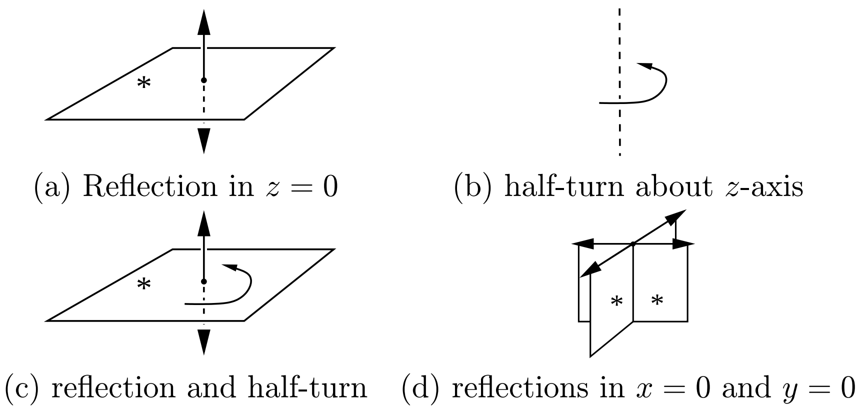


Figure 6. The point groups $*$, 22, $2*$ and $*22$ each determine a unique canonical direction.

We are left with the point groups $*222$ and 222, which always have three fibrations in orthogonal directions. A permutation of the three axes might lead to an isotopic group. If the number of such permutations is:

1, we have three distinct asymmetric names, say $\{A, A', A''\}$,

2, we have two distinct names, one symmetric and one asymmetric, say $\{S, A\}$,

3, we have just one asymmetric name, say $\{A\}$, or

6, we have just one symmetric name, $\{S\}$,

where an axis, or the corresponding group name, is called *symmetric* if there is a symmetry interchanging the other two axes. This can be detected from the fibrifold notation, for example, it is apparent from Figure 7 that $*{\cdot}2{\cdot}2{:}2{:}2$ is symmetrical and $*{\cdot}2{:}2{\cdot}2{:}2$ is not.

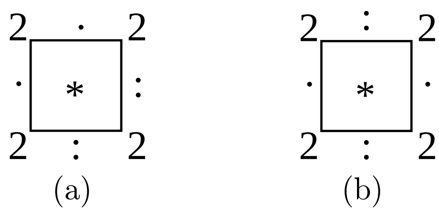


Figure 7. Diagram (a) possesses a symmetry that interchanges the x and y axes, whereas diagram (b) does not.

Table 2b provisionally lists all names that have the same invariant values in a single line. We now show that the corresponding alias-sets

$$\{A, A', A''\}, \{S, A\}, \{A\}, \{S\}$$

are correct. For if not, some $\{A, A', A''\}$ or $\{S, A\}$ would correspond to two or more groups, one of which would have a single asymmetric name. But we show that there is only one such group, whose name $(2_1 2\bar{*}\,:)$ was not in fact in a set of type $\{A, A', A''\}$ or $\{S, A\}$.

A group Γ with a single asymmetric name must be isotopic to that obtained by cyclically permuting x, y, z: this isotopy will become an automorphism if we suitably rescale the axes. Adjoining this automorphism leads to an irreducible group in which Γ has index 3. But inspection of Table 2a reveals that the only asymmetric name for which this happens is $(2_1 2\bar{*}\,:)$.

So indeed the primary and secondary names in any given line of Table 2b are aliases for the same group. Our rules for selecting the primary name are:

1. A unique name is the primary one.
2. Of two names, the primary name is the symmetrical one.
3. Otherwise, the primary name is that of a fibration over $22*$.
4. Finally, we prefer $(2_0 2\bar{*}{\cdot})$ to $[2_1 2_1 *{:}]$ and $(2_0 2\bar{*}{:})$ to $(2_1 2\bar{*}_1)$.

These conventions work well because the three-name cases all involve a fibration over $22*$ (which was likely because the x and y axes can be distinguished for $22*$). The groups in the last rule are those with two such fibrations.

6. Completing the embellishments

We now ask what further embellishments are needed to specify a fibration up to isotopy? We will find that it suffices to add subscripts 0 or 1 to some of the symbols $\circ$, $*$ and $\times$. This section can be omitted on a first reading, since for many groups in the tables these subtle embellishments are not needed. We shall show as we introduce them that they determine the coupling maps up to isotopy, so that no further embellishments are needed.

6.1. Ring symbol $\circ$

Here if X and Y couple to $x+$ and $y+$, then the relation $[X,Y] = \alpha$ shows that α is automatically coupled to $0+$. So the space groups obtained for arbitrary values of x and y are isotopic and we need no further embellishment.

If, however, there are one or more embellishments to $\bar{\circ}$, say those in the relations

$$^{\alpha_1}\circ^{XY} \ldots {}^{\alpha_n}\circ^{UV}$$

then the coupling of the global relation will have the form

$$\underbrace{\alpha_1}_{\displaystyle\downarrow \atop \displaystyle 2(x-y)+} \quad \ldots \quad \underbrace{\alpha_n}_{\displaystyle\downarrow \atop \displaystyle 2(u-v)+} \quad \underbrace{(\gamma\ldots\omega)}_{\displaystyle\downarrow \atop \displaystyle k+} = 1$$

which restricts the variables $x, y, \ldots, u, v$ only by the condition that $2(x - y + \ldots + u - v)$ be congruent to $-k$ (modulo 1).

So if k is already determined this leaves just two values (modulo 1) for $x - y + \ldots + u - v$, namely $\frac{i-k}{2}$ ($i = 0$ or 1); we distinguish when necessary by adding i as a subscript. Once again, the individual values of $x, y, \ldots, u, v$ do not matter since they can be continuously varied in any way that preserves the truth of

$$x - y + \ldots + u - v = \frac{i-k}{2}.$$

6.2. Gyration symbol *G*

A gyration γ can only be coupled to $0+$, $\frac{1}{2}+$ or $c-$. If we suppose that the minus-coupled gyrations are $\gamma_1 \mapsto c_1-, \ldots, \gamma_n \mapsto c_n-$, then the values of the c_i are unimportant for the local relations, while the global relation involves only $c_1 - c_2 + \ldots \pm c_n$, and the c_i can be varied in any way that preserves this sum. So all solutions are isotopic and no more subtle embellishment is needed.

6.3. Kaleidoscope symbol $*AB\ldots C$

The simple embellishments suffice for the relations

$$1 = P^2 = (PQ)^A = Q^2 = \ldots = R^2 = (RS)^C = S^2,$$

so we need only discuss the relation $\lambda^{-1}P\lambda = S$ and the global relation. We have already embellished $*$ to $*$ or $\bar{*}$ according as λ couples to an element $l+$ or $l-$.

Specifying l more precisely is difficult, because we need two rather complicated conventions for circular fibrations, neither of which seems appropriate for interval ones.

For interval fibrations we simply embellish $*$ to $*_0$ or $*_1$ according as λ couples to $0+$ or $\frac{1}{2}+$ (the only two possibilities).

What does the relation $\lambda^{-1}P\lambda = S$ tell us about the number l in the circular-fibration case?

The answer turns out to be 'nothing', if any one of the Latin generators $P, \ldots, S$ couples to a translation $k+$. To see this, it suffices to suppose that P couples to $0+$ or $\frac{1}{2}+$, and then S, being conjugate to P, must couple to the *same* translation $0+$ or $\frac{1}{2}+$. But since these two translations are central they conjugate to themselves by *any* element $l\pm$, so the relation $\lambda^{-1}P\lambda = S$ is automatically satisfied, and we need no further embellishment.

The only remaining case is when all of $P, \ldots, S$ couple to reflections $p-, q-, \ldots, s-$. In this case we have already embellished $A\,B \ldots C$ to $A_a B_b \ldots C_c$ and we have

$$q = p - \frac{a}{A}, \ldots, s = r - \frac{c}{C},$$

so

$$s = p - (\frac{a}{A} + \frac{b}{B} + \ldots \frac{c}{C}) = p - \Sigma, \text{ say.}$$

Then according as λ couples to $l+$ or $l-$, the relation $\lambda^{-1}P\lambda = S$ now tells us that (modulo 1)

$$-l + p - (l + z) = p - \Sigma - z \quad \text{or} \quad l - (p - (l - z)) = p - \Sigma - z,$$

whence (again modulo 1)

$$2l = \Sigma, \quad \text{or} \quad 2l = 2p - \Sigma,$$

and so finally

$$l = \frac{\Sigma + i}{2} \quad \text{or} \quad p - l = \frac{\Sigma + i}{2} \quad (i = 0 \text{ or } 1).$$

In other words, our embellishments so far determine $2l$, but not l itself (modulo 1). We therefore further embellish $*$ or $\bar{*}$ to $*_i$ or $\bar{*}_i$ to distinguish these two cases. Their topological interpretation is rather complicated. Two adjacent generators P and Q correspond to reflections of the circle that each have two fixed points: let p_0 and p_1 be the fixed points for P. Then if the product of the two reflections has rotation number $\frac{a}{A}$, the fixed points for Q are rotated by $\frac{a}{2A}$ and $\frac{a+A}{2A}$ from p_0; call these q_0 and q_1 respectively. In this way the fractions $\frac{a}{A}, \frac{b}{B}, \ldots$ enable us to continue the naming of fixed points all around the circle. The subscripts 0 and 1 tell us whether when we get back to p_0 and p_1 they are in this order or the reverse.

6.4. Cross symbol $\times$

We have embellished a cross symbol to $\times$ or $\bar{\times}$ according as Z couples to $z+$ or $z-$.

What about the value of z? The only relations involving Z and ω are

$$Z^2 = \omega \quad \text{and} \quad 1 = \alpha \ldots \omega.$$

In the case $\bar{\times}$ the first of these implies that $\omega \mapsto 0+$ for any z, and so these relations will remain true if z is continuously varied. In other words, the space groups we obtain here for different values of z are mutually isotopic.

When there are $\times$ symbols, say ${}^{\psi}\!\times^{Y} \cdots {}^{\omega}\!\times^{Z}$, not embellished to $\bar{\times}$ the situation is different, because we have

$$Y \mapsto y+, \ldots, Z \mapsto z+$$

and so the relations $Y^2 = \psi, \ldots, Z^2 = \omega$ imply

$$\psi \mapsto 2y+, \ldots, \omega \mapsto 2z+$$

and now the coupling of the global relation must take the form

$$\underbrace{(\gamma \ldots \lambda \ldots)}_{\substack{\downarrow \\ k+}} \quad \underbrace{\psi}_{\substack{\downarrow \\ 2y+}} \quad \ldots \quad \underbrace{\omega}_{\substack{\downarrow \\ 2z+}} \quad = 1$$

showing that $2(y + \ldots + z)$ must be congruent to $-k$ (modulo 1), where k may be already determined. A subscript $i = 0$ or 1 on such a string of $\times$ symbols will indicate that $y + \ldots + z = \frac{-k+i}{2}$.

However, it may be that the remaining relations allow k to be continuously varied, in which case no subscript is necessary.

Appendix I: Crystallographic groups and Bieberbach's theorem

A *d-dimensional crystallographic group* Γ is a discrete co-compact group of isometries of d-dimensional Euclidean space $\mathbb{E}^d$. In other words, Γ consists of isometries (distance preserving maps), any compact region contains at most finitely many Γ-images of a given point and the Γ-images of some compact region cover $\mathbb{E}^d$.

Every isometry of $\mathbb{E}^d$ can be written as a pair (A, v) consisting of a d-dimensional orthogonal matrix A and a d-dimensional vector v. The result of applying (A, v) to w is $(A, v)w := Aw + v$ and thus the product of two pairs (A, v) and (B, w) is $(AB, Aw + v)$. The matrix A does not depend on the choice of origin in $\mathbb{E}^d$.

An isometry (A, v) is a pure translation exactly if A is the identity matrix. The translations in a given isometry group Γ form a normal subgroup $T(\Gamma)$, namely the kernel of the homomorphism

$$\begin{array}{rccc} \rho\colon & \Gamma & \to & O(d) \\ & (A, v) & \mapsto & A, \end{array}$$

and so there is an exact sequence

$$0 \longrightarrow T(\Gamma) \stackrel{\subseteq}{\longrightarrow} \Gamma \stackrel{\rho}{\longrightarrow} \rho(\Gamma) \to 0.$$

The image $\rho(\Gamma)$ is called the point group of Γ. Since $T(\Gamma)$ is abelian, the conjugation action of Γ on $T(\Gamma)$ is preserved by ρ, and the point group acts in a natural way on the group of translations.

If $T(\Gamma)$ spans (i.e. contains a base for) $\mathbb{R}^d$, then $T(\Gamma)$ is a maximal abelian subgroup of Γ. To see this, consider an element $a = (I, v)$ in $T(\Gamma)$ and an element $b = (A, w)$ in Γ. If a and b commute, we have $w + v = Av + w$, so $v = Av$. Assume that b commutes with every element in $T(\Gamma)$. Since $T(\Gamma)$ spans $\mathbb{R}^d$, it follows that A is the identity matrix and that b is a translation.

The translations subgroup of a crystallographic group is discrete and therefore isomorphic to $\mathbb{Z}^m$ for some $m \leq d$. A famous theorem by Bieberbach published in 1911 states (among other things) that for a crystallographic group Γ, the point group $\rho(\Gamma)$ is finite and $T(\Gamma)$ has full rank. Consequently, $\rho(\Gamma)$ is isomorphic to a finite subgroup of $\mathrm{GL}(d, \mathbb{Z})$.

Bieberbach also proved that in each dimension, there are only finitely many crystallographic groups and that any two such groups are abstractly isomorphic if and only if they are conjugate to each other by an affine map. For a modern proof of Bieberbach's results, see [14].

The point group of a 3-dimensional crystallographic group is a finite subgroup of $O(3)$ and thus the fundamental group of a spherical 2-dimensional orbifold, which contains no rotation of order other than 2, 3, 4 or 6 (the so-called *crystallographic restriction*). All such groups are readily enumerated, for example using the two-dimensional orbifold notation [4].

Appendix II: On elements of order 3

We show that any irreducible space group contains elements of order 3. For the five irreducible point groups all contain 332, which is generated by the two operations:

$$r : (x, y, z) \mapsto (y, z, x) \text{ and } s : (x, y, z) \mapsto (x, -y, -z).$$

We may therefore suppose that our space group contains the operations:

$$R : (x, y, z) \mapsto (a + y, b + z, c + x) \text{ and } S : (x, z, y) \mapsto (d + x, e - y, f - z).$$

We now compute the product $SR^3S^{-1}R$:

$$\begin{array}{c} (x, y, z) \\ \downarrow S \\ (d + x, e - y, f - z) \\ \downarrow R^3 \\ (d + x + a + b + c, e - y + a + b + c, f - z + a + b + c) \\ \downarrow S^{-1} \\ (x + a + b + c, y - a - b - c, z - a - b - c) \\ \downarrow R \\ (y - b - c, z - a - c, x + a + b + 2c). \end{array}$$

Modulo translations, this becomes the order 3 element $(x, y, z) \mapsto (y, z, x)$ of the point group, but it has a fixed point, namely $(-b - c, 0, a + c)$, and so is itself of order 3.

Now we show that there are only two possibilities for the odd subgroup T of an irreducible space group. The axes of the order 3 elements are in the four directions

$$(1, 1, 1),\ (1, -1, -1),\ (-1, 1, -1),\ \text{ and } (-1, -1, 1).$$

The simplest case is when no two axes intersect. In this case we select a closest pair of non-parallel axes, and since any two such pairs are geometrically similar, we may suppose that these are the lines in directions

$$(1,1,1) \text{ through } (0,0,0) \text{ and } (1,-1,-1) \text{ through } (0,1,0),$$

whose shortest distance is $\sqrt{\frac{1}{2}}$. Now the rotations of order three about these two lines generate the group T_2, whose order 3 axes - we call them the "old" axes - are shown in Figure 3. We now show that $T = T_2$; for if not, there must be another order 3 axis not intersecting any of the ones of T_2. However (see Figure 3), the entire space is partitioned into $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$ cubes which each have a pair of opposite vertices on axes of T_2, and any such cube is covered by the two spheres of radius $\sqrt{\frac{1}{2}}$ around these opposite vertices. Any "new" axis must intersect one of these cubes, and therefore its distance from one of the old axes must be less than $\sqrt{\frac{1}{2}}$, a contradiction.

If two axes intersect, then each axis will contain infinitely many such intersection points; we may suppose that a closest pair of these are the points $(0,0,0)$ and $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$, in the direction $(1,1,1)$. Then the group contains the order 3 rotations about the four axes through each of these points, which generate T_1, the odd subgroup of the body centered cubic (bcc) lattice.

We call the order 3 axes of T_1 the "old" axes: they consist of all the lines in directions $(\pm 1, \pm 1, \pm 1)$ through integer points. We now show that these are all the order 3 rotations in the group, so that $T = T_1$. For if not, there would be another order 3 rotation about a "new" axis, and this would differ only by a translation from a rotation of T_1 about a parallel old axis.

However, we show that any translation in the group must be a translation of the bcc lattice, which preserves the set of old axes. Our assumptions imply the minimality condition that if (t,t,t) is a translation in the group, then t must be a multiple of $\frac{1}{2}$.

For since the point group contains the above element r, if there is a translation through (a,b,c), there are others through (b,c,a) and (c,a,b) and so one through $(a+b+c, a+b+c, a+b+c)$, showing that $a+b+c$ must be a multiple of $\frac{1}{2}$. Similarly, $\pm a \pm b \pm c$ must be a multiple of $\frac{1}{2}$ for all choices of sign, and $2a$, $2b$ and $2c$ must also be multiples of $\frac{1}{2}$.

Now transfer the origin to the nearest point of the bcc lattice to (a,b,c) and then change the signs of the coordinates axes, if necessary, to make a, b and c positive. Up to permutations of the coordinates, (a,b,c) is now one of

$$(0,0,0),\ (0,0,\tfrac{1}{2}),\ (\tfrac{1}{4},\tfrac{1}{4},0),\ (\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{2}),\ (\tfrac{1}{2},\tfrac{1}{2},0) \text{ or } (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}),$$

the first and last of which are in the bcc lattice and so preserve the set of old axes. If any of the other four corresponded to a translation in the group, there would be a new axis through it in the direction $(1,1,-1)$, but this line would pass through the point

$$(\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4}),\ (\tfrac{1}{8},\tfrac{1}{8},\tfrac{1}{8}),\ (\tfrac{3}{8},\tfrac{3}{8},\tfrac{3}{8}), \text{ or } (\tfrac{1}{4},\tfrac{1}{4},\tfrac{1}{4}),$$

respectively, contradicting the minimality condition.

Appendix III: The orbifold notation for two-dimensional groups

The *orbifold* of a two-dimensional symmetry group of the Euclidean plane (or the sphere or hyperbolic plane) is "the surface divided by the group". The *orbit* of a point p under a group Γ is the set of all images of p under elements of Γ.

Our *orbifold symbol*

$$\circ \dots \circ GHJ \dots * AB \dots C * \alpha\beta \dots \times \dots \times$$

indicates the features of the orbifold. Here the letters represent numbers: these numbers together with the symbols $\circ$, $*$ and $\times$ we call the *characters* of the orbifold symbol.

Here we can freely permute the numbers G, H, J that represent gyrations and also the parts $*AB \dots C$, $*\alpha\beta$, $\dots$, that represent boundaries, and cyclically permute the numbers A, B, C that represent corners on any given boundary. Finally, we can always reverse the cyclic orders for all boundaries simultaneously, and individually if a $\times$ character is present.

We shall now explain the meanings of the different parts of our symbol. Like all connected two-dimensional manifolds, the orbifold can be obtained from a sphere by possibly punching some holes so as to yield boundary curves (indicated by $*$), and maybe adjoining a number of handles ($\circ$) or crosscaps ($\times$). However, an orbifold is slightly more than a topological manifold, because it inherits a metric from the original surface, which means in particular that angles are defined on it.

Numbers $A, B, \dots, C$ added after a star indicate *corner points*, that is points on the corresponding boundary curve at which the angles are $\frac{\pi}{A}, \frac{\pi}{B}, \dots, \frac{\pi}{C}$. Finally, numbers $G, H, J \dots$ not after any star represent *cone-points*, that is non-boundary points at which the total angles are $\frac{2\pi}{G}, \frac{2\pi}{H}, \frac{2\pi}{J} \dots$.

It is a well-known principle that if a simply-connected manifold is divided by a group Γ to obtain another manifold, then the fundamental group of the quotient manifold is isomorphic to Γ. What happens is that a path from the base point to itself in the quotient manifold lifts to a path in the original manifold that might not return to the base point, in which case it corresponds to a non-trivial element of Γ.

This principle applies also when the quotient space is a more general orbifold, except that some care is required for the definitions. The important point is that a path that bounces off a mirror boundary in the orbifold should be lifted to a path that goes through the corresponding mirror in the original surface.

Figure 8 shows the paths in the two-dimensional orbifold whose lifts are the generators for the corresponding group. We chose a base point in the upper half plane and for each of the features

$$\circ \dots A \dots * abc \dots \times$$

we have one Greek generator corresponding to a path that circumnavigates that feature in the positive direction, and maybe some Latin generators.

For a $\circ$ symbol, represented in the figure by a bridge, the two Latin generators X, Y are homology generators for the handle so formed. They satisfy the relations:

$$X^{-1}Y^{-1}XY = [X, Y] = \alpha.$$

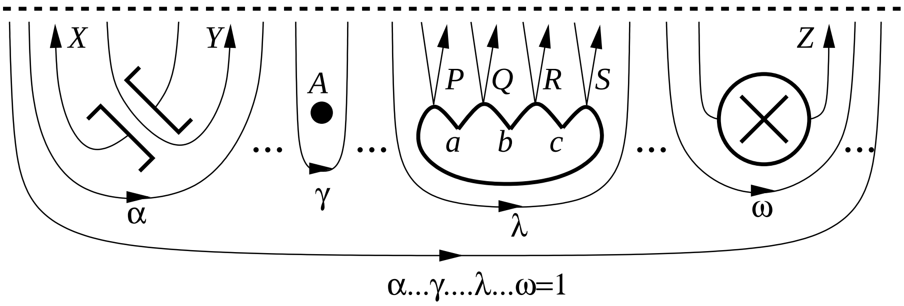


Figure 8. Paths in the orbifold that lift to generators of the corresponding group.

For a gyration symbol G, there is no Latin generator, but the corresponding Greek generator γ satisfies the relation

$$\gamma^G = 1.$$

For a mirror boundary with n corners, there are $n+1$ Latin generators $P, Q, \ldots, S$ corresponding to paths that bounce off the boundary and are separated by the corners. These correspond to reflections in the group that satisfy the relations

$$1 = P^2 = (PQ)^A = Q^2 = (QR)^B = R^2 = (RS)^C = S^2 \text{ and } \lambda^{-1}P\lambda = S.$$

Finally, for a crosscap $\times$, represented in our figure by a cross inside a circle whose opposite points are to be identified, the Latin generator Z corresponds to a path "through" the crosscap and satisfies the relation:

$$Z^2 = \omega.$$

A complete presentation for the group is obtained by combining the generators and relations that we have described for each feature, and adjoining the *global relation*:

$$\alpha \ldots \gamma \ldots \lambda \ldots \omega = 1$$

which asserts that the product of all Greek generators is 1.

We propose the following notation for this set of generators and relations:

$$\circ^{XY} \ldots G^{\gamma} \ldots {}^{\lambda}{*}^{P}A^{Q}B^{R}C^{S} \ldots \times^{Z}.$$

Appendix IV: Details of the enumeration

The enumeration proceeds by assigning coupling maps $k\pm$ to the operations in all possible ways that yield homomorphisms modulo K. The following remarks are helpful:

1. We often use *reduced* names, obtained by omitting subscripts, when their values are unimportant or determined. We have already remarked that the subscript on the digit A between two punctuation marks is 0 or $\frac{A}{2}$ according as these are the same or different. In the reduced name $(*A_aB_b \ldots C_c)$ the omitted subscript on $*$ is necessarily $\frac{a}{A} + \frac{b}{B} + \ldots + \frac{c}{C}$.

2. In ${}^{\lambda}{*}^{P}A^{Q}B^{R}\dots C^{S}$ if A (say) is odd, then P and Q are conjugate. This entails that the punctuation marks (if any) on the two sides of A are equal. So for example, in the case of ${*}^{P}6^{Q}3^{R}2$ (where it is B that is odd) and assuming that all maps couple to $+$ elements we get only 4 cases ∗·6·3·2, ∗·6:3:2, ∗:6·3·2 and ∗:6:3:2, rather than 8.
3. Often certain parameters can be freely varied without effecting the truth of the relations. For example, this happens for $\circ^{XY}$: $1 = [X, Y]$ when $X \mapsto x+$ and $Y \mapsto y+$, since all $k+$ maps commute. So this leads to a single case, for which we choose the couplings $X \mapsto 0+$ and $Y \mapsto 0+$ in Table 1.
4. It suffices to work up to symmetry. We note several symmetries:
   (a) Changing the sign of the z coordinates: this negates all the constants in the maps $k+$ and $k-$. This has obvious effects on our names, for example $(*3_1 3_1 3_1) = (*3_2 3_2 3_2)$, since $\frac{1}{3}$ and $\frac{2}{3}$ are negatives modulo 1.
   (b) Permuting certain generators: E.g. in ${*}^{P}4^{Q}4^{R}2$ we can interchange P and R, in ${*}^{P}3^{Q}3^{R}3$ we can apply any permutation, and in ${*}^{P}2^{Q}2^{R}2^{S}2$ we can cyclically permute P, Q, R, S or reverse their order. So we have the equalities:

$$\begin{aligned}(*{\cdot}4{\cdot}4{:}2) &= (*{:}4{\cdot}4{\cdot}2)\\ (*3_0 3_1 3_2) &= (*3_1 3_2 3_0)\\ (*2_0 2_0 2_1 2_1) &= (*2_0 2_1 2_1 2_0).\end{aligned}$$

   (c) Gyrations can be listed in any order. So in the interval-fiber case [2222] (where all maps couple to $0+$ or $\frac{1}{2}+$), the relation $1 = \gamma\delta\epsilon\zeta$ entails that in fact an even number couple to $\frac{1}{2}+$, leaving just three cases:

$$\begin{aligned}0+\,0+\,0+\,0+ &= [2_0 2_0 2_0 2_0]\\ 0+\,0+\,\tfrac{1}{2}+\,\tfrac{1}{2}+ &= [2_0 2_0 2_1 2_1]\\ \tfrac{1}{2}+\,\tfrac{1}{2}+\,\tfrac{1}{2}+\,\tfrac{1}{2}+ &= [2_1 2_1 2_1 2_1].\end{aligned}$$

   Names that differ only by the obvious symmetries we have described so far will be regarded as equal.
   (d) There are more subtle cases: in ${}^{\gamma}2{*}^{P}2^{Q}2^{R}$, we eliminated R as a generator, since $R = \gamma P \gamma^{-1}$. But to explore the symmetry it is best to include both R and $S = \gamma Q \gamma^{-1}$. We will study the case when all generators couple to $-$ elements, say $\gamma \mapsto 0-$, $P \mapsto p-$, $Q \mapsto q-$, $R \mapsto r-$ and $S \mapsto s-$.
   Then we have $r = -p$, $s = -q$ (modulo 1) and the symmetry permutes p, q, r, s cyclically, giving the equivalences and reduced names below:

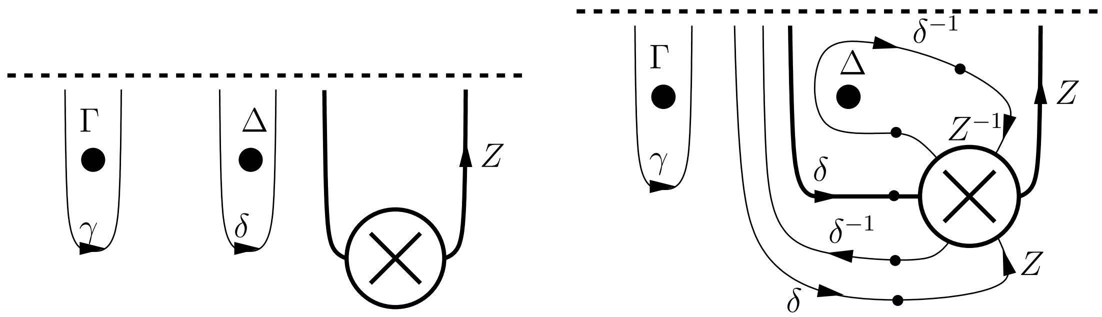
$$\begin{array}{lll}
0-\;0-\;0-\;0- & (2\bar{*}_0 2_0 2_0) & \\
\tfrac{1}{2}-\;\tfrac{1}{2}-\;\tfrac{1}{2}-\;\tfrac{1}{2}- & (2\bar{*}_1 2_0 2_0) & \\
\left.\begin{array}{ll}\tfrac{1}{2}-\;0-\;\tfrac{1}{2}-\;0- & (2\bar{*}_0 2_1 2_1)\\ 0-\;\tfrac{1}{2}-\;0-\;\tfrac{1}{2}- & (2\bar{*}_1 2_1 2_1)\end{array}\right\} & & (2\bar{*} 2_1 2_1)\\
\left.\begin{array}{ll}\tfrac{1}{4}-\;\tfrac{1}{4}-\;\tfrac{3}{4}-\;\tfrac{3}{4}- & (2\bar{*}_0 2_0 2_1)\\ \tfrac{1}{4}-\;\tfrac{3}{4}-\;\tfrac{3}{4}-\;\tfrac{1}{4}- & (2\bar{*}_0 2_1 2_0)\\ \tfrac{3}{4}-\;\tfrac{3}{4}-\;\tfrac{1}{4}-\;\tfrac{1}{4}- & (2\bar{*}_1 2_0 2_1)\\ \tfrac{3}{4}-\;\tfrac{1}{4}-\;\tfrac{1}{4}-\;\tfrac{3}{4}- & (2\bar{*}_1 2_1 2_0)\end{array}\right\} & & (2\bar{*} 2_0 2_1)
\end{array}$$

The case ${}^{\gamma}4{*}^{P}2^{Q}$ is similar, with a symmetry interchanging P and $Q = \gamma P \gamma^{-1}$. The solution in which the generators γ, P, Q all couple to $-$ elements $0-$, $p-$ and $q-$ are:

$$\begin{array}{lll} 0-\ 0- & (4\bar{*}_0 2_0) & \\ \frac{1}{2}-\ \frac{1}{2}- & (4\bar{*}_1 2_0) & \\ \frac{1}{4}-\ \frac{3}{4}- & (4\bar{*}_0 2_1) & \multirow{2}{*}{$\left.\right\}$ (4\bar{*}2_1)} \\ \frac{3}{4}-\ \frac{1}{4}- & (4\bar{*}_1 2_1) & \end{array}$$

(e) A still more subtle symmetry arises in the presence of a cross cap.

We draw the (reduced) set of generators for ${}^{\gamma}2^{\delta}2\times^{Z}$ in Figure 9(a). Figure 9(b) shows that an alternative set of generators is $\gamma' = \gamma$, $\delta' = \delta Z \delta^{-1} Z^{-1} \delta^{-1}$ and $Z' = \delta Z$.



$1 = \gamma^2 = \delta^2 = \gamma\delta Z^2$

(a) (b)

Figure 9. ${}^{\gamma}2^{\delta}2\times^{Z}$ possesses a symmetry that transforms the set of generators indicated in (a) to the set of generators depicted in (b).

This is because the path Z does not separate the plane, so the second gyration point Δ can be pulled through it. So the paths $\delta Z \delta^{-1} Z^{-1} \delta^{-1}$ and δZ of the figure are topologically like δ and Z.

The significant fact here is that Z gets multiplied by δ; the replacement of δ by δ' usually has no effect since δ' is conjugate to δ^{-1}. So in the presence of a 2_1, $\times_0$ is equivalent to $\times_1$ and in the presence of a 2, $\bar{\times}$ is equivalent to $\times$.

This observation reduces the preliminary list of 14 fibrations of $22\times$ to 10:

$$\begin{array}{lll} [2_0 2_0 \times_0] & (2_0 2_0 \times_0) & (2_0 2_0 \bar{\times}) \\ [2_0 2_0 \times_1] & (2_0 2_0 \times_1) & (2_1 2_1 \bar{\times}) \\ \left.\begin{array}{l}[2_1 2_1 \times_0] \\ [2_1 2_1 \times_1]\end{array}\right\} [2_1 2_1 \times] & \left.\begin{array}{l}(2_0 2_1 \times_0) \\ (2_0 2_1 \times_1)\end{array}\right\} (2_0 2_1 \times) & \left.\begin{array}{l}(22\times) \\ (22\bar{\times})\end{array}\right\} (22\times) \\ & \left.\begin{array}{l}(2_1 2_1 \times_0) \\ (2_1 2_1 \times_1)\end{array}\right\} (2_1 2_1 \times) & \end{array}$$

It turns out that these are all distinct.

Main tables

Table 1. This table consists of 17 blocks. Each is headed by a plane group H and a set of relations for H. This is followed by a number of lines corresponding to different fibrations over H. In each such line, we list the fibrifold name for the resulting space group Γ (first column), the appropriate couplings for the generators of H (second column), the point group of Γ (third column) and finally, the IT number of Γ (fourth column).

Table 1

Plane group: $*632$
Relations $*{}^P6{}^Q3{}^R2$:
$1 = P^2 = (PQ)^6 = Q^2 = (QR)^3 = R^2 = (RP)^2$

Fibrifold name	Couplings for $P\ Q\ R\ (I)$	Point group	Int. no.
[*·6·3·2]	$0+0+0+0-$	*226	191
[*:6·3·2]	$\frac12+0+0+0-$	*226	194
[*·6:3:2]	$0+\frac12+\frac12+0-$	*226	193
[*:6:3:2]	$\frac12+\frac12+\frac12+0-$	*226	192
(*·6·3·2)	$0+0+0+$	*66	183
(*:6·3·2)	$\frac12+0+0+$	*66	186
(*·6:3:2)	$0+\frac12+\frac12+$	*66	185
(*:6:3:2)	$\frac12+\frac12+\frac12+$	*66	184
(*6·3·2)	$0-0+0+$	2*3	164
(*6:3:2)	$0-\frac12+\frac12+$	2*3	165
$(*{\cdot}6\,3_0 2)$	$0+0-0-$	2*3	162
$(*{\cdot}6\,3_1 2)$	$0+\frac13-0-$	2*3	166
$(*{:}6\,3_0 2)$	$\frac12+0-0-$	2*3	163
$(*{:}6\,3_1 2)$	$\frac12+\frac13-0-$	2*3	167
$(*6_0 3_0 2_0)$	$0-0-0-$	226	177
$(*6_1 3_1 2_1)$	$\frac12-\frac13-0-$	226	178
$(*6_2 3_2 2_0)$	$0-\frac23-0-$	226	180
$(*6_3 3_0 2_1)$	$\frac12-0-0-$	226	182

Plane group: 632
Relations ${}^\gamma6{}^\delta3{}^\epsilon2$:
$1 = \gamma^6 = \delta^3 = \epsilon^2 = \gamma\delta\epsilon$

Fibrifold name	Couplings for $\gamma\ \delta\ \epsilon\ (I)$	Point group	Int. no.
$[6_0 3_0 2_0]$	$0+0+0+0-$	6*	175
$[6_3 3_0 2_1]$	$\frac12+0+\frac12+0-$	6*	176
$(6_0 3_0 2_0)$	$0+0+0+$	66	168
$(6_1 3_1 2_1)$	$\frac16+\frac13+\frac12+$	66	169
$(6_2 3_2 2_0)$	$\frac13+\frac23+0+$	66	171
$(6_3 3_0 2_1)$	$\frac12+0+\frac12+$	66	173
$(6\,3_0 2)$	$0-0+0-$	3×	147
$(6\,3_1 2)$	$\frac13-\frac13+0-$	3×	148

Plane group: $*442$
Relations $*{}^P4{}^Q4{}^R2$:
$1 = P^2 = (PQ)^4 = Q^2 = (QR)^4 = R^2 = (RP)^2$

Fibrifold name	Couplings for $P\ Q\ R\ (I)$	Point group	Int. no.
[*·4·4·2]	$0+0+0+0-$	*224	123
[*·4·4:2]	$0+0+\frac12+0-$	*224	139
[*·4:4·2]	$0+\frac12+0+0-$	*224	131
[*·4:4:2]	$0+\frac12+\frac12+0-$	*224	140
[*:4·4:2]	$\frac12+0+\frac12+0-$	*224	132
[*:4:4:2]	$\frac12+\frac12+\frac12+0-$	*224	124
(*·4·4·2)	$0+0+0+$	*44	99
(*·4·4:2)	$0+0+\frac12+$	*44	107
(*·4:4·2)	$0+\frac12+0+$	*44	105
(*·4:4:2)	$0+\frac12+\frac12+$	*44	108
(*:4·4:2)	$\frac12+0+\frac12+$	*44	101
(*:4:4:2)	$\frac12+\frac12+\frac12+$	*44	103
(*4·4·2)	$0-0+0+$	*224	129
(*4·4:2)	$0-0+\frac12+$	*224	137
(*4:4·2)	$0-\frac12+0+$	*224	138
(*4:4:2)	$0-\frac12+\frac12+$	*224	130
(*·4 4·2)	$0+0-0+$	2*2	115
(*·4 4:2)	$0+0-\frac12+$	2*2	121
(*:4 4:2)	$\frac12+0-\frac12+$	2*2	116
$(*4_0 4{\cdot}2)$	$0-0-0+$	*224	125
$(*4_1 4{\cdot}2)$	$\frac14-0-0+$	*224	141
$(*4_2 4{\cdot}2)$	$\frac12-0-0+$	*224	134
$(*4_0 4{:}2)$	$0-0-\frac12+$	*224	126
$(*4_1 4{:}2)$	$\frac14-0-\frac12+$	*224	142
$(*4_2 4{:}2)$	$\frac12-0-\frac12+$	*224	133
$(*4{\cdot}4\,2_0)$	$0-0+0-$	2*2	111
$(*4{\cdot}4\,2_1)$	$\frac12-0+0-$	2*2	119

Table 1 continued

$(*4{:}4\,2_0)$	$0-\frac{1}{2}+0-$	$2{*}2$	112
$(*4{:}4\,2_1)$	$\frac{1}{2}-\frac{1}{2}+0-$	$2{*}2$	120
$(*4_04_02_0)$	$0-0-0-$	224	89
$(*4_14_12_1)$	$\frac{1}{2}-\frac{1}{4}-0-$	224	91
$(*4_24_22_0)$	$0-\frac{1}{2}-0-$	224	93
$(*4_24_02_1)$	$\frac{1}{2}-0-0-$	224	97
$(*4_34_12_0)$	$0-\frac{1}{4}-0-$	224	98

Plane group: $4{*}2$
Relations ${}^{\gamma}4{*}{}^{P}2$:
$1=\gamma^4=P^2=[P,\gamma]^2$

Fibrifold name	Couplings for γ P (I)	Point group	Int. no.
$[4_0{*}{\cdot}2]$	$0+0+0-$	$*224$	127
$[4_2{*}{\cdot}2]$	$\frac{1}{2}+0+0-$	$*224$	136
$[4_0{*}{:}2]$	$0+\frac{1}{2}+0-$	$*224$	128
$[4_2{*}{:}2]$	$\frac{1}{2}+\frac{1}{2}+0-$	$*224$	135
$(4_0{*}{\cdot}2)$	$0+0+$	$*44$	100
$(4_1{*}{\cdot}2)$	$\frac{1}{4}+0+$	$*44$	109
$(4_2{*}{\cdot}2)$	$\frac{1}{2}+0+$	$*44$	102
$(4_0{*}{:}2)$	$0+\frac{1}{2}+$	$*44$	104
$(4_1{*}{:}2)$	$\frac{1}{4}+\frac{1}{2}+$	$*44$	110
$(4_2{*}{:}2)$	$\frac{1}{2}+\frac{1}{2}+$	$*44$	106
$(4\bar{*}{\cdot}2)$	$0-0+$	$2{*}2$	113
$(4\bar{*}{:}2)$	$0-\frac{1}{2}+$	$2{*}2$	114
$(4_0{*}2_0)$	$0+0-$	224	90
$(4_1{*}2_1)$	$\frac{1}{4}+0-$	224	92
$(4_2{*}2_0)$	$\frac{1}{2}+0-$	224	94
$(4\bar{*}_02_0)$	$0-0-$	$2{*}2$	117
$(4\bar{*}_12_0)$	$0-\frac{1}{2}-$	$2{*}2$	118
$(4\bar{*}2_1)$	$0-\frac{1}{4}-$	$2{*}2$	122

Plane group: 442
Relations ${}^{\gamma}4\,{}^{\delta}4\,{}^{\epsilon}2$:
$1=\gamma^4=\delta^4=\epsilon^2=\gamma\delta\epsilon$

Fibrifold name	Couplings for γ δ ϵ (I)	Point group	Int. no.
$[4_04_02_0]$	$0+0+0+0-$	$4*$	83
$[4_24_22_0]$	$\frac{1}{2}+\frac{1}{2}+0+0-$	$4*$	84
$[4_24_02_1]$	$\frac{1}{2}+0+\frac{1}{2}+0-$	$4*$	87
$(4_04_02_0)$	$0+0+0+$	44	75
$(4_14_12_1)$	$\frac{1}{4}+\frac{1}{4}+\frac{1}{2}+$	44	76
$(4_24_22_0)$	$\frac{1}{2}+\frac{1}{2}+0+$	44	77
$(4_24_02_1)$	$\frac{1}{2}+0+\frac{1}{2}+$	44	79
$(4_34_12_0)$	$\frac{3}{4}+\frac{1}{4}+0+$	44	80
$(4\,4_02)$	$0+0-0-$	$4*$	85
$(4\,4_12)$	$\frac{1}{4}+0-\frac{1}{4}-$	$4*$	88
$(4\,4_22)$	$\frac{1}{2}+0-\frac{1}{2}-$	$4*$	86
$(4\,4\,2_0)$	$0-0-0+$	$2\times$	81
$(4\,4\,2_1)$	$\frac{1}{2}-0-\frac{1}{2}+$	$2\times$	82

Plane group: $*333$
Relations $*{}^{P}3\,{}^{Q}3\,{}^{R}3$:
$1=P^2=(PQ)^3=Q^2=(QR)^3=R^2=(RP)^3$

Fibrifold name	Couplings for P Q R (I)	Point group	Int. no.
$[*{\cdot}3{\cdot}3{\cdot}3]$	$0+0+0+0-$	$*223$	187
$[*{:}3{:}3{:}3]$	$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+0-$	$*223$	188
$(*{\cdot}3{\cdot}3{\cdot}3)$	$0+0+0+$	$*33$	156
$(*{:}3{:}3{:}3)$	$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+$	$*33$	158
$(*3_03_03_0)$	$0-0-0-$	223	149
$(*3_13_13_1)$	$\frac{2}{3}-\frac{1}{3}-0-$	223	151
$(*3_03_13_2)$	$\frac{1}{3}-\frac{1}{3}-0-$	223	155

Plane group: $3{*}3$
Relations ${}^{\gamma}3{*}{}^{P}3$:
$1=\gamma^3=P^2=[P,\gamma]^3$

Fibrifold name	Couplings for γ P (I)	Point group	Int. no.
$[3_0{*}{\cdot}3]$	$0+0+0-$	$*223$	189
$[3_0{*}{:}3]$	$0+\frac{1}{2}+0-$	$*223$	190
$(3_0{*}{\cdot}3)$	$0+0+$	$*33$	157
$(3_1{*}{\cdot}3)$	$\frac{1}{3}+0+$	$*33$	160
$(3_0{*}{:}3)$	$0+\frac{1}{2}+$	$*33$	159
$(3_1{*}{:}3)$	$\frac{1}{3}+\frac{1}{2}+$	$*33$	161
$(3_0{*}3_0)$	$0+0-$	223	150
$(3_1{*}3_1)$	$\frac{1}{3}+0-$	223	152

Plane group: 333
Relations ${}^{\gamma}3\,{}^{\delta}3\,{}^{\epsilon}3$:
$1=\gamma^3=\delta^3=\epsilon^3=\gamma\delta\epsilon$

Fibrifold name	Couplings for γ δ ϵ (I)	Point group	Int. no.

Table 1 continued

$[3_03_03_0]$	$0+0+0+0-$	$3*$	174
$(3_03_03_0)$	$0+0+0+$	33	143
$(3_13_13_1)$	$\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+$	33	144
$(3_03_13_2)$	$0+\frac{1}{3}+\frac{2}{3}+$	33	146

Plane group: $*2222$
Relations $*^P2^Q2^R2^S2$:
$1 = P^2 = (PQ)^2 = Q^2 = (QR)^2 = R^2 = (RS)^2 = S^2 = (SP)^2$

Fibrifold name	Couplings for $P\ Q\ R\ S\ (I)$	Point group	Int. no.
$[*{\cdot}2{\cdot}2{\cdot}2{\cdot}2]$	$0+0+0+0+0-$	$*222$	47
$[*{\cdot}2{\cdot}2{\cdot}2{:}2]$	$0+0+0+\frac{1}{2}+0-$	$*222$	65
$[*{\cdot}2{\cdot}2{:}2{:}2]$	$0+0+\frac{1}{2}+\frac{1}{2}+0-$	$*222$	69
$[*{\cdot}2{:}2{\cdot}2{:}2]$	$0+\frac{1}{2}+0+\frac{1}{2}+0-$	$*222$	51
$[*{\cdot}2{:}2{:}2{:}2]$	$0+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+0-$	$*222$	67
$[*{:}2{:}2{:}2{:}2]$	$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+0-$	$*222$	49
$(*{\cdot}2{\cdot}2{\cdot}2{\cdot}2)$	$0+0+0+0+$	$*22$	25
$(*{\cdot}2{\cdot}2{\cdot}2{:}2)$	$0+0+0+\frac{1}{2}+$	$*22$	38
$(*{\cdot}2{\cdot}2{:}2{:}2)$	$0+0+\frac{1}{2}+\frac{1}{2}+$	$*22$	42
$(*{\cdot}2{:}2{\cdot}2{:}2)$	$0+\frac{1}{2}+0+\frac{1}{2}+$	$*22$	26
$(*{\cdot}2{:}2{:}2{:}2)$	$0+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+$	$*22$	39
$(*{:}2{:}2{:}2{:}2)$	$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+$	$*22$	27
$(*2{\cdot}2{\cdot}2{\cdot}2)$	$0+0+0+0-$	$*222$	51
$(*2{\cdot}2{\cdot}2{:}2)$	$0+0+\frac{1}{2}+0-$	$*222$	63
$(*2{\cdot}2{:}2{\cdot}2)$	$0+\frac{1}{2}+0+0-$	$*222$	55
$(*2{\cdot}2{:}2{:}2)$	$0+\frac{1}{2}+\frac{1}{2}+0-$	$*222$	64
$(*2{:}2{\cdot}2{:}2)$	$\frac{1}{2}+0+\frac{1}{2}+0-$	$*222$	57
$(*2{:}2{:}2{:}2)$	$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+0-$	$*222$	54
$(*2_02{\cdot}2{\cdot}2)$	$0-0-0+0+$	$*222$	67
$(*2_12{\cdot}2{\cdot}2)$	$\frac{1}{2}-0-0+0+$	$*222$	74
$(*2_02{\cdot}2{:}2)$	$0-0-0+\frac{1}{2}+$	$*222$	72
$(*2_12{\cdot}2{:}2)$	$\frac{1}{2}-0-0+\frac{1}{2}+$	$*222$	64
$(*2_02{:}2{:}2)$	$0-0-\frac{1}{2}+\frac{1}{2}+$	$*222$	68
$(*2_12{:}2{:}2)$	$\frac{1}{2}-0-\frac{1}{2}+\frac{1}{2}+$	$*222$	73
$(*2{\cdot}2\,2{\cdot}2)$	$0-0+0-0+$	$2*$	10
$(*2{\cdot}2\,2{:}2)$	$0-0+0-\frac{1}{2}+$	$2*$	12
$(*2{:}2\,2{:}2)$	$0-\frac{1}{2}+0-\frac{1}{2}+$	$2*$	13
$(*2_02_02{\cdot}2)$	$0-0-0-0+$	$*222$	49
$(*2_02_12{\cdot}2)$	$0-0-\frac{1}{2}-0+$	$*222$	66
$(*2_12_12{\cdot}2)$	$\frac{1}{2}-0-\frac{1}{2}-0+$	$*222$	53
$(*2_02_02{:}2)$	$0-0-0-\frac{1}{2}+$	$*222$	50

Table 1 continued

$(*2_02_12{:}2)$	$0-0-\frac{1}{2}-\frac{1}{2}+$	$*222$	68
$(*2_12_12{:}2)$	$\frac{1}{2}-0-\frac{1}{2}-\frac{1}{2}+$	$*222$	54
$(*2_02_02_02_0)$	$0-0-0-0-$	222	16
$(*2_02_02_12_1)$	$0-0-0-\frac{1}{2}-$	222	21
$(*2_02_12_02_1)$	$0-0-\frac{1}{2}-\frac{1}{2}-$	222	22
$(*2_12_12_12_1)$	$0-\frac{1}{2}-0-\frac{1}{2}-$	222	17

Plane group: $2*22$
Relations ${}^\gamma2*^P2^Q2$:
$1 = \gamma^2 = P^2 = (PQ)^2 = Q^2 = (Q\gamma P\gamma^{-1})^2$

Fibrifold name	Couplings for $\gamma\ P\ Q\ (I)$	Point group	Int. no.
$[2_0*{\cdot}2{\cdot}2]$	$0+0+0+0-$	$*222$	65
$[2_1*{\cdot}2{\cdot}2]$	$\frac{1}{2}+0+0+0-$	$*222$	71
$[2_0*{\cdot}2{:}2]$	$0+0+\frac{1}{2}+0-$	$*222$	74
$[2_1*{\cdot}2{:}2]$	$\frac{1}{2}+0+\frac{1}{2}+0-$	$*222$	63
$[2_0*{:}2{:}2]$	$0+\frac{1}{2}+\frac{1}{2}+0-$	$*222$	66
$[2_1*{:}2{:}2]$	$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+0-$	$*222$	72
$(2_0*{\cdot}2{\cdot}2)$	$0+0+0+$	$*22$	35
$(2_1*{\cdot}2{\cdot}2)$	$\frac{1}{2}+0+0+$	$*22$	44
$(2_0*{\cdot}2{:}2)$	$0+0+\frac{1}{2}+$	$*22$	46
$(2_1*{\cdot}2{:}2)$	$\frac{1}{2}+0+\frac{1}{2}+$	$*22$	36
$(2_0*{:}2{:}2)$	$0+\frac{1}{2}+\frac{1}{2}+$	$*22$	37
$(2_1*{:}2{:}2)$	$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+$	$*22$	45
$(2_0*2{\cdot}2)$	$0+0+0-$	$*222$	53
$(2_1*2{\cdot}2)$	$\frac{1}{2}+0+0-$	$*222$	58
$(2_0*2{:}2)$	$0+\frac{1}{2}+0-$	$*222$	52
$(2_1*2{:}2)$	$\frac{1}{2}+\frac{1}{2}+0-$	$*222$	60
$(2\bar{*}{\cdot}2{\cdot}2)$	$0-0+0+$	$*222$	59
$(2\bar{*}{\cdot}2{:}2)$	$0-0+\frac{1}{2}+$	$*222$	62
$(2\bar{*}{:}2{:}2)$	$0-\frac{1}{2}+\frac{1}{2}+$	$*222$	56
$(2\bar{*}2{\cdot}2)$	$0-0-0+$	$2*$	12
$(2\bar{*}2{:}2)$	$0-0-\frac{1}{2}+$	$2*$	15
$(2_0*2_02_0)$	$0+0-0-$	222	21
$(2_1*2_02_0)$	$\frac{1}{2}+0-0-$	222	23
$(2_0*2_12_1)$	$0+\frac{1}{2}-0-$	222	24
$(2_1*2_12_1)$	$\frac{1}{2}+\frac{1}{2}-0-$	222	20
$(2\bar{*}_02_02_0)$	$0-0-0-$	$*222$	50
$(2\bar{*}_12_02_0)$	$0-\frac{1}{2}-\frac{1}{2}-$	$*222$	48
$(2\bar{*}2_02_1)$	$0-\frac{1}{4}-\frac{1}{4}-$	$*222$	70

Table 1 continued

$(2\bar{*}2_12_1)$	$0-0-\frac{1}{2}-$	$*222$	52

Plane group: $22*$
Relations ${}^{\gamma}2\,{}^{\delta}2{*}\,{}^{P}$:
$1=\gamma^2=\delta^2=P^2=[P,\gamma\delta]$

Fibrifold name	Couplings for $\gamma\ \ \delta\ \ P\ \ (I)$	Point group	Int. no.
$[2_02_0{*}\cdot]$	$0+0+0+0-$	$*222$	51
$[2_02_1{*}\cdot]$	$0+\frac{1}{2}+0+0-$	$*222$	63
$[2_12_1{*}\cdot]$	$\frac{1}{2}+\frac{1}{2}+0+0-$	$*222$	59
$[2_02_0{*}{:}]$	$0+0+\frac{1}{2}+0-$	$*222$	53
$[2_02_1{*}{:}]$	$0+\frac{1}{2}+\frac{1}{2}+0-$	$*222$	64
$[2_12_1{*}{:}]$	$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+0-$	$*222$	57
$(2_02_0{*}\cdot)$	$0+0+0+$	$*22$	28
$(2_02_1{*}\cdot)$	$0+\frac{1}{2}+0+$	$*22$	40
$(2_12_1{*}\cdot)$	$\frac{1}{2}+\frac{1}{2}+0+$	$*22$	31
$(2_02_0{*}{:})$	$0+0+\frac{1}{2}+$	$*22$	30
$(2_02_1{*}{:})$	$0+\frac{1}{2}+\frac{1}{2}+$	$*22$	41
$(2_12_1{*}{:})$	$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+$	$*22$	29
(2_02_0*)	$0+0+0-$	222	17
(2_02_1*)	$0+\frac{1}{2}+0-$	222	20
(2_12_1*)	$\frac{1}{2}+\frac{1}{2}+0-$	222	18
$(2_02\bar{*}\cdot)$	$0+0-0+$	$*222$	57
$(2_12\bar{*}\cdot)$	$\frac{1}{2}+0-0+$	$*222$	62
$(2_02\bar{*}{:})$	$0+0-\frac{1}{2}+$	$*222$	60
$(2_12\bar{*}{:})$	$\frac{1}{2}+0-\frac{1}{2}+$	$*222$	61
$(2_02\bar{*}_0)$	$0+0-0-$	$*222$	54
$(2_02\bar{*}_1)$	$0+\frac{1}{2}-0-$	$*222$	52
$(2_12\bar{*}_0)$	$\frac{1}{2}+\frac{1}{2}-0-$	$*222$	56
$(2_12\bar{*}_1)$	$\frac{1}{2}+0-0-$	$*222$	60
$(2\,2{*}\cdot)$	$0-0-0+$	$2*$	11
$(2\,2{*}{:})$	$0-0-\frac{1}{2}+$	$2*$	14
$(2\,2*_0)$	$0-0-0-$	$2*$	13
$(2\,2*_1)$	$0-\frac{1}{2}-0-$	$2*$	15

Plane group: $22\times$
Relations ${}^{\gamma}2\,{}^{\delta}2\times{}^{Z}$:
$1=\gamma^2=\delta^2=\gamma\delta Z^2$

Fibrifold name	Couplings for $\gamma\ \ \delta\ \ Z\ \ (I)$	Point group	Int. no.
$[2_02_0\times_0]$	$0+0+0+0-$	$*222$	55
$[2_02_0\times_1]$	$0+0+\frac{1}{2}+0-$	$*222$	58
$[2_12_1\times]$	$\frac{1}{2}+\frac{1}{2}+0+0-$	$*222$	62
$(2_02_0\times_0)$	$0+0+0+$	$*22$	32
$(2_02_0\times_1)$	$0+0+\frac{1}{2}+$	$*22$	34
$(2_02_1\times)$	$0+\frac{1}{2}+\frac{1}{4}+$	$*22$	43
$(2_12_1\times)$	$\frac{1}{2}+\frac{1}{2}+0+$	$*22$	33
$(2_02_0\bar{\times})$	$0+0+0-$	222	18
$(2_12_1\bar{\times})$	$\frac{1}{2}+\frac{1}{2}+0-$	222	19
$(2\,2\times)$	$0-0-0+$	$2*$	14

Plane group: 2222
Relations ${}^{\gamma}2\,{}^{\delta}2\,{}^{\epsilon}2\,{}^{\zeta}2$:
$1=\gamma^2=\delta^2=\epsilon^2=\zeta^2=\gamma\delta\epsilon\zeta$

Fibrifold name	Couplings for $\gamma\ \ \delta\ \ \epsilon\ \ \zeta\ \ (I)$	Point group	Int. no.
$[2_02_02_02_0]$	$0+0+0+0+0-$	$2*$	10
$[2_02_02_12_1]$	$0+0+\frac{1}{2}+\frac{1}{2}+0-$	$2*$	12
$[2_12_12_12_1]$	$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+0-$	$2*$	11
$(2_02_02_02_0)$	$0+0+0+0+$	22	3
$(2_02_02_12_1)$	$0+0+\frac{1}{2}+\frac{1}{2}+$	22	5
$(2_12_12_12_1)$	$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+$	22	4
$(2_02_0\,2\,2)$	$0+0+0-0-$	$2*$	13
$(2_02_1\,2\,2)$	$0+\frac{1}{2}+0-\frac{1}{2}-$	$2*$	15
$(2_12_1\,2\,2)$	$\frac{1}{2}+\frac{1}{2}+0-0-$	$2*$	14
$(2\,2\,2\,2)$	$0-0-0-0-$	$\times$	2

Plane group: $**$
Relations ${}^{\lambda}{*}\,{}^{P}{*}\,{}^{Q}$:
$1=P^2=Q^2=[\lambda,P]=[\lambda,Q]$

Fibrifold name	Couplings for $\lambda\ \ P\ \ Q\ \ (I)$	Point group	Int. no.
$[*_0\cdot*_0\cdot]$	$0+0+0+0-$	$*22$	25
$[*_1\cdot*_1\cdot]$	$\frac{1}{2}+0+0+0-$	$*22$	38
$[*_0\cdot*_0{:}]$	$0+0+\frac{1}{2}+0-$	$*22$	35
$[*_1\cdot*_1{:}]$	$\frac{1}{2}+0+\frac{1}{2}+0-$	$*22$	42
$[*_0{:}*_0{:}]$	$0+\frac{1}{2}+\frac{1}{2}+0-$	$*22$	28
$[*_1{:}*_1{:}]$	$\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+0-$	$*22$	39
$({*}\cdot{*}\cdot)$	$0+0+0+$	$*$	6
$({*}\cdot{*}{:})$	$0+0+\frac{1}{2}+$	$*$	8
$({*}{:}{*}{:})$	$0+\frac{1}{2}+\frac{1}{2}+$	$*$	7
$(\bar{*}\cdot\bar{*}\cdot)$	$0-0+0+$	$*22$	26

Table 1 continued

$(\bar{*}{\cdot}\bar{*}{:})$	$0-0+\frac{1}{2}+$	$*22$	36
$(\bar{*}{:}\bar{*}{:})$	$0-\frac{1}{2}+\frac{1}{2}+$	$*22$	29
$(*{\cdot}*_0)$	$0+0+0-$	$*22$	28
$(*{\cdot}*_1)$	$\frac{1}{2}+0+0-$	$*22$	40
$(*{:}*_0)$	$0+\frac{1}{2}+0-$	$*22$	32
$(*{:}*_1)$	$\frac{1}{2}+\frac{1}{2}+0-$	$*22$	41
$(\bar{*}{\cdot}\bar{*}_0)$	$0-0+0-$	$*22$	39
$(\bar{*}{\cdot}\bar{*}_1)$	$\frac{1}{2}-0+0-$	$*22$	46
$(\bar{*}{:}\bar{*}_0)$	$0-\frac{1}{2}+0-$	$*22$	45
$(\bar{*}{:}\bar{*}_1)$	$\frac{1}{2}-\frac{1}{2}+0-$	$*22$	41
$(*_0*_0)$	$0+0-0-$	22	3
$(*_1*_1)$	$\frac{1}{2}+0-0-$	22	5
$(\bar{*}_0\bar{*}_0)$	$0-0-0-$	$*22$	27
$(\bar{*}_0\bar{*}_1)$	$0-0-\frac{1}{2}-$	$*22$	37
$(\bar{*}_1\bar{*}_1)$	$0-\frac{1}{2}-\frac{1}{2}-$	$*22$	30

Plane group: $*\times$
Relations $*^P\times^Z$:
$1 = P^2 = [P, Z^2]$

Fibrifold name	Couplings for $P\ \ Z\ \ (I)$	Point group	Int. no.
$[*{\cdot}\times_0]$	$0+0+0-$	$*22$	38
$[*{\cdot}\times_1]$	$0+\frac{1}{2}+0-$	$*22$	44
$[*{:}\times_0]$	$\frac{1}{2}+0+0-$	$*22$	46
$[*{:}\times_1]$	$\frac{1}{2}+\frac{1}{2}+0-$	$*22$	40
$(*{\cdot}\times)$	$0+0+$	$*$	8
$(*{:}\times)$	$\frac{1}{2}+0+$	$*$	9
$(*{\cdot}\bar{\times})$	$0+0-$	$*22$	31
$(*{:}\bar{\times})$	$\frac{1}{2}+0-$	$*22$	33
$(*_0\times_0)$	$0-0+$	$*22$	30
$(*_0\times_1)$	$0-\frac{1}{2}+$	$*22$	34
$(*_1\times)$	$0-\frac{1}{4}+$	$*22$	43
$(*\bar{\times})$	$0-0-$	22	5

Plane group: $\times\times$
Relations $\times^Y\times^Z$:
$1 = Y^2Z^2$

Fibrifold name	Couplings for $Y\ \ Z\ \ (I)$	Point group	Int. no.
$[\times_0\times_0]$	$0+0+0-$	$*22$	26

Table 1 continued

$[\times_0\times_1]$	$0+\frac{1}{2}+0-$	$*22$	36
$[\times_1\times_1]$	$\frac{1}{2}+\frac{1}{2}+0-$	$*22$	31
$(\times\times_0)$	$0+0+$	$*$	7
$(\times\times_1)$	$0+\frac{1}{2}+$	$*$	9
$(\bar{\times}\times_0)$	$0-0+$	$*22$	29
$(\bar{\times}\times_1)$	$0-\frac{1}{2}+$	$*22$	33
$(\bar{\times}\bar{\times})$	$0-0-$	22	4

Plane group: $\circ$
Relations $\circ^{X\,Y}$:
$1 = [X, Y]$

Fibrifold name	Couplings for $X\ \ Y\ \ (I)$	Point group	Int. no.
$[\circ_0]$	$0+0+0-$	$*$	6
$[\circ_1]$	$\frac{1}{2}+\frac{1}{2}+0-$	$*$	8
$(\circ)$	$0+0+$	1	1
$(\bar{\circ}_0)$	$0-0-$	$*$	7
$(\bar{\circ}_1)$	$0-\frac{1}{2}-$	$*$	9

Table 2a. Here we list the 35 irreducible groups, organized by their point groups. Each line describes one such group Γ, listing its primary name (column one), its international number and name (column two) and its secondary name(s) (column three). (‡) indicates that a group has two enantiomorphous forms and we report only the lower IT number.

Table 2a

Prim. name	Internat. no. and name	Secondary names
Point Group $*432$,	nos. 221-230	
$8^{\circ}:2$	$229.Im\bar{3}m$	$[*\cdot 4\cdot 4:2]:3$, $[2_1*\cdot 2\cdot 2]:6$
8°	$223.Pm\bar{3}n$	$[*\cdot 4:4\cdot 2]:3$, $[*\cdot 2\cdot 2\cdot 2\cdot 2]:6$
$8^{\circ\circ}$	$222.Pn\bar{3}n$	$(*4_0 4:2):3$, $(2\bar{*}_1 2_0 2_0):6$
$4^{-}:2$	$221.Pm\bar{3}m$	$[*\cdot 4\cdot 4\cdot 2]:3$, $[*\cdot 2\cdot 2\cdot 2\cdot 2]:6$
$4^{+}:2$	$224.Pn\bar{3}m$	$(*4_2 4\cdot 2):3$, $(2\bar{*}_1 2_0 2_0):6$
4^{--}	$226.Fm\bar{3}c$	$[*\cdot 4:4:2]:3$, $[*\cdot 2\cdot 2:2:2]:6$
4^{++}	$228.Fd\bar{3}c$	$(*4_1 4:2):3$, $(2\bar{*}2_0 2_1):6$
$2^{-}:2$	$225.Fm\bar{3}m$	$[*\cdot 4\cdot 4:2]:3$, $[*\cdot 2\cdot 2:2:2]:6$
$2^{+}:2$	$227.Fd\bar{3}m$	$(*4_1 4\cdot 2):3$, $(2\bar{*}2_0 2_1):6$
$8^{\circ}/4$	$230.Ia\bar{3}d$	$(*4_1 4:2):3$, $(*2_1 2:2:2):6$
Point Group 432,	nos. 207-214	
$8^{+\circ}$	$211.I432$	$(*4_2 4_0 2_1):3$, $(2_1*2_0 2_0):6$
4^{+}	$208.P4_2 32$	$(*4_2 4_2 2_0):3$, $(*2_0 2_0 2_0 2_0):6$
$4^{\circ -}$	$207.P432$	$(*4_0 4_0 2_0):3$, $(*2_0 2_0 2_0 2_0):6$
2^{+}	$210.F4_1 32$	$(*4_3 4_1 2_0):3$, $(*2_0 2_1 2_0 2_1):6$
$2^{\circ -}$	$209.F432$	$(*4_2 4_0 2_1):3$, $(*2_0 2_1 2_0 2_1):6$
$4^{+}/4$	$214.I4_1 32$	$(*4_3 4_1 2_0):3$, $(2_0*2_1 2_1):6$
$2^{+}/4$	‡ $212.P4_3 32$	$(4_1*2_1):3$, $(2_1 2_1\bar{\times}):6$
Point Group $3*2$,	nos. 200-206	
$8^{-\circ}$	$204.Im\bar{3}$	$[2_1*\cdot 2\cdot 2]:3$
4^{-}	$200.Pm\bar{3}$	$[*\cdot 2\cdot 2\cdot 2\cdot 2]:3$
$4^{\circ +}$	$201.Pn\bar{3}$	$(2\bar{*}_1 2_0 2_0):3$
2^{-}	$202.Fm\bar{3}$	$[*\cdot 2\cdot 2:2:2]:3$
$2^{\circ +}$	$203.Fd\bar{3}$	$(2\bar{*}2_0 2_1):3$
$4^{-}/4$	$206.Ia\bar{3}$	$(*2_1 2:2:2):3$
$2^{-}/4$	$205.Pa\bar{3}$	$(2_1 2\bar{*}:):3$
Point Group $*332$,	nos. 215-220	
$4^{\circ}:2$	$217.I\bar{4}3m$	$(*.4\,4:2):3$, $(2_1*2_0 2_0):6$
4°	$218.P\bar{4}3n$	$(*4:4\,2_0):3$, $(*2_0 2_0 2_0 2_0):6$
$2^{\circ}:2$	$215.P\bar{4}3m$	$(*4\cdot 4\,2_0):3$, $(*2_0 2_0 2_0 2_0):6$
$2^{\circ\circ}$	$219.F\bar{4}3c$	$(*4:4\,2_1):3$, $(*2_0 2_1 2_0 2_1):6$
$1^{\circ}:2$	$216.F\bar{4}3m$	$(*4\cdot 4\,2_1):3$, $(*2_0 2_1 2_0 2_1):6$

Table 2a continued

Prim. name	Internat. no. and name	Secondary names
$4^{\circ}/4$	$220.I\bar{4}3d$	$(4\bar{*}2_1):3$, $(2_0*2_1 2_1):6$
Point Group 332,	nos. 195-199	
$4^{\circ\circ}$	$197.I23$	$(2_1*2_0 2_0):3$
2°	$195.P23$	$(*2_0 2_0 2_0 2_0):3$
1°	$196.F23$	$(*2_0 2_1 2_0 2_1):3$
$2^{\circ}/4$	$199.I2_1 3$	$(2_0*2_1 2_1):3$
$1^{\circ}/4$	$198.P2_1 3$	$(2_1 2_1\bar{\times}):3$

Table 2b. Here we list the 184 reducible groups, organized by their point groups. Each line describes one such group Γ, listing its primary name (column one), its international number and name (column two) and, its secondary name(s), if it has any (column three). (‡) indicates that a group has two enantiomorphous forms and we report only the lower IT number.

Table 2b

Primary name	International no. and name	Secondary names
Point Group ∗226, nos. 191-194		
[∗·6·3·2]	191.$P6/mmm$	
[∗:6·3·2]	194.$P6_3/mmc$	
[∗·6:3:2]	193.$P6_3/mcm$	
[∗:6:3:2]	192.$P6/mcc$	
Point Group 226, nos. 177-182		
(∗$6_0 3_0 2_0$)	177.$P622$	
(∗$6_1 3_1 2_1$)	‡ 178.$P6_122$	
(∗$6_2 3_2 2_0$)	‡ 180.$P6_222$	
(∗$6_3 3_0 2_1$)	182.$P6_322$	
Point Group ∗66, nos. 183-186		
(∗·6·3·2)	183.$P6mm$	
(∗:6·3·2)	186.$P63mc$	
(∗·6:3:2)	185.$P6_3cm$	
(∗:6:3:2)	184.$P6cc$	
Point Group 6∗, nos. 175-176		
[$6_0 3_0 2_0$]	175.$P6/m$	
[$6_3 3_0 2_1$]	176.$P6_3/m$	
Point Group 66, nos. 168-173		
($6_0 3_0 2_0$)	168.$P6$	
($6_1 3_1 2_1$)	‡ 169.$P6_1$	
($6_2 3_2 2_0$)	‡ 171.$P6_2$	
($6_3 3_0 2_1$)	173.$P6_3$	
Point Group 2∗3, nos. 162-167		
(∗6·3·2)	164.$P\bar{3}m1$	
(∗6:3:2)	165.$P\bar{3}c1$	
(∗·6 $3_0$2)	162.$P\bar{3}1m$	
(∗:6 $3_0$2)	163.$P\bar{3}1c$	
(∗·6 $3_1$2)	166.$R\bar{3}m$	
(∗:6 $3_1$2)	167.$R\bar{3}c$	

Table 2b continued		
Point Group 3×, nos. 147-148		
(6 $3_0$2)	147.$P\bar{3}$	
(6 $3_1$2)	148.$R\bar{3}$	
Point Group ∗224, nos. 123-142		
[∗·4·4·2]	123.$P4/mmm$	
[∗·4·4:2]	139.$I4/mmm$	
[∗·4:4·2]	131.$P4_2/mmc$	
[∗·4:4:2]	140.$I4/mcm$	
[∗:4·4:2]	132.$P4_2/mcm$	
[∗:4:4:2]	124.$P4/mcc$	
(∗4·4·2)	129.$P4/nmm$	
(∗4·4:2)	137.$P4_2/nmc$	
(∗4:4·2)	138.$P4_2/ncm$	
(∗4:4:2)	130.$P4/ncc$	
(∗$4_0$4·2)	125.$P4/nbm$	
(∗$4_0$4:2)	126.$P4/nnc$	
(∗$4_1$4·2)	141.$I4_1/amd$	
(∗$4_1$4:2)	142.$I4_1/acd$	
(∗$4_2$4·2)	134.$P4_2/nnm$	
(∗$4_2$4:2)	133.$P4_2/nbc$	
[4_0∗·2]	127.$P4/mbm$	
[4_0∗:2]	128.$P4/mnc$	
[4_2∗·2]	136.$P4_2/mnm$	
[4_2∗:2]	135.$P4_2/mbc$	
Point Group 224, nos. 89-98		
(∗$4_0 4_0 2_0$)	89.$P422$	
(∗$4_1 4_1 2_1$)	‡ 91.$P4_122$	
(∗$4_2 4_2 2_0$)	93.$P4_222$	
(∗$4_2 4_0 2_1$)	97.$I422$	
(∗$4_3 4_1 2_0$)	98.$I4_122$	
(4_0∗2_0)	90.$P42_12$	
(4_1∗2_1)	‡ 92.$P4_12_12$	
(4_2∗2_0)	94.$P4_22_12$	
Point Group ∗44, nos. 99-110		
(∗·4·4·2)	99.$P4mm$	
(∗·4·4:2)	107.$I4mm$	
(∗·4:4·2)	105.$P4_2mc$	
(∗·4:4:2)	108.$I4cm$	
(∗:4·4:2)	101.$P4_2cm$	
(∗:4:4:2)	103.$P4cc$	

Table 2b continued		
$(4_0{*}{\cdot}2)$	100.$P4bm$	
$(4_0{*}{:}2)$	104.$P4nc$	
$(4_1{*}{\cdot}2)$	109.$I4_1md$	
$(4_1{*}{:}2)$	110.$I41cd$	
$(4_2{*}{\cdot}2)$	102.$P4_2nm$	
$(4_2{*}{:}2)$	106.$P4_2bc$	
Point Group 4∗, nos. 83-88		
$[4_04_02_0]$	83.$P4/m$	
$[4_24_22_0]$	84.$P4_2/m$	
$[4_24_02_1]$	87.$I4/m$	
$(4\,4_02)$	85.$P4/n$	
$(4\,4_12)$	88.$I4_1/a$	
$(4\,4_22)$	86.$P4_2/n$	
Point Group 44, nos. 75-80		
$(4_04_02_0)$	75.$P4$	
$(4_14_12_1)$	‡ 76.$P4_1$	
$(4_24_22_0)$	77.$P4_2$	
$(4_24_02_1)$	79.$I4$	
$(4_34_12_0)$	80.$I4_1$	
Point Group 2∗2, nos. 111-122		
$({*}{\cdot}4\,4{\cdot}2)$	115.$P\bar{4}m2$	
$({*}.4\,4{:}2)$	121.$I\bar{4}2m$	
$({*}{:}4\,4{:}2)$	116.$P\bar{4}c2$	
$({*}4{\cdot}4\,2_0)$	111.$P\bar{4}2m$	
$({*}4{:}4\,2_0)$	112.$P\bar{4}2c$	
$({*}4{\cdot}4\,2_1)$	119.$I\bar{4}m2$	
$({*}4{:}4\,2_1)$	120.$I\bar{4}c2$	
$(4\bar{*}{\cdot}2)$	113.$P\bar{4}2_1m$	
$(4\bar{*}{:}2)$	114.$P\bar{4}2_1c$	
$(4\bar{*}_02_0)$	117.$P\bar{4}b2$	
$(4\bar{*}_12_0)$	118.$P\bar{4}n2$	
$(4\bar{*}2_1)$	122.$I\bar{4}2d$	
Point Group 2×, nos. 81-82		
$(4\,4\,2_0)$	81.$P\bar{4}$	
$(4\,4\,2_1)$	82.$I\bar{4}$	
Point Group ∗223, nos. 187-190		
$[{*}{\cdot}3{\cdot}3{\cdot}3]$	187.$P\bar{6}m2$	
$[{*}{:}3{:}3{:}3]$	188.$P\bar{6}c2$	

Table 2b continued		
$[3_0{*}{\cdot}3]$	189.$P\bar{6}2m$	
$[3_0{*}{:}3]$	190.$P\bar{6}2c$	
Point Group 223, nos. 149-155		
$({*}3_03_03_0)$	149.$P312$	
$({*}3_13_13_1)$	‡ 151.$P3_112$	
$({*}3_03_13_2)$	155.$R32$	
$(3_0{*}3_0)$	150.$P321$	
$(3_1{*}3_1)$	‡ 152.$P3_121$	
Point Group ∗33, nos. 156-161		
$({*}{\cdot}3{\cdot}3{\cdot}3)$	156.$P3m1$	
$({*}{:}3{:}3{:}3)$	158.$P3c1$	
$(3_0{*}{\cdot}3)$	157.$P31m$	
$(3_0{*}{:}3)$	159.$P31c$	
$(3_1{*}{\cdot}3)$	160.$R3m$	
$(3_1{*}{:}3)$	161.$R3c$	
Point Group 3∗, no. 174		
$[3_03_03_0]$	174.$P\bar{6}$	
Point Group 33, nos. 143-146		
$(3_03_03_0)$	143.$P3$	
$(3_03_13_2)$	146.$R3$	
$(3_13_13_1)$	‡ 144.$P3_1$	
Point Group ∗222, nos. 47-74		
$[{*}{\cdot}2{\cdot}2{\cdot}2{\cdot}2]$	47.$Pmmm$	
$[{*}{\cdot}2{\cdot}2{:}2{:}2]$	69.$Fmmm$	
$[{*}{:}2{:}2{:}2{:}2]$	49.$Pccm$	$({*}2_02_02{\cdot}2)$
$({*}2_02{\cdot}2{\cdot}2)$	67.$Cmma$	$[{*}{\cdot}2{:}2{:}2{:}2]$
$({*}2_02{:}2{:}2)$	68.$Ccca$	$({*}2_02_12{:}2)$
$({*}2_12{\cdot}2{\cdot}2)$	74.$Imma$	$[2_0{*}{\cdot}2{:}2]$
$({*}2_12{:}2{:}2)$	73.$Ibca$	
$[2_0{*}{\cdot}2{\cdot}2]$	65.$Cmmm$	$[{*}{\cdot}2{\cdot}2{\cdot}2{:}2]$
$[2_0{*}{:}2{:}2]$	66.$Cccm$	$({*}2_02_12{\cdot}2)$
$[2_1{*}{\cdot}2{\cdot}2]$	71.$Immm$	
$[2_1{*}{:}2{:}2]$	72.$Ibam$	$({*}2_02{\cdot}2{:}2)$
$(2\bar{*}{\cdot}2{\cdot}2)$	59.$Pmmn$	$[2_12_1{*}{\cdot}]$
$(2\bar{*}{:}2{:}2)$	56.$Pccn$	$(2_12\bar{*}_0)$
$(2\bar{*}_02_02_0)$	50.$Pban$	$({*}2_02_02{:}2)$
$(2\bar{*}_12_02_0)$	48.$Pnnn$	

Table 2b continued		
$(2\bar{*}2_02_1)$	70.$Fddd$	
$[2_02_0\times_0]$	55.$Pbam$	$(*2{\cdot}2{:}2{\cdot}2)$
$[2_02_0\times_1]$	58.$Pnnm$	$(2_1*2{\cdot}2)$
$[2_02_0*{\cdot}]$	51.$Pmma$	$[*{\cdot}2{:}2{\cdot}2{:}2]$, $(*2{\cdot}2{\cdot}2{\cdot}2)$
$[2_02_0*{:}]$	53.$Pmna$	$(*2_12_12{\cdot}2)$, $(2_0*2{\cdot}2)$
$[2_02_1*{\cdot}]$	63.$Cmcm$	$(*2{\cdot}2{\cdot}2{:}2)$, $[2_1*{\cdot}2{:}2]$
$[2_02_1*{:}]$	64.$Cmca$	$(*2{\cdot}2{:}2{:}2)$, $(*2_12{\cdot}2{:}2)$
$(2_02\bar{*}{\cdot})$	57.$Pbcm$	$(*2{:}2{\cdot}2{:}2)$, $[2_12_1*{:}]$
$(2_02\bar{*}{:})$	60.$Pbcn$	$(2_1*2{:}2)$, $(2_12\bar{*}_1)$
$(2_12\bar{*}{\cdot})$	62.$Pnma$	$(2\bar{*}{\cdot}2{:}2)$, $[2_12_1\times]$
$(2_12\bar{*}{:})$	61.$Pbca$	
$(2_02\bar{*}_0)$	54.$Pcca$	$(*2{:}2{:}2{:}2)$, $(*2_12_12{:}2)$
$(2_02\bar{*}_1)$	52.$Pnna$	$(2_0*2{:}2)$, $(2\bar{*}2_12_1)$
Point Group 222, nos. 16-24		
$(*2_02_02_02_0)$	16.$P222$	
$(*2_02_12_02_1)$	22.$F222$	
$(*2_12_12_12_1)$	17.$P222_1$	(2_02_0*)
$(2_0*2_02_0)$	21.$C222$	$(*2_02_02_12_1)$
$(2_1*2_02_0)$	23.$I222$	
$(2_0*2_12_1)$	24.$I2_12_12_1$	
$(2_1*2_12_1)$	20.$C222_1$	(2_02_1*)
$(2_02_0\bar{\times})$	18.$P2_12_12$	(2_12_1*)
$(2_12_1\bar{\times})$	19.$P2_12_12_1$	
Point Group $*22$, nos. 25-46		
$(*{\cdot}2{\cdot}2{\cdot}2{\cdot}2)$	25.$Pmm2$	$[*_0{\cdot}*_0{\cdot}]$
$(*{\cdot}2{\cdot}2{\cdot}2{:}2)$	38.$Amm2$	$[*_1{\cdot}*_1{\cdot}]$, $[*{\cdot}\times_0]$
$(*{\cdot}2{\cdot}2{:}2{:}2)$	42.$Fmm2$	$[*_1{\cdot}*_1{:}]$
$(*{\cdot}2{:}2{\cdot}2{:}2)$	26.$Pmc2_1$	$(\bar{*}{\cdot}\bar{*}{\cdot})$, $[\times_0\times_0]$
$(*{\cdot}2{:}2{:}2{:}2)$	39.$Abm2$	$[*_1{:}*_1{:}]$, $(\bar{*}{\cdot}\bar{*}_0)$
$(*{:}2{:}2{:}2{:}2)$	27.$Pcc2$	$(\bar{*}_0\bar{*}_0)$
$(2_0*{\cdot}2{\cdot}2)$	35.$Cmm2$	$[*_0{\cdot}*_0{:}]$
$(2_0*{\cdot}2{:}2)$	46.$Ima2$	$(\bar{*}{\cdot}\bar{*}_1)$, $[*{:}\times_0]$
$(2_0*{:}2{:}2)$	37.$Ccc2$	$(\bar{*}_0\bar{*}_1)$
$(2_1*{\cdot}2{\cdot}2)$	44.$Imm2$	$[*{\cdot}\times_1]$
$(2_1*{\cdot}2{:}2)$	36.$Cmc2_1$	$(\bar{*}{\cdot}\bar{*}{:})$, $[\times_1\times_1]$
$(2_1*{:}2{:}2)$	45.$Iba2$	$(\bar{*}{:}\bar{*}_0)$
$(2_02_0*{\cdot})$	28.$Pma2$	$[*_0{:}*_0{:}]$, $(*{\cdot}*_0)$
$(2_02_0*{:})$	30.$Pnc2$	$(\bar{*}_1\bar{*}_1)$, $(*_0\times_0)$

Table 2b continued		
$(2_02_1*{\cdot})$	40.$Ama2$	$(*{\cdot}*_1)$, $[*{:}\times_1]$
$(2_02_1*{:})$	41.$Aba2$	$(*{:}*_1)$, $(\bar{*}{:}\bar{*}_1)$
$(2_12_1*{\cdot})$	31.$Pmn2_1$	$(*{\cdot}\bar{\times})$, $[\times_0\times_1]$
$(2_12_1*{:})$	29.$Pca2_1$	$(\bar{*}{:}\bar{*}{:})$, $(\bar{\times}\times_0)$
$(2_02_0\times_0)$	32.$Pba2$	$(*{:}*_0)$
$(2_02_0\times_1)$	34.$Pnn2$	$(*_0\times_1)$
$(2_02_1\times)$	43.$Fdd2$	$(*_1\times)$
$(2_12_1\times)$	33.$Pa2_1$	$(*{:}\bar{\times})$, $(\bar{\times}\times_1)$
Point Group $2*$, nos. 10-15		
$[2_02_02_02_0]$	10.$P2/m$	$(*2{\cdot}2\,2{\cdot}2)$
$[2_02_02_12_1]$	12.$C2/m$	$(*2{\cdot}2\,2{:}2)$, $(2\bar{*}2{\cdot}2)$
$[2_12_12_12_1]$	11.$P2_1/m$	$(2\,2*{\cdot})$
$(2_02_02\,2)$	13.$P2/c$	$(*2{:}2\,2{:}2)$, $(2\,2*_0)$
$(2_02_12\,2)$	15.$C2/c$	$(2\bar{*}2{:}2)$, $(2\,2*_1)$
$(2_12_12\,2)$	14.$P2_1/c$	$(2\,2*{:})$, $(2\,2\times)$
Point Group 22, nos. 3-5		
$(2_02_02_02_0)$	3.$P2$	$(*_0*_0)$
$(2_02_02_12_1)$	5.$C2$	$(*_1*_1)$, $(*\bar{\times})$
$(2_12_12_12_1)$	4.$P2_1$	$(\bar{\times}\bar{\times})$
Point Group $\times$, no. 2		
$(2\,2\,2\,2)$	2.$P\bar{1}$	
Point Group $*$, nos. 6-9		
$[\circ_0]$	6.Pm	$(*{\cdot}*{\cdot})$
$[\circ_1]$	8.Cm	$(*{\cdot}*{:})$, $(*{\cdot}\times)$
$(\bar{\circ}_0)$	7.Pc	$(*{:}*{:})$, $(\times\times_0)$
$(\bar{\circ}_1)$	9.Cc	$(*{:}\times)$, $(\times\times_1)$
Point Group 1, no. 1		
$(\circ)$	1.$P1$	

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