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On a new class of optimal eighth-order derivative-free methods

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Abstract. Making use of last derivative approximation and weight function approach, we construct an eighth-order class of three-step methods, which are consistent with the optimality conjecture of Kung-Traub for constructing multi-point methods without memory. Per iteration, any method of the developed class is totally free from derivative evaluation. Such classes of schemes are more practical when the calculation of derivatives is hard. Error analysis will also be studied. Finally, numerical comparisons are made to reveal the reliability of the proposed class.

1 Introduction

The theoretical thorough study of iterative processes for simple roots goes back at least to the book of Traub [19]. Among questions and ideas which have been addressed, the problem of computing simple roots by multi-point without memory methods emerged. To illustrate further in [4], the authors have given two classes of n-step methods without memory; one including derivative calculation, also known as derivative-involved methods; and one derivative-free

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class. As an example, they gave the following family of one-parameter methods

$$\begin{cases} y_n = x_n + \beta f(x_n), \\ z_n = y_n - \beta \frac{f(x_n)f(y_n)}{f(y_n)-f(x_n)}, \\ w_n = z_n - \frac{f(x_n)f(y_n)}{f(z_n)-f(x_n)}[\frac{1}{f[y_n,x_n]} - \frac{1}{f[z_n,y_n]}], \\ x_{n+1} = w_n - \frac{f(x_n)f(y_n)f(z_n)}{f(w_n)-f(x_n)}[\frac{1}{f(w_n)-f(y_n)}\{\frac{1}{f[w_n,z_n]} - \frac{1}{f[z_n,y_n]}\} \\ -\frac{1}{f(z_n)-f(x_n)}\{\frac{1}{f[z_n,y_n]} - \frac{1}{f[y_n,x_n]}\}], \end{cases} \tag{1}$$

wherein $\beta \in \mathbb{R} - \{0\}$, by using inverse interpolation for annihilating the new-appeared first derivatives of the function in the Steffensen-Newton-Newton structure. They also conjectured that a multi-point iteration without memory can achieve the maximum order of convergence $2^{(n-1)}$, by consuming n, functional evaluations per full cycle. Therefore, (1)'s order end efficiency index are optimal.

Different methods of various orders have been introduced and improved by many authors. A complete review on the published papers in this field for the works from 2000 to 2010 have been given in the book of Iliev and Kyurkchiev [2]. In [8], the authors considered weight function approach to give some new classes of optimal Jarratt-type fourth-order methods. Authors in [13] studied a combination of last derivative approximation and weight function approach to furnish optimal eighth-order derivative-involved methods. Discussion on new multiple zero finders when the multiplicity of the roots is available have been recently introduced by Sharifi et al. in [5]. Note that Soleymani and Hosseinabadi in [9] presented a sixth-order derivative-free method including three steps. The references [10-12] also contain new derivative-free developments in this active topic of study. For more information, one may consult [15-18].

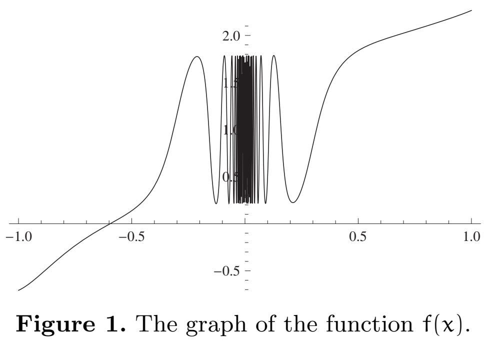

Figure 1. The graph of the function $f(x)$.

Derivative-free methods are important when we deal with complicated functions, such as $f(x) = \cos(\sin(x^2\sqrt{x})) \times \cos(x^3) \times \arctan(\sin(x^5 + x^{-1})) + x^3 + 1$, where its plot is given in Figure 1, or we try to find multiple roots of nonlinear equations (in the case that multiplicity is unknown), [1].

For this cause, this work is devoted to find an optimal three-step class of iterations without memory in which any method includes four function evaluations per cycle to obtain the eighth order of convergence and possess the optimal efficiency index 1.682. Toward this end, we make use of weight function approach alongside an approximation for the first derivative of the function for the quotients of a Steffensen-Newton-Newton structure. The efficiency of our class is then compared with those available in the literature to show better or equal results. Some methods from the suggested class are tested numerically in Section 3 to support the theoretical results given in Section 2. Section 4 includes a short conclusion of the article.

2 Main contribution

To construct a high-order class of methods for solving nonlinear scalar equations, we take into account the following three-step Steffensen-Newton-Newton structure

$$y_n = x_n - \frac{\beta f(x_n)^2}{f(w_n) - f(x_n)}, \qquad z_n = y_n - \frac{f(y_n)}{f'(y_n)}, \qquad x_{n+1} = z_n - \frac{f(z_n)}{f'(z_n)}, \quad (2)$$

wherein $f(w_n) = f(x_n + \beta f(x_n))$; that is to say $w_n = x_n + \beta f(x_n)$, $\beta \in \mathbb{R} - \{0\}$. This structure possesses the eighth order of convergence, while it is inefficient. Because it includes 6 evaluations per step and its efficiency index therefore will be 1.4142, which is the same as Steffensen's and Newton's methods. Thus, in order to contribute and hit the assigned target, we should eliminate the existent of the derivative calculations without lowering the order, i.e. obtaining a class (family) of order eight with four evaluations of the function per full cycle only. There are many ways to do so. Among all, we first consider an approximation for the new-appeared first derivatives $f'(y_n)$ and $f'(z_n)$, and second make use of the approach of weight functions.

Let $f'(y_n) \approx (f(w_n) - f(x_n))/(\beta f(x_n))$, that is the same approximation as Steffensen used in the first step of (2). Then, an estimation of the function $f(t)$, in the open domain D, is taken into consideration as follows: $f(t) \approx w(t) = a_0 + a_1(t - y_n)$, which its first derivative is $w'(t) = a_1$. We suppose this estimation passes the points y_n and z_n. By substituting the known values $f(t)\mid_{y_n} = f(y_n), f(t)\mid_{z_n} = f(z_n)$, we could easily obtain the unknown parameters. Thus, we have $a_0 = f(y_n)$ and $a_1 = (f(y_n) - f(z_n))/(y_n - z_n) = f[y_n, z_n]$.

Consequently, we have $f'(z_n) \approx f[y_n, z_n]$. Therefore, we suggest the following iteration

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]}, w_n = x_n + \beta f(x_n), \beta \in \mathbb{R} - \{0\}, \\ z_n = y_n - \frac{f(y_n)}{f[x_n,w_n]} P(t), \\ x_{n+1} = z_n - \frac{f(z_n)}{f[y_n,z_n]} \big(G(\gamma) + H(t) + K(\zeta) \big), \end{cases} \tag{3}$$

where $t = f(y)/f(w)$, $\gamma = f(y)/f(x)$, and $\zeta = f(z)/f(w)$. $P(t)$, $G(\gamma)$, $H(t)$ and $K(\zeta)$ are four real-valued weight functions that should be chosen such that the order of convergence attains the value eight, this is the role of weight function approach. Taylor's series expansion around the solution for the first two steps of (3) gives us

$$\begin{aligned} e_{n+1} =& -((c_2(1 + c_1\beta)(-1 + P(0))e_n^2)/c_1) + \\ & + 1/c_1^2(-c_1c_3(1 + c_1\beta)(2 + c_1\beta)(-1 + P(0)) + \\ & + c_2^2(-2 + 4P(0) + c_1\beta(-2 + 5P(0) + \\ & + c_1\beta(-1 + 2P(0)) - P'(0)) - P'(0)))e_n^3 + O(e_n^4), \end{aligned} \tag{4}$$

where $c_j = f^{(j)}(\alpha)/j!$, $j \geq 1$, α is the solution. This shows that $P(0) = 1$, $P'(0) = 2 + \beta f[x_n, w_n]$ should be selected in order to attain at fourth-order convergence. By taking into account this, and similar expansion up to the seventh term, we obtain for (3) now that $G(0) = 1$, $G'(0) = H(0) = K(0) = H'(0) = H''(0) = G^{(3)}(0) = 0$, $K'(0) = 2 + \beta f[x_n, w_n]$ and $G''(0) = 2/(1 + \beta f[x_n, w_n])$ should be chosen in order to arrive at seventh-order convergence as follows

$$\begin{aligned} e_{n+1} =& \frac{-1}{12c_1^6}((1 + \beta c_1)c_2^4(-2c_1(1 + \beta c_1)c_3 + c_2^2(10 + 2\beta c_1(5 + \beta c_1) - \\ & - P''(0)))(3(2 + \beta c_1)(6 + 2\beta c_1(3 + \beta c_1) - \\ & - P''(0)) + H^{(3)}(0)))e_n^7 + O(e_n^8). \end{aligned} \tag{5}$$

Obviously, now to gain the optimal order eight with using only four evaluations of the function we should find $H^{(3)}(0)$ in such a way that order goes up to eight. This is summarized in Theorem 1.

Theorem 1 Let $\alpha \in D$, be a simple zero of sufficiently differentiable function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and let $c_j = f^{(j)}(\alpha)/j!$, $j \geq 1$. If x_0 is sufficiently close to α, then, (i): the order of convergence of the solution by the three-step class of iterations without memory methods defined in (3) is eight, when $P(0) = 1$, $P'(0) = 2 + \beta f[x_n, w_n]$, $|P''(0)| < \infty$, $|P^{(3)}(0)| < \infty$, and

$$\begin{cases} G(0) = 1, G'(0) = G^{(3)}(0) = 0, G''(0) = \frac{2}{1+\beta f[x_n,w_n]}, \text{ and } |G^{(4)}(0)| < \infty, \\ H(0) = H'(0) = H''(0) = 0, \text{ and } |H^{(4)}(0)| < \infty, \\ H^{(3)}(0) = -3(2 + \beta f[x_n, w_n])(6 + 2\beta f[x_n, w_n](3 + \beta f[x_n, w_n]) - P''(0)), \\ K(0) = 0, K'(0) = 2 + \beta f[x_n, w_n], \end{cases} \tag{6}$$

and (ii): this solution reads the error equation

$$\begin{aligned} e_{n+1} = \frac{-1}{48c_1^7}&((1 + \beta c_1)c_2^2(-2c_1(1 + \beta c_1)c_3 + c_2^2(10 + 2\beta c_1(5 + \beta c_1) - P''(0))) \\ &\times(96\beta c_1(1 + \beta c_1)^2 c_2 c_3 - 24c_1^2(1 + \beta c_1)^2 c_4 + c_2^3(-168 + 48P''(0) - 8P^{(3)}(0) \\ &+G^{(4)}(0) + c_1\beta(-4(84 + 3c_1\beta(16 + 2c_1\beta - P''(0)) - 12P''(0) + P^{(3)}(0)) \\ &+(2 + c_1\beta)(2 + c_1\beta(2 + c_1\beta))G^{(4)}(0)) + H^{(4)}(0))))e_n^8 + O(e_n^9). \end{aligned} \tag{7}$$

Proof. We expand any term of (3) around the solution α in the nth iterate by considering (6). Thus, we write

$$f(x_n) = c_1 e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + c_6 e_n^6 + c_7 e_n^7 + c_8 e_n^8 + O(e_n^9). \tag{8}$$

Accordingly, we attain

$$y_n = \alpha + (\beta + \frac{1}{c_1})c_2 e_n^2 + \frac{(-(2 + (2 + \beta c_1)\beta c_1)c_2^2 + \beta c_1(1 + \beta c_1)(2 + \beta c_1)c_3)}{c_1^2} e_n^3 + \ldots + O(e_n^9). \tag{9}$$

Now we should expand $f(y_n)$ around the simple root by using (9). We obtain

$$\begin{aligned} f(y_n) = (1 + \beta c_1)c_2 e_n^2 + (&-\frac{(2 + \beta c_1(2 + \beta c_1))c_2^2}{c_1} + (1 + \beta c_1)(2 + \beta c_1)c_3)e_n^3 \\ &+\frac{1}{c_1^2}(5 + \beta c_1(7 + \beta c_1(4 + \beta c_1)))c_2^3 - c_1 c_2 c_3(7 + \beta c_1(10 + \beta c_1(7 + 2\beta c_1))) \\ &+c_1^2(1 + \beta c_1)(3 + \beta c_1(3 + \beta c_1))c_4)e_n^4 + \ldots + O(e_n^8). \end{aligned} \tag{10}$$

Using (10) and the second step of (3), we attain

$$y_n - \frac{f(y_n)}{f[x_n, w_n]} = \alpha + \frac{(1 + \beta c_1)(2 + \beta c_1)c_2^2}{c_1^2} e_n^3 + \ldots + O(e_n^9). \tag{11}$$

Additionally, we attain that

$$z_n = \alpha + \frac{((1+\beta c_1)c_2(-2c_1(1+\beta c_1)c_3 + c_2^2(10+2\beta c_1(5+\beta c_1) - P''(0))))}{2c_1^3} e_n^4$$

$$+ \ldots + O(e_n^9). \tag{12}$$

Moreover, we obtain now

$$f(z_n) = \frac{(1+\beta c_1)c_2(-2c_1(1+\beta c_1)c_3 + c_2^2(10+2\beta c_1(5+\beta c_1) - P''(0)))}{2c_1^2} e_n^4$$

$$-\frac{1}{6c_1^3}(6c_1^2(1+\beta c_1)^2(2+\beta c_1)c_3^2 + 6c_1^2(1+\beta c_1)^2(2+\beta c_1)c_2c_4 - 3c_1(1+\beta c_1)c_2^2c_3(64$$

$$+2\beta c_1(46+\beta c_1(22+3\beta c_1)) - 3(2+\beta c_1)P''(0)) + c_2^4(6(36+c_1\beta(80+3c_1\beta(22+\beta c_1(8$$

$$+\beta c_1)))) - 3(10+3\beta c_1(5+2\beta c_1))P''(0) + (1+\beta c_1)P^{(3)}(0)))e_n^5 + \ldots + O(e_n^9). \tag{13}$$

Using (9)-(13), we have

$$z_n - \frac{f(z_n)}{f[y_n, z_n]} = \alpha$$

$$+\frac{((1+\beta c_1)^2 c_2^3(-2c_1(1+\beta c_1)c_3 + c_2^2(10+2\beta c_1(5+\beta c_1) - P''(0))))e_n^6}{2c_1^5}$$

$$+ \ldots + O(e_n^9). \tag{14}$$

Now we also for the last step have (without considering (6)) $x_{n+1} - \alpha = -\frac{1}{2c_1^3}(c_2(1+c_1\beta)(-1+G(0)+H(0)+K(0))(-2c_1c_3(1+c_1\beta)+c_2^2(10+2c_1\beta(5+c_1\beta)-P''(0))))e_n^4 + \frac{1}{6c_1^4}(6c_1^2c_3^2(1+c_1\beta)^2(2+c_1\beta)(-1+G(0)+H(0)+K(0)) + 6c_1^2c_2c_4(1+c_1\beta)^2(2+c_1\beta)(-1+G(0)+H(0)+K(0)) - 3c_1c_2^2c_3(1+c_1\beta)(2(-32+32G(0)+32H(0)+32K(0)-G'(0)+c_1\beta(-46+46G(0)+46H(0)+46K(0)+c_1\beta((22+3c_1\beta)(-1+G(0)+H(0)+K(0))-G'(0))-2G'(0)-H'(0))-H'(0))-3(2+c_1\beta)(-1+G(0)+H(0)+K(0))P''(0))+c_2^4(6c_1^4\beta^4(3(-1+G(0)+H(0)+K(0))-G'(0))+6(-36+36G(0)+36H(0)+36K(0)-5G'(0)-5H'(0))+6c_1^3\beta^3(-24+24G(0)+24H(0)+24K(0)-7G'(0)-H'(0))+3(10-10G(0)-10H(0)-10K(0)+G'(0)+H'(0))P''(0)+3c_1^2\beta^2(4(-33+33G(0)+33H(0)+33K(0)-8G'(0)-3H'(0))+(-6(-1+G(0)+H(0)+K(0))+G'(0))P''(0))+(-1+G(0)+H(0)+K(0))P^{(3)}(0)+c_1\beta(30(16G(0)+16H(0)+16K(0)-3G'(0)-2(8+H'(0)))+3(15-15G(0)-15H(0)-15K(0)+2G'(0)+H'(0))P''(0)+(-1+G(0)+H(0)+K(0))P^{(3)}(0))))e_n^5 + \cdots + O(e_n^9)$. Therefore, by combining this, (14) and the terms of (6) in the last step of (3), we have the error equation (7). This completes the proof and shows that our multi-point class of methods arrives at optimal eighth-order convergence by using only four pieces of information and considering (6). □

Clearly, any method from our class of derivative-free methods reaches the optimal efficiency index $8^{1/4} \approx 1.682$, which is greater than that of Newton's and Steffensen's $2^{1/2} \approx 1.414$, $6^{1/4} \approx 1.565$ of the sixth-order methods given in [3, 9], $4^{1/3} \approx 1.587$ of method given in [14], and is equal to that of (1) and the classes of methods in [6, 7].

To provide the simplest case of our class of methods; by considering (6), we suggest the following method without memory including three steps

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]}, w_n = x_n + f(x_n), \\ z_n = y_n - \frac{f(y_n)}{f[x_n,w_n]}\{1 + (2 + f[x_n, w_n])\frac{f(y_n)}{f(w_n)}\}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[y_n,z_n]}\{1 + \frac{1}{1+f[x_n,w_n]}(\frac{f(y_n)}{f(x_n)})^2 - ((2 + f[x_n, w_n])(3 \\ +f[x_n, w_n](3 + f[x_n, w_n])))(\frac{f(y_n)}{f(w_n)})^3 + (2 + f[x_n, w_n])\frac{f(z_n)}{f(w_n)}\}, \end{cases} \tag{15}$$

where its error equation satisfies

$$\begin{aligned} e_{n+1} = {} & (1/(c_1^7))(1 + c_1)^2 c_2^2((5 + c_1(5 + c_1))c_2^2 \\ & - c_1(1 + c_1)c_3)((7 + c_1(7 + c_1))c_2^3 - 4c_1(1 + c_1)c_2c_3 + \\ & + c_1^2(1 + c_1)c_4)e_n^8 + O(e_n^9). \end{aligned} \tag{16}$$

Remark 1. In order to implement and code the methods from the class (3), we should be careful that after computing $f[x_n, w_n]$ in the first step, its value will be used throughout the iteration step, which in fact does not increase the computational load of the novel optimal eighth-order derivative-free methods.

A very efficient but complicated optimal three-step method from the proposed class (3) can be

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]}, w_n = x_n + f(x_n), \\ z_n = y_n - \frac{f(y_n)}{f[x_n,w_n]}\{1 + (2 + f[x_n, w_n])\frac{f(y_n)}{f(w_n)} + (5 + f[x_n, w_n](5 \\ +f[x_n, w_n]))\left(\frac{f(y_n)}{f(w_n)}\right)^2\}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[y_n,z_n]}\{1 + \frac{1}{1+f[x_n,w_n]}(\frac{f(y_n)}{f(x_n)})^2 - ((2 + f[x_n, w_n])(3 + f[x_n, w_n](3 \\ +f[x_n, w_n]) - 5 - f[x_n, w_n](5 + f[x_n, w_n])))(\frac{f(y_n)}{f(w_n)})^3 - (13 + f[x_n, w_n](26 \\ +f[x_n, w_n](21 + f[x_n, w_n](8 + f[x_n, w_n]))))(\frac{f(y_n)}{f(w_n)})^4 + (2 + f[x_n, w_n])\frac{f(z_n)}{f(w_n)}\}, \end{cases} \tag{17}$$

where its error equation satisfies

$$e_{n+1} = \frac{(1 + c_1)^4 c_2^2 c_3(4c_2c_3 - c_1c_4)}{c_1^5} e_n^8 + O(e_n^9). \tag{18}$$

We can easily now observe that the error equation (18) is very small. In fact, we have obtained the finest error equations for optimal three-step derivative-free methods without memory by introducing (17).

Note that if we choose very small value for the nonzero parameter β in (3), the error equations will be mostly refined and the numerical results will be better, for example we can have

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]}, w_n = x_n + 0.01f(x_n), \\ z_n = y_n - \frac{f(y_n)}{f[x_n,w_n]}\{1 + (2 + 0.01f[x_n, w_n])\frac{f(y_n)}{f(w_n)}\}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[y_n,z_n]}\{1 + \frac{1}{1+0.01f[x_n,w_n]}(\frac{f(y_n)}{f(x_n)})^2 - ((2 + 0.01f[x_n, w_n])(3 \\ +0.01f[x_n, w_n](3 + 0.01f[x_n, w_n])))(\frac{f(y_n)}{f(w_n)})^3 + (2 + 0.01f[x_n, w_n])\frac{f(z_n)}{f(w_n)}\}. \end{cases} \tag{19}$$

Notice that if we use backward finite difference approximation in the first step of our cycle (2), by changing the weight functions suitably, we can give another class which is similar to (3), i.e. we can have

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]}, w_n = x_n - \beta f(x_n), \beta \in \mathbb{R} - \{0\}, \\ z_n = y_n - \frac{f(y_n)}{f[x_n,w_n]}P(t), \\ x_{n+1} = z_n - \frac{f(z_n)}{f[y_n,z_n]}\big(G(\gamma) + H(t) + K(\zeta)\big), \end{cases} \tag{20}$$

where $t = f(y)/f(w)$, $\gamma = f(y)/f(x)$, and $\zeta = f(z)/f(w)$. And $P(t)$, $G(\gamma)$, $H(t)$ and $K(\zeta)$ are four real-valued weight functions that should be chosen such that the order of convergence arrives at eight. This is illustrated in Theorem 2.

Theorem 2 Let $\alpha \in D$, be a simple zero of sufficiently differentiable function $f : D \subseteq \mathbb{R} \to \mathbb{R}$ and let that $c_j = f^{(j)}(\alpha)/j!$, $j \geq 1$. If x_0 is sufficiently close to α, then, (i): the local order of convergence of the solution by the three-step class of without memory methods defined in (20) is eight, when $P(0) = 1$, $P'(0) = 2 - \beta f[x_n, w_n]$, $|P''(0)| < \infty$, $|P^{(3)}(0)| < \infty$, and

$$\begin{cases} G^{(3)}(0) = 0, \ |G(0)| < \infty, \ |G'(0)| < \infty, \ |G''(0)| < \infty, \ \text{and} \ |G^{(4)}(0)| < \infty, \\ H(0) = 1 - G(0) - K(0), \ \text{and} \ H'(0) = G'(0)(-1 + \beta f[x_n, w_n]), \\ H''(0) = -(-1 + \beta f[x_n, w_n])(2 + (-1 + \beta f[x_n, w_n])G''(0)), \ |H^{(4)}(0)| < \infty, \\ H^{(3)}(0) = 3(-2 + \beta f[x_n, w_n])(6 + 2\beta f[x_n, w_n](-3 + \beta f[x_n, w_n]) - P''(0)), \\ K'(0) = 2 - \beta f[x_n, w_n], \ |K(0)| < \infty, \end{cases} \tag{21}$$

and (ii): this solution reads the error equation

$$e_{n+1} = \frac{1}{48c_1^7}c_2^2(-1 + c_1\beta)(2c_1c_3(-1 + c_1\beta) + c_2^2(10 + 2c_1\beta(-5 + c_1\beta) - P''(0)))$$

$$\times(96c_1c_2c_3(-1+c_1\beta)^2-24c_1^2c_4(-1+c_1\beta)^2+c_2^3(-168+48P''(0)-8P^{(3)}(0)$$
$$+G^{(4)}(0)+c_1\beta(4(84-12P''(0)+3c_1\beta(-16+2c_1\beta+P''(0))+P^{(3)}(0))$$
$$+(-2+c_1\beta)(2+c_1\beta(-2+c_1\beta))G^{(4)}(0))+H^{(4)}(0)))e_n^8+O(e_n^9). \quad (22)$$

Proof. The proof of this theorem is similar to the previous one, hence it is omitted. □

An example from this new class using backward finite difference in this first step can be

$$\begin{cases} y_n = x_n - \frac{f(x_n)}{f[x_n,w_n]}, w_n = x_n - f(x_n), \\ z_n = y_n - \frac{f(y_n)}{f[x_n,w_n]}\{1+(2-f[x_n,w_n])\frac{f(y_n)}{f(w_n)}\}, \\ x_{n+1} = z_n - \frac{f(z_n)}{f[y_n,z_n]}\{1-(-1+f[x_n,w_n])(\frac{f(y_n)}{f(w_n)})^2-((-2+f[x_n,w_n])(3 \\ +f[x_n,w_n](-3+f[x_n,w_n])))(\frac{f(y_n)}{f(w_n)})^3+(2-f[x_n,w_n])\frac{f(z_n)}{f(w_n)}\}, \end{cases} \quad (23)$$

where its error equation satisfies

$$\begin{aligned} e_{n+1} = {} & (1/(c_1^7))(-1+c_1)^2c_2^2((5+c_1(-5+c_1))c_2^2 \\ & -c_1(-1+c_1)c_3)((7+c_1(-7+c_1))c_2^3+4c_1(-1+c_1)c_2c_3 \\ & -c_1^2(-1+c_1)c_4)e_n^8+O(e_n^9). \end{aligned} \quad (24)$$

3 Numerical results

We check the effectiveness of the novel derivative-free method (15), (17) and (19) of our proposed class of iterative methods (3) here. Due to this, we have compared them with the optimal eighth-order family of Kung and Traub (1), where $\beta = 1$, using the examples given below. The reason that we do not include other root solvers for comparisons is that, the derivative-involved methods consists of derivative calculation, which is not mostly easy-to-calculate for hard test functions as well as the other existing derivative-free methods of lower orders do not have any dominance to the optimal 8th-order methods for sufficiently close initial guess.

$f_1(x) = \cos(\sin(x^2\sqrt{x})) \times \cos(x^3) \times \arctan(\sin(x^5+x^{-1})) + x^3 + 1$,
$\alpha_1 \approx -0.59\ldots$ $\quad x_0 = -0.65$,

$f_2(x) = \operatorname{arccot}(x^{-2}) + x^2 + x\sin(x^2) + x^3 - 6$,
$\alpha_2 \approx 1.27\ldots$ $\quad x_0 = 1.38$.

The results of comparisons are given in Tables 1 and 2 in terms of the number significant digits for each test function after the specified number of iterations, that is, e.g. $0.8e-3949$ shows that the absolute value of the given nonlinear function f_1, after four iterations is zero up to 3949 decimal places. For numerical comparisons, the stopping criterion is $|f(x_n)| < 1.E{-}6000$. MATLAB 7.6 has been used in all computations using VPA command. As can be seen, numerical results are in concordance with the theory developed in this paper.

In the examples, the new methods improve the corresponding classical methods. The new methods inherit the merit of the optimal fourth-order two-step methods with regards to application of divided differences, weight function and high efficiency index, which is confirmed by the results in Tables 1 and 2. According to Tables 1 and 2, under a fair comparison structure, our proposed methods from the optimal class (3) perform well.

We mention that our primary aim was to construct a general class of very efficient multi-point methods and to check the Kung-Traub conjecture for the value $n = 4$, not to show off with thousands of accurate decimal digits. The achieved accuracy of calculated approximations is certainly exceptional, maybe provocative. Nonetheless, it may initiate a new challenge for constructing more efficient methods.

Table 1. Convergence study for the test function f_1

Methods	$\|f_1(x_1)\|$	$\|f_1(x_2)\|$	$\|f_1(x_3)\|$	$\|f_1(x_4)\|$
(1)	$0.1e-6$	$0.6e-56$	$0.1e-450$	$0.1e-3608$
(15)	$0.3e-7$	$0.3e-61$	$0.3e-493$	$0.8e-3949$
(17)	$0.1e-6$	$0.2e-55$	$0.3e-446$	$0.8e-3573$
(19)	$0.2e-8$	$0.5e-71$	$0.7e-573$	$0.5e-4588$

Table 2. Convergence study for the test function f_2

Methods	$\|f_2(x_1)\|$	$\|f_2(x_2)\|$	$\|f_2(x_3)\|$	$\|f_2(x_4)\|$
(1)	$0.7e-5$	$0.1e-49$	$0.3e-407$	$0.4e-3268$
(15)	$0.3e-5$	$0.1e-51$	$0.4e-422$	$0.5e-3387$
(17)	$0.6e-5$	$0.7e-50$	$0.3e-409$	$0.3e-3284$
(19)	$0.1e-9$	$0.8e-91$	$0.1e-740$	$0.8e-5938$

Constructing with memory methods according to the main class (3) in this paper, by introducing an iteration for the nonzero parameter β can be considered for future works in this field.

4 Concluding remarks

In order to approximate the simple roots of uni-variate nonlinear equations, we have developed a class of four-point three-step methods in which no derivative evaluations per full iteration is required. Per cycle, any method of our class, such as (17), needs only four pieces of information to reach the convergence rate eight. Therefore, this class satisfies the conjecture of Kung-Traub for constructing optimal high-order multi-point without memory methods for solving nonlinear equations. Numerical examples were considered to reveal the accuracy of the methods from the class.

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