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On generalized Hurwitz–Lerch Zeta distributions occuring in statistical inference

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Abstract. The object of the present paper is to define certain new incomplete generalized Hurwitz–Lerch Zeta functions and incomplete generalized Gamma functions. Further, we introduce two new statistical distributions named as, generalized Hurwitz–Lerch Zeta Beta prime distribution and generalized Hurwitz–Lerch Zeta Gamma distribution and investigate their statistical functions, such as moments, distribution and survivor function, characteristic function, the hazard rate function and the mean residue life functions. Finally, Moment Method parameter estimators are given by means of a statistical sample of size n. The results obtained provide an elegant extension of the work reported earlier by Garg *et al.* [3] and others.

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1 Introduction and preliminaries

A generalized Hurwitz–Lerch Zeta function $\Phi(z, s, a)$ is defined [1, p. 27, Eq. 1.11.1] as the power series

$$\Phi(z, s, a) = \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s}, \tag{1}$$

where $a \in \mathbb{C}\backslash\mathbb{Z}_0^-$; $\Re\{s\} > 1$ when $|z| = 1$ and $s \in \mathbb{C}$ when $|z| < 1$ and continues meromorphically to the complex s–plane, except for the simple pole at $s = 1$, with its residue equal to 1.

The function $\Phi(z, s, a)$ has many special cases such as Riemann Zeta [1], Hurwitz–Zeta [23] and Lerch Zeta function [27, p. 280, Example 8]. Some other special cases involve the polylogarithm (or Jonqière's function) and the generalized Zeta function [27, p. 280, Example 8], [23, p. 122, Eq. 2.5] discussed for the first time by Lipschitz and Lerch.

Lin and Srivastava investigated [12, p. 727, Eq. 8] the Hurwitz–Lerch Zeta function in the following form

$$\Phi_{\mu,\nu}^{(\rho,\sigma)}(z, s, a) = \sum_{n=0}^{\infty} \frac{(\mu)_{\rho n}}{(\nu)_{\sigma n}} \frac{z^n}{(n+a)^s}, \tag{2}$$

where $\mu \in \mathbb{C}$; $a, \nu \in \mathbb{C}\backslash\mathbb{Z}_0^-$; $\rho, \sigma \in \mathbb{R}_+$; $\rho < \sigma$ for $s, z \in \mathbb{C}$; $\rho = \sigma$ for $z \in \mathbb{C}$; $\rho = \sigma$, $s \in \mathbb{C}$ for $|z| < 1$; $\rho = \sigma$, $\Re\{s - \mu + \nu\} > 1$ for $|z| = 1$. Here $(\theta)_{\kappa n} = \Gamma(\theta + \kappa n)/\Gamma(\theta)$ denotes the generalized Pochhammer symbol, with the convention $(\theta)_0 = 1$.

Recently, Srivastava *et al.* [24] studied a new family of the Hurwitz–Lerch Zeta function

$$\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, a) = \sum_{n=0}^{\infty} \frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}} \frac{z^n}{(n+a)^s n!}, \tag{3}$$

where $\lambda, \mu \in \mathbb{C}$; $a, \nu \in \mathbb{C}\backslash\mathbb{Z}_0^-$; $\rho, \sigma, \kappa > 0$; for $|z| < 1$ and $\Re\{s + \nu - \lambda - \mu\} > 1$ for $|z| = 1$. Function (3) is a generalization of Hurwitz–Lerch Zeta function $\Phi_{\lambda,\mu,\nu}(z, s, a) := \Phi_{\lambda,\mu,\nu}^{(1,1,1)}(z, s, a)$ which has been studied by Garg *et al.* [2]. Special attention will be given to the special case of (3) (studied earlier by Goyal and Laddha [4, p. 100, Eq. (1.5)])

$$\Phi_\mu^*(z, s, a) := \Phi_{1,\mu,1}^{(1,1,1)}(z, s, a) = \sum_{n=1}^{\infty} \frac{(\mu)_n}{(n+a)^s} \frac{z^n}{n!}\,. \tag{4}$$

Another case of the Hurwitz–Lerch Zeta function (3), which differs in the choice of parameters, have been considered in [24] as well. Moreover, the article [24] contains the integral representation

$$\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}\,{}_2\Psi_1^*\left[\begin{array}{c}(\lambda,\rho),(\mu,\sigma)\\(\nu,\kappa)\end{array}\Big|\, ze^{-t}\right]dt, \tag{5}$$

valid for all $a, s \in \mathbb{C}, \Re\{a\} > 0, \Re\{s\} > 0$, when $|z| \le 1$, $z \ne 1$; and $\Re\{s\} > 1$ for $z = 1$. Here

$${}_p\Psi_q^*\left[\begin{array}{c}(a,A)_p\\(b,B)_q\end{array}\Big|\, z\right] = \sum_{n=0}^\infty \frac{\prod_{j=1}^p (a_j)_{A_j n}}{\prod_{j=1}^q (b_j)_{B_j n}}\frac{z^n}{n!} \tag{6}$$

stands for the *unified variant of the Fox–Wright generalized hypergeometric function* with p upper and q lower parameters; $(a,A)_p$ denotes the parameter p–tuple $(a_1,A_1),\cdots,(a_p,A_p)$ and $a_j \in \mathbb{C}$, $b_i \in \mathbb{C}\setminus\mathbb{Z}_0^-$, $A_i, B_j > 0$ for all $j = \overline{1,p}, i = \overline{1,q}$, while the series converges for suitably bounded values of $|z|$ when

$$\Delta := 1 + \sum_{j=1}^q B_j - \sum_{j=1}^p A_j > 0\,.$$

In the case $\Delta = 0$, the converegence holds in the open disc $|z| < \beta = \prod_{j=1}^q B_j^{B_j} \cdot \prod_{j=1}^p A_j^{-A_j}$.

Remark 1 *Let us point out that the original definition of the Fox–Wright function ${}_p\Psi_q[z]$ (consult monographs [1, 11, 15]) contains Gamma functions instead of the here used generalized Pochhammer symbols. However, these two functions differ only up to constant multiplying factor, that is*

$${}_p\Psi_q\left[\begin{array}{c}(a,A)_p\\(b,B)_q\end{array}\Big|\, z\right] = \frac{\prod_{j=1}^p\Gamma(a_j)}{\prod_{j=1}^q\Gamma(b_j)}\,{}_p\Psi_q^*\left[\begin{array}{c}(a,A)_p\\(b,B)_q\end{array}\Big|\, z\right].$$

The unification's motivation is clear - for $A_1 = \cdots = A_p = B_1 = \cdots = B_q = 1$, ${}_p\Psi_q^*[z]$ one reduces exactly to the generalized hypergeometric function ${}_pF_q[z]$, see recent articles [12, 24].

Finally, we recall the integral expression for function (3), derived by Srivastava *et al.* [24]:

$$\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a) = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)}\int_0^\infty \frac{t^{\lambda-1}}{(1+t)^\nu}\Phi_{\mu,\nu-\lambda}^{(\sigma,\kappa-\rho)}\Big(\frac{zt^\rho}{(1+t)^\kappa}, s, a\Big)dt, \tag{7}$$

where $\Re\{\nu\} > \Re\{\lambda\} > 0,\ \kappa \geq \rho > 0,\ \sigma > 0, s \in \mathbb{C}$.

Now, we study generalized incomplete functions and the associated statistical distributions based mainly on integral expressions (5) and (7).

2 Families of incomplete φ and ξ functions

By virtue of integral (7), we define the *lower incomplete generalized Hurwitz–Lerch Zeta function* as

$$\varphi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x) = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)} \int_0^x \frac{t^{\lambda-1}}{(1+t)^\nu}\, \Phi_{\mu,\nu-\lambda}^{(\sigma,\kappa-\rho)}\Big(\frac{zt^\rho}{(1+t)^\kappa}, s, a\Big)\mathrm{d}t, \quad (8)$$

and the *upper (complementary) generalized Hurwitz–Lerch Zeta function* in the form

$$\overline{\varphi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x) = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)} \int_x^\infty \frac{t^{\lambda-1}}{(1+t)^\nu}\, \Phi_{\mu,\nu-\lambda}^{(\sigma,\kappa-\rho)}\Big(\frac{zt^\rho}{(1+t)^\kappa}, s, a\Big)\mathrm{d}t. \quad (9)$$

In both cases one requires $\Re(\nu), \Re(\lambda) > 0,\ \kappa \geq \rho > 0;\ \sigma > 0,\ s \in \mathbb{C}$.

From (8) and (9) readily follows that

$$\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a) = \lim_{x\to\infty} \varphi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x) = \lim_{x\to 0+} \overline{\varphi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x)\,, \quad (10)$$

$$\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a) = \varphi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x) + \overline{\varphi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a|x), \qquad x \in \mathbb{R}_+. \quad (11)$$

In view of the integral expression (5), the *lower incomplete generalized Gamma function* and the *upper (complementary) incomplete generalized Gamma function* are defined respectively by

$$\xi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a,b|x) = \frac{b^s}{\Gamma(s)} \int_0^x t^{s-1}\mathrm{e}^{-at}\, {}_2\Psi_1^*\Big[\begin{array}{c}(\alpha,\rho),(\mu,\sigma)\\(\nu,\kappa)\end{array}\Big|\, z\mathrm{e}^{-bt}\Big]\mathrm{d}t \quad (12)$$

and

$$\overline{\xi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa);x,\infty}(z,s,a,b|x) = \frac{b^s}{\Gamma(s)} \int_x^\infty t^{s-1}\mathrm{e}^{-at}{}_2\Psi_1^*\Big[\begin{array}{c}(\alpha,\rho),(\mu,\sigma)\\(\nu,\kappa)\end{array}\Big|\, z\mathrm{e}^{-bt}\Big]\mathrm{d}t, \quad (13)$$

where $\Re\{a\}, \Re\{s\} > 0$, when $|z| \leq 1\,(z \neq 1)$ and $\Re\{s\} > 1$, when $z = 1$, provided that each side exists. By virtue of (12) and (13) we easily conclude the properties:

$$\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\rho)}(z,s,a) = \lim_{x\to\infty} \xi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a,b|x) = \lim_{x\to 0+} \overline{\xi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a,b|x)\,, \quad (14)$$

$$\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a/b) = \xi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a,b|x) + \overline{\xi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a,b|x), \qquad x \in \mathbb{R}_+. \quad (15)$$

3 Generalized Hurwitz–Lerch Zeta Beta prime distribution

Special functions and integral transforms are useful in the development of the theory of probability density functions (PDF). In this connection, one can refer to the books e.g. by Mathai and Saxena [14, 15] or by Johnson and Kotz [8, 9]. Hurwitz–Lerch Zeta distributions are studied by many mathematicians such as Dash, Garg, Gupta, Kalla, Saxena, Srivastava etc. (see e.g. [2, 3, 6, 7, 18, 19, 20, 21, 25]). Due to usefulness and popularity of Hurwitz–Lerch Zeta distribution in reliability theory, statistical inference etc. the authors are motivated to define a generalized Hurwitz–Lerch Zeta distribution and to investigate its important properties.

Let the random variable X be defined on some fixed standard probability space $(\Omega, \mathfrak{F}, P)$. The r.v. X such that possesses PDF

$$f(x) = \begin{cases} \dfrac{\Gamma(\nu)\, x^{\lambda-1}}{\Gamma(\lambda)\Gamma(\nu-\lambda)(1+x)^{\nu}} \dfrac{\Phi^{(\sigma,\kappa-\rho)}_{\mu,\nu-\lambda}\Big(\dfrac{zx^{\rho}}{(1+x)^{\kappa}}, s, a\Big)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)} & x > 0, \\ 0 & x \le 0, \end{cases} \tag{16}$$

we call *generalized Hurwitz–Lerch Zeta Beta prime* and write $X \sim \mathrm{HLZB}'$. Here μ, λ are shape parameters, and z stands for the scale parameter which satisfy $\Re\{\nu\} > \Re\{\lambda\} > 0$, $s \in \mathbb{C}$, $\kappa \ge \rho > 0$, $\sigma > 0$.

The behaviour of the PDF $f(x)$ at $x = 0$ depends on λ in the manner that $f(0) = 0$ for $\lambda > 1$, while $\lim_{x\to 0+} f(x) = \infty$ for all $0 < \lambda < 1$.

Now, let us mention some interesting special cases of PDF (16).

(i) For $\sigma = \rho = \kappa = 1$ we get the following Hurwitz–Lerch Zeta Beta prime distribution discussed by Garg *et al.* [3]:

$$f_1(x) = \begin{cases} \dfrac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)\Phi_{\lambda,\mu,\nu}(z,s,a)} \dfrac{x^{\lambda-1}}{(1+x)^{\nu}} \Phi^{*}_{\mu}\Big(\dfrac{zx}{1+x}, s, a\Big) & x > 0, \\ 0 & \text{elsewhere} \end{cases}$$

where $a \notin \mathbb{Z}_0^-$, $\Re\{\nu\} > \Re\{\lambda\} > 0$, $x \in \mathbb{R}$, $s \in \mathbb{C}$ when $|z| < 1$ and $\Re\{s-\mu\} > 0$, when $|z| = 1$. Here $\Phi^{*}_{\mu}(\cdot, s, a)$ stands for the Goyal–Laddha type generalized Hurwitz–Lerch Zeta function described in (4).

(ii) If we set $\sigma = \rho = \kappa = \lambda = 1$ it gives a new probability distribution function, defined by

$$f_2(x) = \begin{cases} \dfrac{\nu - 1}{(1+x)^\nu \, \Phi_{1,\mu,\nu}(z,s,a)} \, \Phi^*_\mu\Big(\dfrac{zx}{1+x}, s, a\Big) & x > 0, \\ 0 & x \le 0, \end{cases} \tag{17}$$

where $a \notin \mathbb{Z}_0^-$, $\Re\{\lambda\} > 0$, $x \in \mathbb{R}$, $s \in \mathbb{C}$ when $|z| < 1$ and $\Re\{s - \mu\} > 0$, when $|z| = 1$.

(iii) When $\sigma = \rho = \kappa = 1$ and $\nu = \mu$, from (16) it follows

$$f_3(x) = \begin{cases} \dfrac{\Gamma(\mu)}{\Gamma(\lambda)\Gamma(\mu-\lambda)\Phi^*_\lambda(z,s,a)} \, \dfrac{x^{\lambda-1}}{(1+x)^\mu} \, \Phi^*_\mu\Big(\dfrac{zx}{1+x}, s, a\Big) & x > 0, \\ 0 & x \le 0, \end{cases} \tag{18}$$

with $a \notin \mathbb{Z}_0^-$, $\Re\{\mu\} > \Re\{\lambda\} > 0$, $x \in \mathbb{R}$, $s \in \mathbb{C}$ when $|z| < 1$ and $\Re\{s-\mu\} > 0$, when $|z| = 1$.

(iv) For $\sigma = \rho = \kappa = 1$ and $\mu = 0$, we obtain the Beta prime distribution (or the Beta distribution of the second kind).

(v) For Fischer's F–distribution, which is a Beta prime distribution, we set $\sigma = \rho = \kappa = 1$ and replace $x = mx/n$, $\lambda = m/2$, $\nu = (m+n)/2$, where m and n are positive integers.

4 Statistical functions for the HLZB$'$ distribution

In this section we would introduce some classical statistical functions for the HLZB$'$ distributed random variable having the PDF given with (16). These characteristics are *moments* of positive, fractional order $m_r, r \in \mathbb{R}$, being the Mellin transform of order $r+1$ of the PDF; the *generating function* $G_X(t)$ which equals to the Laplace transform and the *characteristic function* (CHF) $\phi_X(t)$ which coincides with the Fourier transform of the PDF (16).

We point out that all three highly important characteristics of the probability distributions can be uniquely expressed *via* the operator of the mathematical expectation E. However, it is well–known that for any Borel function ψ there holds

$$\mathsf{E}\psi(X) = \int_{\mathbb{R}} \psi(x) f(x)\, dx. \tag{19}$$

To obtain explicitely $m_r, G_X(t), \phi_X(t)$ we also need in the sequel the unified Hurwitz–Lerch Zeta function, recently introduced by Srivastava *et al.* [24]. According to [24] we consider nonnegative integer parameters $p, q \in \mathbb{N}_0 = \{0, 1, 2, \cdots\}$; $\lambda_j \in \mathbb{C}$, $\mu_k \in \mathbb{C} \setminus \mathbb{Z}_0^-$; $\sigma_j, \rho_k > 0$, $j = \overline{1,p}$, $k = \overline{1,q}$. Then the *Unified Hurwitz–Lerch Zeta Function* with $p + q$ upper and $p + q + 2$ lower parameters, reads as follows

$$\Phi^{(\rho;\sigma)}_{\lambda;\mu}(z, s, a) := \Phi^{(\rho_1,\cdots,\rho_p;\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z, s, a) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{\prod_{j=1}^{q}(\mu_j)_{n\sigma_j}} \frac{z^n}{(n + a)^s\, n!}\,, \tag{20}$$

where $s, \Re\{a\} > 0$ and the empty product is taken to be unity. The series (20) converges

1. for all $z \in \mathbb{C} \setminus \{0\}$ if $\Upsilon > -1$;
2. in the open disc $|z| < \nabla$ if $\Upsilon = -1$;
3. on the circle $|z| = \nabla$, for $\Upsilon = -1$, $\Re\{\Theta\} > 1/2$,

where

$$\nabla := \prod_{j=1}^{q} \sigma_j^{\sigma_j} \prod_{j=1}^{p} \rho_j^{-\rho_j}, \quad \Upsilon := \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j + s, \quad \Theta := \sum_{j=1}^{q} \mu_j - \sum_{j=1}^{p} \lambda_j + \frac{p-q}{2}.$$

Theorem 1 *Let* $X \sim$ HLZB$'$ *be a r.v. defined on a standard probability space* $(\Omega, \mathfrak{F}, P)$ *and let* $r \in \mathbb{R}_+$. *Then the* r*th fractional order moment of* X *reads as follows*

$$m_r = \frac{(\lambda)_r \sin \pi(\nu - \lambda)}{(1 - \nu + \lambda)_r \sin \pi(\nu - \lambda - r)} \, \frac{\Phi^{(\sigma,\rho,\kappa-\rho;\,\kappa,\kappa-\rho)}_{\mu,\lambda+r,\nu-\lambda-r;\nu,\nu-\lambda}(z, s, a)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s, a)}\,. \tag{21}$$

Proof. The fractional moment m_r of the r.v. $X \sim$ HLZB$'$ is given by

$$m_r = \mathsf{E}X^r = \frac{A\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu - \lambda)} \int_0^\infty \frac{x^{\lambda+r-1}}{(1 + x)^\kappa} \Phi^{(\sigma,\kappa-\rho)}_{\mu,\nu-\lambda}\Big(\frac{zx^\rho}{(1 + x)^\kappa}, s, a\Big)\, dx \quad r \in \mathbb{R}_+\,,$$

where A is the related normalizing constant.

Expressing the Hurwitz–Lerch Zeta function in initial power series form, and interchanging the order of summation and integration, we find that:

$$\begin{aligned} m_r &= \frac{A\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu - \lambda)} \sum_{n=0}^{\infty} \frac{(\mu)_{\sigma n}}{(\nu - \lambda)_{(\kappa-\rho)n}} \frac{z^n}{(n + a)^s\, n!} \int_0^\infty \frac{x^{\lambda+r+\rho n-1}}{(1 + x)^{\nu+\kappa n}}\, dx \\ &= \frac{A\Gamma(\lambda + r)\Gamma(\nu - \lambda - r)}{\Gamma(\lambda)\Gamma(\nu - \lambda)} \sum_{n=0}^{\infty} \frac{(\mu)_{\sigma n}(\lambda + r)_{\rho n}}{(\nu)_{\kappa n}} \cdot \frac{(\nu - \lambda - r)_{(\kappa-\rho)n}}{(\nu - \lambda)_{(\kappa-\rho)n}} \frac{z^n}{(n + a)^s\, n!}\,. \end{aligned}$$

By the Euler's reflection formula we get

$$\begin{aligned} m_r &= \frac{A(\lambda)_r \Gamma(1-\nu+\lambda)\sin\pi(\nu-\lambda)}{\Gamma(1-\nu+\lambda+r)\sin\pi(\nu-\lambda-r)} \sum_{n=0}^{\infty} \frac{(\mu)_{\sigma n}(\lambda+r)_{\rho n}(\nu-\lambda-r)_{(\kappa-\rho)n}\, z^n}{(\nu)_{\kappa n}(\nu-\lambda)_{(\kappa-\rho)n}\,(n+a)^s\, n!} \\ &= \frac{A(\lambda)_r \sin\pi(\nu-\lambda)}{(1-\nu+\lambda)_r \sin\pi(\nu-\lambda-r)} \sum_{n=0}^{\infty} \frac{(\mu)_{\sigma n}(\lambda+r)_{\rho n}(\nu-\lambda-r)_{(\kappa-\rho)n}\, z^n}{(\nu)_{\kappa n}(\nu-\lambda)_{(\kappa-\rho)n}(n+a)^s\, n!}\,, \end{aligned}$$

which is same as (21). □

We point out that for the integer $r \in \mathbb{N}$, the moment (21) it reduces to

$$m_r = \frac{(-1)^r(\lambda)_r}{(1-\nu+\lambda)_r}\, \frac{\Phi^{(\sigma,\rho,\kappa-\rho;\,\kappa,\kappa-\rho)}_{\mu,\lambda+r,\nu-\lambda-r;\nu,\nu-\lambda}(z,s,a)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)}\,. \tag{22}$$

Theorem 2 *The generating function* $G_X(t)$ *and the* CHF $\phi_X(t)$, $t \in \mathbb{R}$ *for the r.v.* $X \sim \mathrm{HLZB}'$ *are represented in the form*

$$G_X(t) = \mathsf{E}\, e^{-tX} = \frac{1}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)} \sum_{r=0}^{\infty} \frac{(\lambda)_r}{(1+\lambda-\nu)_r}\frac{t^r}{r!}\Phi^{(\rho,\sigma,\kappa)}_{\lambda+r,\mu,\nu}(z,s,a), \tag{23}$$

$$\phi_X(t) = \mathsf{E}\, e^{itX} = \frac{1}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)} \sum_{r=0}^{\infty} \frac{(\lambda)_r}{(1+\lambda-\nu)_r}\frac{(-it)^r}{r!}\Phi^{(\rho,\sigma,\kappa)}_{\lambda+r,\mu,\nu}(z,s,a)\,. \tag{24}$$

Proof. Setting $\psi(X) = e^{-tX}$ in (19) respectively, then expanding the Laplace kernel into Maclaurin series, by legitimate interchange the order of summation and integration we obtain the generating function $G_X(t)$ in terms of (22). Because $\phi_X(t) = G_X(-it), t \in \mathbb{R}$, the proof is completed. □

The second set of important statistical functions concers the reliability applications of the newly introduced generalized Hurwitz–Lech Zeta Beta prime distribution. The functions associated with r.v. X are the cumulative distribution function (CDF) F, the survivor function $S = 1-F$, the hazard rate function $h = f/(1-F)$, and the mean residual life function $K(x) = \mathsf{E}(X-x|X \geq x)$. Their explicit formulæ are given in terms of lower and upper incomplete (complementary) φ–functions.

Theorem 3 *Let r.v.* $X \sim \mathrm{HLZB}'$. *Then we have:*

$$h(x) = \frac{f(x)}{S(x)} = \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)}\,\frac{x^{\lambda-1}}{(1+x)^{\nu}}\,\frac{\Phi^{(\sigma,\kappa-\rho)}_{\mu,\nu-\lambda}\Big(\frac{zx^{\rho}}{(1+x)^{\kappa}}, s, a\Big)}{\overline{\varphi}^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a|x)}, \tag{25}$$

$$\begin{aligned} K(x) = {} & \frac{\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)\,\overline{\varphi}^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a|x)} \sum_{n=0}^{\infty} \frac{(\mu))_{\sigma n}}{(\nu-\lambda)_{(\kappa-\rho)n}}\,\frac{z^n}{(n+a)^s\, n!} \\ & \times \mathrm{B}_{(1+x)^{-1}}\big(\nu-\lambda-1+(\kappa-\rho)n, \lambda+1+\rho n\big) - x, \end{aligned} \tag{26}$$

where

$$\mathrm{B}_z(a,b) = \int_0^z t^{a-1}(1-t)^{b-1}\,\mathrm{d}t, \qquad \min\big(\Re\{a\},\Re\{b\}\big) > 0,\ |z| < 1$$

represents the incomplete Beta*–function.*

Proof. The CDF and the survivor functions of the r.v. X are

$$F(x) = \frac{\varphi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a|x)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)}, \qquad S(x) = \frac{\overline{\varphi}^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a|x)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)} \qquad x > 0\,,$$

and vanishes elsewhere. Therefore, being $h(x) = f(x)/S(x)$, (25) is proved.

It is well–known that for the mean residual life function there holds [5]

$$K(x) = \frac{1}{S(x)} \int_x^{\infty} tf(t)\mathrm{d}t - x\,.$$

The integral will be

$$\mathcal{J} = \int_x^{\infty} tf(t)\mathrm{d}t = \frac{A\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)} \sum_{n=0}^{\infty} \frac{(\mu)_{\sigma n}\,(n+a)^{-s}\,z^n}{(\nu-\lambda)_{(\kappa-\rho)n}\,n!} \int_x^{\infty} \frac{t^{\lambda+\rho n}}{(1+t)^{\nu+\kappa n}}\,\mathrm{d}t\,,$$

where the innermost t–integral reduces to the incomplete Beta function in the following way:

$$\int_x^{\infty} \frac{t^{p-1}}{(1+t)^q}\,\mathrm{d}t = \int_0^{(1+x)^{-1}} t^{q-p-1}t^{p-1}\,\mathrm{d}t = \mathrm{B}_{(1+x)^{-1}}\big(p, q-p\big)\,.$$

Therefore we conlude

$$\mathcal{J} = \frac{A\Gamma(\nu)}{\Gamma(\lambda)\Gamma(\nu-\lambda)} \sum_{n=0}^{\infty} \frac{(\mu)_{\sigma n}\,(n+a)^{-s}\,z^n}{(\nu-\lambda)_{(\kappa-\rho)n}\,n!}\,\mathrm{B}_{(1+x)^{-1}}\big(\nu-\lambda-1+(\kappa-\rho)n, \lambda+1+\rho n\big)\,.$$

After some simplification it leads to the stated formula (26). □

5 Generalized Hurwitz–Lerch Zeta Gamma distribution

Gamma–type distributions, associated with certain special functions of science and engineering, are studied by several researchers, such as Stacy [26]. In this section a new probability density function is introduced, which extends both the well–known Gamma distribution [21, 28] and Planck distribution [9].

Consider the r.v. X defined on a standard probability space $(\Omega, \mathfrak{F}, P)$, defined by the PDF

$$f(x) = \begin{cases} \dfrac{b^s x^{s-1} e^{-ax}}{\Gamma(s)} \dfrac{{}_2\Psi_1^*\left[\begin{array}{c} (\lambda,\rho),(\mu,\sigma) \\ (\nu,\kappa) \end{array} \Big| ze^{-bx} \right]}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,s,a/b)} & x > 0, \\ 0, & x \le 0\,; \end{cases} \tag{27}$$

where a, b are scale parameters and s is shape parameter. Further $\Re\{a\}, \Re\{s\} > 0$ when $|z| \le 1\, (z \ne 1)$ and $\Re\{s\} > 1$ when $z = 1$. Such distribution we call by convention *generalized Hurwitz–Lerch Zeta Gamma distribution* and write $X \sim$ HLZG. Notice that behavior of $f(x)$ near to the origin depends on s in the manner that $f(0) = 0$ for $s > 1$, and for $s = 1$ we have

$$f(0) = \frac{b\ {}_2\Psi_1^*\left[\begin{array}{c} (\lambda,\rho),(\mu,\sigma) \\ (\nu,\kappa) \end{array} \Big| z \right]}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z,1,a/b)}\,,$$

and $\lim_{x\to 0+} f(x) = \infty$ when $0 < s < 1$.

Now, we list some important special cases of the HLZG distribution.

(a) For $\sigma = \rho = \kappa = 1$ we obtain the following PDF discussed by Garg *et al.* [3]:

$$f_1(x) = \frac{b^s x^{s-1} e^{-ax}}{\Gamma(s)} \frac{{}_2F_1\left[\begin{array}{c} \lambda,\mu \\ \nu \end{array} \Big| ze^{-bx} \right]}{\Phi_{\lambda,\mu,\nu}(z,s,a/b)}, \tag{28}$$

where $\Re\{a\}, \Re\{b\}, \Re\{s\} > 0$ and $|z| < 1$ or $|z| = 1$ with $\Re\{\nu - \lambda - \mu\} > 0$.

(b) If we set $\sigma = \rho = \kappa = 1$, $b = a$, $\lambda = 0$, then (27) reduces to the Gamma distribution [9, p. 32] and

(c) for $\sigma = \rho = \kappa = 1$, $\mu = \nu$, $\lambda = 1$ it reduces to the generalized Planck distribution defined by Nadarajah and Kotz [16], which is a generalization of the Planck distribution [9, p. 273].

6 Statistical functions for the HLZG distribution

In this section we will derive the statistical functions for the r.v. $X \sim$ HLZG distribution associated with PDF (27). For the moments $\mathfrak{m}_r$ of fractional order $r \in \mathbb{R}_+$ we derive by definition

$$\mathfrak{m}_r = \int_0^\infty x^r f(x)\mathrm{d}x = \frac{(s)_r}{b^r}\frac{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s+r, a/b)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, a/b)}. \tag{29}$$

Next we present the Laplace and the Fourier transforms of the probability density function (27), that is the generating function $G_Y(t)$ and the related CHF $\phi_Y(t)$:

$$G_X(t) = \mathsf{E}\, e^{-tY} = \frac{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, (a+t)/b)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, a/b)}, \tag{30}$$

$$\phi_X(t) = G_Y(-it) = \mathsf{E}\, e^{itY} = \frac{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, (a-it)/b)}{\Phi_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, a/b)}, \qquad t \in \mathbb{R}. \tag{31}$$

The second set of the statistical functions include the hazard function h and the mean residual life function K.

Theorem 4 *Let* $X \sim$ HLZG*. Then we have:*

$$h(x) = \frac{b^s x^{s-1} e^{-ax}}{\Gamma(s)} \frac{{}_2\Psi_1^*\left[\begin{array}{c}(\lambda,\rho),(\mu,\sigma)\\(\nu,\kappa)\end{array}\Big| ze^{-bx}\right]}{\overline{\xi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, a/b, b|x)} \tag{32}$$

$$K(x) = \frac{1}{b\,\Gamma(s)\overline{\xi}_{\lambda,\mu,\nu}^{(\rho,\sigma,\kappa)}(z, s, a/b, b|x)} \sum_{n=0}^\infty \frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}} \frac{\Gamma(s+1, (a+bn)x)}{(n+a/b)^{s+1}} \frac{z^n}{n!} - x\,. \tag{33}$$

Here

$$\Gamma(p, z) = \int_z^\infty t^{p-1} e^{-t}\,\mathrm{d}t, \qquad \Re\{p\} > 0\,,$$

stands for the upper incomplete Gamma function.

Proof. From the hazard function formula a simple calculation gives:

$$K(x) = \frac{b^s}{\Gamma(s)\overline{\xi}^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a/b,b|x)} \int_x^\infty t^s e^{-at}\, {}_2\Psi_1^*\Big[\begin{array}{c} (\lambda,\rho),\,(\mu,\sigma) \\ (\nu,\kappa) \end{array} \Big| z e^{-bt} \Big]\, dt - x$$

$$= \frac{b^s}{\Gamma(s)\overline{\xi}^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a/b,b|x)} \sum_{n=0}^\infty \frac{(\lambda)_{\rho n}(\mu)_{\sigma n}}{(\nu)_{\kappa n}} \frac{z^n}{n!} \int_x^\infty t^s e^{-(a+bn)t}\, dt - x\,.$$

Further simplification leads to the asserted formula (33). □

7 Statistical parameter estimation in HLZB′ and HLZG distribution models

The statistical parameter estimation becomes one of the main tools in random model identification procedures. In the study of HLZB′ and HLZG distributions the PDFs (16) and (27) are built by higher transcendental functions such as generalized Hurwitz–Lerch Zeta function $\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)$ and Fox–Wright generalized hypergeometric function ${}_2\Psi_1^*[z]$. The power series definitions of these functions does not enable the successful implementation of the popular and efficient Maximum Likelihood (ML) parameter estimation, only the numerical system solving can reach any result for HLZB′, while ML cannot be used for HLZG distribution case, being the extrema of the likelihood function out of the parameter space.

Therefore, we consider the Moment Method estimators, such that are weakly consistent (by the Khinchin's Law of Large Numbers), also strongly consistent (by the Kolmogorov LLN) and asymptotically unbiased.

7.1 Parameter estimation in HLZB′ model

Assume that the considered statistical population possesses HLZB′ distribution, that is the r.v. $X \sim f(x)$, (16) generates n independent, identically distributed replicæ $\Xi = \big(X_j\big)_{j=\overline{1,n}}$ which forms a statistical sample of the size n. We are now interested in estimating the 9-dimensional parameter

$$\theta_9 = (a, \sigma, \kappa, \rho, \lambda, \mu, \nu, z, s)$$

or some of its coordinates by means of the sample Ξ.

First we consider the PDF (16) for small $z \to 0$. For such values we get asymptotics

$$f(x) \sim \frac{\Gamma(\nu)\, x^{\lambda-1}}{\Gamma(\lambda)\Gamma(\nu-\lambda)(1+x)^\nu} \qquad x > 0, \tag{34}$$

which is the familiar Beta distribution of the second kind (or Beta prime) $B'(\lambda, \nu)$. The moment method estimators for the remaining parameters $\lambda > 0, \nu > 2$ read:

$$\widetilde{\lambda} = \frac{\overline{X}_n(\overline{X_n^2} + \overline{X}_n)}{\overline{S}_n^2}, \quad \widetilde{\nu} = \frac{\overline{X_n^2} + \overline{X}_n}{\overline{S}_n^2}(\overline{X}_n + 1) + 1\,, \tag{35}$$

where

$$\overline{X}_n = \frac{1}{n}\sum_{j=1}^n X_j, \qquad \overline{S}_n^2 = \frac{1}{n}\sum_{j=1}^n (X_j - \overline{X}_n)^2$$

expressing the sample mean and the sample variance respectively. Let us mention that for $\nu < 2$, the variance of a r.v. $X \sim B'(\lambda, \nu)$ does not exists, so for these range of parameters MM is senseless.

The case of full range parameter estimation is highly complicated. The moment method estimator can be reached by virtue of the positive integer order moments formula (22) substituting

$$\overline{X_n^r} = \frac{1}{n}\sum_{j=1}^n X_j^r \mapsto m_r,$$

where $\overline{X_n^r}$ is the rth sample moment. Thus, numerical solution of the system

$$\frac{(-1)^r(\lambda)_r}{(1-\nu+\lambda)_r}\,\frac{\Phi^{(\sigma,\rho,\kappa-\rho;\,\kappa,\kappa-\rho)}_{\mu,\lambda+r,\nu-\lambda-r;\nu,\nu-\lambda}(z,s,a)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z,s,a)} = \overline{X_n^r} \qquad r = \overline{1,9} \tag{36}$$

which results in the vectorial moment estimator $\widetilde{\theta}_9 = (\widetilde{a}, \widetilde{\sigma}, \widetilde{\kappa}, \widetilde{\rho}, \widetilde{\lambda}, \widetilde{\mu}, \widetilde{\nu}, \widetilde{z}, \widetilde{s})$.

7.2 Parameter estimation in HLZG distribution

To achieve Gamma distribution's PDF from the density function (27) of HLZG in a way different then $(\mathfrak{b})$ in Section 6, it is enough to consider the PDF (27) for $a = b$ and small $z \to 0$. Indeed, we have

$$\lim_{z\to 0} f(x) = \begin{cases} \dfrac{b^s x^{s-1} e^{-bx}}{\Gamma(s)} & x > 0, \\ 0, & x \le 0\,; \end{cases} \tag{37}$$

It is well known that the moment method estimators for parameters b, s are

$$\widetilde{b} = \frac{\overline{X}_n}{\overline{S}_n^2}, \qquad \widetilde{s} = \frac{(\overline{X}_n)^2}{\overline{S}_n^2}\,.$$

The general case includes the vectorial parameter

$$\theta_{10} = (a, b, s, \lambda, \rho, \mu, \sigma, \nu, \kappa, z)\,.$$

First we show a kind of recurrence relation for the fractional order moments between distant neighbours.

Theorem 5 *Let* $0 \le t \le r$ *be nonnegative real numbers, and* $\mathfrak{m}_r$ *denotes the fractional positive* r*th order moment of a r.v.* $X \sim \text{HLZG}$. *Then it holds true*

$$\mathfrak{m}_r = \mathfrak{m}_{r-t} \cdot \mathfrak{m}_t. \tag{38}$$

Proof. It is not difficult to prove

$$\begin{aligned} \mathfrak{m}_r &= \frac{(s)_r}{b^r}\,\frac{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s+r, a/b)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s, a/b)} \\ &= \frac{\Gamma(s+r)}{b^{r-t}\,\Gamma(s+t)}\,\frac{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s+r, a/b)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s+t, a/b)}\,\frac{\Gamma(s+t)}{b^t\,\Gamma(s)}\,\frac{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s+t, a/b)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s, a/b)}\,, \end{aligned}$$

which is equivalent to the assertion of the Theorem. □

Remark 2 *Taking the integer order moments* (29)*, that is* $m_r, r \in \mathbb{N}_0$*, the recurrence relation* (38) *becomes a contiguous relation for distant neighbours:*

$$m_\ell = m_{\ell-k} \cdot m_k = \frac{(s+\ell)_{\ell-k}}{b^{\ell-k}}\,\frac{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s+\ell, a/b)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s+k, a/b)}\,m_k \tag{39}$$

for all $0 \le k \le \ell,\ k, \ell \in \mathbb{N}_0$.

Choosing a system of 10 suitable different equations like (38) in which $\mathfrak{m}_r$ is substituted with $\overline{X_n^r} \mapsto \mathfrak{m}_r$, we get

$$\frac{(s+t)_{r-t}}{b^{r-t}}\,\frac{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s+r, a/b)}{\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s+t, a/b)} = \frac{\overline{X_n^r}}{\overline{X_n^t}}\,. \tag{40}$$

However, the at least complicated case of (38) occurs at the contiguous (39) with $k = 0, \ell = \overline{1, 10}$, that is, by virtue of (40) we deduce the system in unknown θ_{10}:

$$(s)_\ell\,\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s+\ell, a/b) = b^\ell\,\Phi^{(\rho,\sigma,\kappa)}_{\lambda,\mu,\nu}(z, s, a/b)\,\overline{X_n^\ell} \qquad \ell = \overline{1, 10}\,. \tag{41}$$

The numerical solution of system (41) with respect to unknown parameter vector θ_{10} we call moment method estimator $\widetilde{\theta}_{10}$.

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