On topological properties of the set of maldistributed sequences

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Abstract. The real sequence \((x_n)\) is maldistributed if for any non-empty interval \(I\), the set \(\{n \in \mathbb{N} : x_n \in I\}\) has upper asymptotic density 1. The main result of this note is that the set of all maldistributed real sequences is a residual set in the set of all real sequences (i.e., the maldistribution is a typical property in the sense of Baire categories). We also generalize this result.

1 Introduction

Following the concept of statistical convergence for real sequences, J. A. Fridy [2] introduced the concept of statistical cluster points of a sequence \((x_n)\). A number \(\alpha\) is called a statistical cluster point of the sequence \((x_n)\) provided that for every \(\varepsilon > 0\) the set \(\{n \in \mathbb{N} : |x_n - \alpha| < \varepsilon\}\) has a positive upper asymptotic density.

G. Myerson [7] calls a sequence \((x_n)\) maldistributed if for any non-empty interval \(I\) the set \(\{n \in \mathbb{N} : x_n \in I\}\) has upper asymptotic density 1. In [12] the maldistribution property is characterized by one-jump distribution functions. Examples of maldistributed sequences are given in [12] and [3]. Using the idea from [4] (Example VII) for the generalization of the concept of statistical

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convergence, we can extend the maldistribution property of sequences with the help of weighted densities.

The concept of weighted density as a generalization of asymptotic density was introduced in [1] and [10]. Let \( f : \mathbb{N} \to (0, \infty) \) be a weight function with the properties

\[
\sum_{n=1}^{\infty} f(n) = \infty, \quad \lim_{n \to \infty} \frac{f(n)}{\sum_{a \leq n} f(a)} = 0. \tag{1}
\]

For \( A \subset \mathbb{N} \) define by

\[
d_r(A) = \liminf_{n \to \infty} \frac{\sum_{a \leq n, a \in A} f(a)}{\sum_{a \leq n} f(a)} \quad \text{and} \quad \overline{d}_r(A) = \limsup_{n \to \infty} \frac{\sum_{a \leq n, a \in A} f(a)}{\sum_{a \leq n} f(a)}
\]

the lower and upper \( f \)-densities of \( A \), respectively. Note that the asymptotic densities correspond to \( f(n) = 1 \) and the logarithmic densities to \( f(n) = \frac{1}{n} \). It is well-known that each set which has asymptotic density also has the logarithmic one but a set may have a logarithmic density without having an asymptotic one.

The main tool to compare weighted densities is the classical result of C. T. Rajagopal (cf. [9], Theorem 3) which, in terms of weighted densities, says the following.

\[\text{Let } f, g : \mathbb{N} \to (0, \infty) \text{ be weight functions with properties (1). If } \frac{f(n)}{g(n)} \text{ is decreasing, then for any } A \subset \mathbb{N} \text{ we have}\]

\[d_g(A) \leq d_r(A) \leq \overline{d}_r(A) \leq \overline{d}_g(A). \tag{2}\]

Now we give a generalization of maldistributed sequences.

**Definition 1** Let \( f : \mathbb{N} \to (0, \infty) \) be a weight function with properties (1). The sequence \((x_n)\) is said to be \( f \)-maldistributed, if for any non-empty interval \( I \) the set \( \{n \in \mathbb{N} : x_n \in I\} \) has upper \( f \)-density 1.

Comparing to asymptotic density, logarithmic density is less sensitive to certain perturbations. For example, if a sequence is maldistributed, then it is not necessary \( f \)-maldistributed for \( f(n) = \frac{1}{n} \) (which defines the logarithmic density).
Let us denote by $\mathcal{M}_f$ the set of all $f$-maldistributed sequences. The purpose of this note is to show that for any weight function $f$ satisfying (1) the set $\mathcal{M}_f$ is residual in the Fréchet metric space of all real sequences.

Let $s$ be the Fréchet metric space of all sequences of real numbers with the metric

$$\rho(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \frac{|x_k - y_k|}{1 + |x_k - y_k|},$$

where $x = (x_k), y = (y_k)$. It is known that $(s, \rho)$ is a complete metric space.

In [5] it was proved that the set of all uniformly distributed sequences is a dense subset of the first Baire category in $s$. The same is true for the set of all statistically convergent sequences of real numbers (cf. [11]).

2 Main results

The main result of this paper is as follows.

**Theorem 1** Let $f : \mathbb{N} \to (0, \infty)$ be a weight function with properties (1). Then the set of all $f$-maildistributed sequences $\mathcal{M}_f$ is residual in the Fréchet metric space of all sequences of real numbers $s$.

For the proof of the theorem we shall use the following lemma.

**Lemma 1** For the interval $I = [a, b]$ denote by $\mathcal{A}(I, \alpha)$ the set of all $x = (x_k) \in s$ for which

$$\mathcal{A}_f^1 \left( \{ n \in \mathbb{N} : x_n \in I \} \right) \leq \alpha,$$

where $\alpha \in (0, 1)$. Then $\mathcal{A}(I, \alpha)$ is a set of the first Baire category in $s$.

**Proof of Lemma 1.** We define a continuous function $h : \mathbb{R} \to [0, 1]$ by

$$h(x) = \begin{cases} \frac{2x - 2a}{b - a} & \text{for } x \in \left[ a, \frac{a + b}{2} \right] \\ \frac{2b - 2x}{b - a} & \text{for } x \in \left[ \frac{a + b}{2}, b \right] \\ 0 & \text{for } x \in \mathbb{R} \setminus [a, b] \end{cases}$$

We choose an arbitrary real number $\beta \in (\alpha, 1)$. Using the function $h$ we define for $x = (x_k) \in s$ and fixed $n$ the function $g_n : s \to [0, 1]$ in the following way:

$$g_n(x) = \max \left\{ \beta, \frac{n}{\sum_{k=1}^{n} f(k)} \right\}.$$
Denote \( A^*(I, \alpha) \) the set of all \( x = (x_k) \in s \) for which there exists the limit \( \lim_{n \to \infty} g_n(x) \).

One can easily check that for each \( x = (x_k) \in s \) and natural number \( n \) we have

\[
\frac{\sum_{k=1}^{n} h(x_k).f(k)}{\sum_{k=1}^{n} f(k)} \leq \frac{\sum_{k=n, x_k \in I} f(k)}{\sum_{k \leq n} f(k)}.
\]

For any \( x \in A(I, \alpha) \), the right hand side of (3) does not exceed \( \alpha \) if \( n \) is large enough. Therefore \( \lim_{n \to \infty} g_n(x) = \beta \), and then \( A(I, \alpha) \subset A^*(I, \alpha) \).

Put \( g(x) = \lim_{n \to \infty} g_n(x) \) for \( x \in A^*(I, \alpha) \). We shall prove that
(a) the function \( g_n \ (n = 1, 2, \ldots) \) is a continuous function on \( s \),
(b) \( g \) is discontinuous at each point of \( A^*(I, \alpha) \).

(a) Let \( x^0 = (x_k^0)_{k=1}^{\infty}, x^{(j)} = (x_k^{(j)})_{k=1}^{\infty} \in s \ (j = 1, 2, \ldots) \) and \( x^{(j)} \to x^0 \) (for \( j \to \infty \)).

Then from the convergence in the space \( s \) for each fixed \( k \) we have \( \lim_{j \to \infty} x_k^{(j)} = x_k^0 \). The continuity of function \( h \) implies \( \lim_{j \to \infty} g_n(x^{(j)}) = g_n(x^0) \). Thus \( g_n \ (n = 1, 2, \ldots) \) is continuous on \( s \).

(b) Let \( y = (y_k) \in A^*(I, \alpha) \). We have the following two possibilities.
(1) \( g(y) < 1 \),
(2) \( g(y) = 1 \).

In case (1) we choose a positive \( \varepsilon \) such that \( \varepsilon < 1 - g(y) \). It is suffice to prove that in each ball \( K(y, \delta) = \{ x \in A^*(I, \alpha), \rho(x, y) < \delta \} \ (\delta > 0) \) of the subspace \( A^*(I, \alpha) \) of \( s \) there exists an element \( x = (x_k) \in s \) with \( g(x) - g(y) > \varepsilon \).

Let \( \delta > 0 \). Choose a positive integer \( m \) such that \( \sum_{k=m+1}^{\infty} 2^{-k} < \delta \), and define the sequence \( x = (x_k) \) in the following way:

\[
x_k = \begin{cases} y_k, & \text{if } k \leq m, \\ \frac{a+b}{2}, & \text{if } k > m. \end{cases}
\]

Hence \( \rho(x, y) < \delta \), further \( h(x_k) = 1 \) for \( k > m \). Then

\[
\frac{\sum_{k=1}^{n} h(x_k).f(k)}{\sum_{k=1}^{n} f(k)} \geq \frac{\sum_{k=m+1}^{n} f(k)}{\sum_{k=1}^{n} f(k)} = 1 - \frac{\sum_{k=1}^{m} f(k)}{\sum_{k=1}^{n} f(k)} \to 1 \text{ for } n \to \infty,
\]
and therefore \( g(x) = \lim_{n \to \infty} g_n(x) = 1 \). Then immediately follows

\[
g(x) - g(y) = 1 - g(y) > \varepsilon.
\]

In case (2) we have \( g(y) = 1 \). Let \( \delta, m, x \) have the previous meaning. Put

\[
x_k = \begin{cases} y_k, & \text{if } k \leq m, \\ a, & \text{if } k > m. \end{cases}
\]

Then, clearly \( \rho(x, y) < \delta \), and \( h(x_k) = 0 \) for \( k > m \). Then

\[
\frac{\sum_{k=1}^{n} h(x_k) f(k)}{\sum_{k=1}^{n} f(k)} \leq \frac{\sum_{k=1}^{m} f(k)}{\sum_{k=1}^{n} f(k)} \to 0 \text{ for } n \to \infty.
\]

So, we have \( g(x) = \lim_{n \to \infty} g_n(x) = \beta \), and therefore \( g(y) - g(x) = 1 - \beta > 0 \). Hence the discontinuity of \( g \) at \( y \in A^*(I, \alpha) \) has been proved.

The function \( g \) is a limit function (on \( A^*(I, \alpha) \)) of the sequence of continuous functions \( (g_n)_{n=1}^{\infty} \) on \( A^*(I, \alpha) \). Then the function \( g \) is a function in the first Baire class on \( A^*(I, \alpha) \). According to the well-known fact that the set of discontinuity points of an arbitrary function of the first Baire class is a set of the first Baire category (cf. [8], p. 32), we see that the set \( A^*(I, \alpha) \) is of the first Baire category in \( A^*(I, \alpha) \) Thus \( A^*(I, \alpha) \) is in \( s \), too. Since \( A(I, \alpha) \subset A^*(I, \alpha) \), the assertion follows.

**Proof of Theorem 1.** Denote by \( \mathbb{Q} \) the set of all rational numbers. Denote by \( \mathcal{H} \) the set of all \( x = (x_k) \in s \) for which there exists an interval \( I \) with

\[
\mathcal{H} \subset \bigcup_{a, b \in \mathbb{Q}, a < b} \bigcup_{i \in \mathbb{N}, i \geq 2} A \left( [a, b], 1 - \frac{1}{i} \right)
\]

for some \( \alpha \in (0, 1) \). Combining Lemma 1 and the fact that for each interval \( I \) there exist rational numbers \( a, b \) such that \( I \subset [a, b] \), we have

\[
\mathcal{H} \subset \bigcup_{a, b \in \mathbb{Q}, a < b} \bigcup_{i \in \mathbb{N}, i \geq 2} A \left( [a, b], 1 - \frac{1}{i} \right)
\]

from which follows at once that \( \mathcal{H} \) is a meager set. But \( M_f = s \setminus \mathcal{H} \) and therefore the assertion of theorem follows. Hence the property of \( f \)-maldistribution is a typical property of real sequences from the topological point of view. \[\square\]

We now introduce the concept of \( f \)-maldistributed integer sequences.
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Definition 2 Let $f : \mathbb{N} \rightarrow (0, \infty)$ be a weight function with properties (1). The sequence $(x_n)$ of positive integers is said to be $f$-maldistributed, if for any positive integers $m \geq 2$ and $j \in \{0, 1, \ldots, m - 1\}$ the set $\{n \in \mathbb{N} : x_n \equiv j \pmod{m}\}$ has upper $f$-density 1.

Let $S$ be the Baire’s space of all sequences of positive integers with the metric $\rho'$ defined in the following way.

Let $x = (x_k) \in S$, and $y = (y_k) \in S$. If $x = y$, then $\rho'(x, y) = 0$, otherwise

$$\rho'(x, y) = \frac{1}{\min\{n : x_n \neq y_n\}}.$$  

The space $(S, \rho')$ is a complete metric space. In [6] the topological properties of the set of all uniformly distributed sequences of positive integers in Baire’s space were investigated.

The following auxiliary result is similar to Lemma 1.

Lemma 2 For the positive integers $m \geq 2$ and $j \in \{0, 1, \ldots, m - 1\}$ denote by $A(j, m, \alpha)$ the set of all $x = (x_k) \in S$ for which

$$d_f(\{n \in \mathbb{N} : x_n \equiv j \pmod{m}\}) \leq \alpha,$$

where $\alpha \in (0, 1)$. Then $A(j, m, \alpha)$ is a set of the first Baire category in $S$.

The proof is analogous to the proof of Lemma 1. The crucial role is played by the function $g_n : S \rightarrow [0, 1]$ given by

$$g_n(x) = \max \left\{ \sqrt{\alpha}, \sum_{k \leq n} \frac{f(k)}{x_k \equiv j \pmod{m}} \right\}.$$  

The following theorem says that the set of all $f$-maldistributed integer sequences form a residual set in Baire’s space.

Theorem 2 Let $f : \mathbb{N} \rightarrow (0, \infty)$ be a weight function with properties (1). Denote by $G$ the set of all $x = (x_k) \in S$ for which there exist $m \geq 2$ and $j \in \{0, 1, \ldots, m - 1\}$ such that

$$d_f(\{n \in \mathbb{N} : x_n \equiv j \pmod{m}\}) \leq \alpha$$

for some $\alpha \in (0, 1)$. Then $G$ is a set of the first Baire category in $S$. 

Proof. Combining Lemma 2 with the fact that

\[ G \subset \bigcup_{m=2}^{+\infty} \bigcup_{j=0}^{m-1} \bigcup_{i=2}^{+\infty} A(j, m, 1 - \frac{1}{i}) \]

it immediately follows that \( G \) is a meager set in \( S \). \( \square \)

References


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