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About a differential inequality

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Abstract.
A differential inequality concerning holomorfic function is generalised and improved. Several other differential inequalities are considered.

1 Introduction

Let $\mathcal{H}(U)$ be the set of holomorfic functions defined on the unit disc

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Y. Komatsu in [2] proved, the following implication:

If $f \in \mathcal{H}(U)$, $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ and $\mathrm{Re}\sqrt{f'(z)} > \frac{1}{2}$, $z \in U$, then $\frac{f(z)}{z} > \frac{1}{2}$, $z \in U$.

The aim of this paper is to generalize this inequality.

In the paper each multiple-valued function is taken with the principal value.

2 Preliminaries

In our study we need the following definitions and lemmas:

Let X be a locally convex linear topological space. For a subset $U \subset X$ the closed convex hull of U is defined as the intersection of all closed convex sets containing U and will be denoted by $\mathrm{co}(U)$. If $U \subset V \subset X$ then U is called an

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extremal subset of V provided that whenever $u = tx + (1-t)y$ where $u \in U$, $x, y \in V$ and $t \in (0,1)$ then $x, y \in U$.

An extremal subset of U consisting of just one point is called an extreme point of U.

The set of the extreme points of U will be denoted by EU.

Lemma 1 ([1], pp. 45) *If $J : \mathcal{H}(U) \to \mathbb{R}$ is a real-valued, continuous convex functional and $\mathcal{F}$ is a compact subset of $\mathcal{H}(U)$, then*

$$\max\{J(f) : f \in \mathrm{co}(\mathcal{F})\} = \max\{J(f) : f \in \mathcal{F}\} = \max\{J(f) : f \in E(\mathrm{co}(\mathcal{F}))\}.$$

In the particular case if J is a linear map then we also have:

$$\min\{J(f) : f \in \mathrm{co}(\mathcal{F})\} = \min\{J(f) : f \in \mathcal{F}\} = \min\{J(f) : f \in E(\mathrm{co}(\mathcal{F}))\}.$$

Suppose that $f, g \in \mathcal{H}(U)$. The function f is subordinate to g if there exists a function $\theta \in \mathcal{H}(U)$ such that $\theta(0) = 0$, $|\theta(z)| < 1$, $z \in U$ and $f(z) = g(\theta(z))$, $z \in U$.

The subordination will be denoted by $f \prec g$.

Remark 1 *Suppose that $f, g \in \mathcal{H}(U)$ and g is univalent. If $f(0) = g(0)$ and $f(U) \subset g(U)$ then $f \prec g$.*

When $F \in \mathcal{H}(U)$ we use the notation

$$s(F) = \{f \in \mathcal{H}(U) : f \prec F\}.$$

Lemma 2 ([1] pp. 51) *Suppose that F_α is defined by the equality*

$$F_\alpha(z) = \left(\frac{1+cz}{1-z}\right)^\alpha, \ |c| \le 1, \ c \ne -1.$$

If $\alpha \ge 1$ then $\mathrm{co}(s(F_\alpha))$ consists of all functions in $\mathcal{H}(U)$ represented by

$$f(z) = \int_0^{2\pi} \left(\frac{1+cze^{-it}}{1-ze^{-it}}\right)^\alpha d\mu(t)$$

where μ is a positive measure on $[0, 2\pi]$ having the property $\mu([0, 2\pi]) = 1$ and

$$E(\mathrm{co}(s(F_\alpha))) = \left\{\frac{1+cze^{-it}}{1-ze^{-it}} \mid t \in [0, 2\pi]\right\}.$$

Remark 2 *If $L : \mathcal{H}(U) \to \mathcal{H}(U)$ is an invertible linear map and $\mathcal{F} \subset \mathcal{H}(U)$ is a compact subset, then $L(\mathrm{co}(\mathcal{F})) = \mathrm{co}(L(\mathcal{F}))$ and the set $E(\mathrm{co}(\mathcal{F}))$ is in one-to-one correspondence with $EL(\mathrm{co}(\mathcal{F}))$.*

3 The main result

Theorem 1 *Let* $f \in \mathcal{H}(U)$ *be a function normalized by the conditions* $f(0) = f'(0) - 1 = 0$ $m, p \in \mathbb{N}^*$; $a_k \in \mathbb{R}$, $k = \overline{1,p}$ *and*

$$\mathrm{Re}\sqrt[m]{f'(z) + a_1 z f''(z) + \cdots + a_p z^p f^{(p+1)}(z)} > \frac{1}{2},\ z \in U, \tag{1}$$

then

$$1 + \inf_{\theta \in (0,2\pi)} \left(\sum_{n=1}^{\infty} \frac{C_{n+m-1}^{m-1}}{P(n+1)} \cos n\theta \right) < \mathrm{Re}\frac{f(z)}{z} < 1 + \\ + \sup_{\theta \in (0,2\pi)} \left(\sum_{n=1}^{\infty} \frac{C_{n+m-1}^{m-1}}{P(n+1)} \cos n\theta \right)$$

$$1 + \inf_{\theta \in (0,2\pi)} \left(\sum_{n=1}^{\infty} \frac{(n+1)C_{n+m-1}^{m-1}}{P(n+1)} \cos n\theta \right) < \mathrm{Re} f'(z) < 1 + \\ + \sup_{\theta \in (0,2\pi)} \left(\sum_{n=1}^{\infty} \frac{(n+1)C_{n+m-1}^{m-1}}{P(n+1)} \cos n\theta \right)$$

where $P(x) = x + a_1 x(x-1) + a_2 x(x-1)(x-2) + \cdots + a_p x(x-1)\ldots(x-p)$.

Proof. The condition (1) is equivalent to:

$$\sqrt[m]{f'(z) + a_1 z f''(z) + \cdots + a_p z^p f^{(p+1)}(z)} \prec \frac{1}{1-z}$$

and this can be rewritten as follows:

$$f'(z) + a_1 z f''(z) + \ldots a_p z^p f^{(p+1)}(z) \prec \frac{1}{(1-z)^m}.$$

According to the Lemma 2,

$$f'(z) + a_1 z f''(z) + \cdots + a_p z^p f^{(p+1)}(z) = \int_0^{2\pi} \frac{1}{(1 - z e^{-it})^m} d\mu(t) = h(z)$$

where $\mu([0, 2\pi]) = 1$.

Denoting

$$f(z) = z + \sum_{n=2}^{\infty} b_n z^n,\ z \in U$$

we get

$$f'(z) + a_1 z f''(z) + \cdots + a_p z^p f^{(p+1)}(z) = 1 + \sum_{n=2}^{\infty} b_n P(n) z^{n-1},$$

on the other hand

$$\int_0^{2\pi} \frac{1}{(1 - ze^{-it})^m} d\mu(t) = 1 + \sum_{n=2}^{\infty} C_{n+m-2}^{m-1} z^{n-1} \cdot \int_0^{2\pi} e^{-i(n-1)t} d\mu(t).$$

The above two developments in power series imply that:

$$1 + \sum_{n=2}^{\infty} b_n P(n) z^{n-1} = 1 + \sum_{n=2}^{\infty} C_{n+m-2}^{m-1} z^{n-1} \int_0^{2\pi} e^{-i(n-1)t} d\mu(t),$$

and

$$b_n = \frac{C_{n+m-2}^{m-1}}{P(n)} \int_0^{2\pi} e^{-i(n-1)t} d\mu(t), \; n \in \mathbb{N}, n \geq 2.$$

Thus

$$f(z) = z + \sum_{n=2}^{\infty} \frac{C_{n+m-2}^{m-1}}{P(n)} z^n \int_0^{2\pi} e^{-i(n-1)t} d\mu(t) \tag{2}$$

and we deduce:

$$\frac{f(z)}{z} = 1 + \sum_{n=1}^{\infty} \frac{C_{n+m-1}^{m-1}}{P(n+1)} z^n \int_0^{2\pi} e^{-int} d\mu(t)$$

$$f'(z) = 1 + \sum_{n=1}^{\infty} \frac{(n+1) C_{n+m-1}^{m-1}}{P(n+1)} z^n \int_0^{2\pi} e^{-int} d\mu(t).$$

If

$$\mathcal{B} = \left\{ h \in \mathcal{H}(U) \mid h(z) = \int_0^{2\pi} \frac{1}{(1 - ze^{-it})^m} d\mu(t), \; z \in U, \; \mu([0, 2\pi]) = 1 \right\},$$

$$\mathcal{C} = \left\{ f \in \mathcal{H}(U) \mid \mathrm{Re} \left(\sqrt[m]{f(z) + a_1 z f'(z) + \cdots + a_p z^p f^{(p)}(z)} \right) > 0, \; z \in U \right\}$$

then the correspondence $L : \mathcal{B} \to \mathcal{C}$, $L(h) = f$ defines an invertible linear map and according to the Observation 2 the extreme points of the class $\mathcal{C}$ are

$$f_t(z) = z + \sum_{n=1}^{\infty} \frac{C_{n+m-1}^{m-1}}{P(n+1)} z^{n+1} e^{-int}.$$

This result, Lemma 1 and the minimum and maximum principle for harmonic functions imply the assertion of Theorem 1. ■

4 Particular cases

Let $\mathcal{A}$ be the class of analytic functions defined by the equality:

$$\mathcal{A} = \{f \in \mathcal{H} : f(0) = f'(0) - 1 = 0\}.$$

If we put $p = 2, a_1 = a_2 = m = 1$ in Theorem 1 then we get:

Corollary 1 (Komatu) [2]) *If* $f \in \mathcal{A}$ *and* $Re\sqrt{f'(z)} > \frac{1}{2}$, $z \in U$, *then* $Re\frac{f(z)}{z} > \frac{1}{2}$, $z \in U$, *and the result is sharp.*

Proof.
We apply Theorem 1 in the particular case $a_1 = 1$, $a_2 = a_3 = \ldots = a_p = 0$ i $m = 2$. We get $P(n+1) = (n+1)^2$ and

$$\mathrm{Re}\frac{f(z)}{z} > 1 + \inf_{z\in U} \mathrm{Re}\sum_{n=1}^{\infty} \frac{C_{n+1}^1}{(n+1)^2} z^n = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} = \ln 2,\ z \in U.$$

The other case can be proved as follows:

$$\mathrm{Re} f'(z) > 1 + \inf_{z\in U} \mathrm{Re}\sum_{n=1}^{\infty} \frac{(n+1)C_{n+1}^1}{(n+1)^2} z^n = 1 + \inf_{z\in U} \mathrm{Re}\frac{z}{1-z} = \frac{1}{2},\ z \in U.$$

■

Corollary 2 *If* $f \in \mathcal{A}$ *and* $Re\sqrt{f'(z) + zf''(z)} > \frac{1}{2,}$ $z \in U$ *then*
1) $Re\frac{f(z)}{z} > \ln 2,\ z \in U$
2) $Re f'(z) > \frac{1}{2}$, $z \in U$ *and the results are sharp.*

Proof. We apply again Theorem 1 in case of $a_1 = 1$, $a_2 = a_3 = \ldots = a_p = 0$ and $m = 2$. It is easily seen that $P(n+1) = (n+1)^2$ and

$$\mathrm{Re}\frac{f(z)}{z} > 1 + \inf_{z\in U} \mathrm{Re}\sum_{n=1}^{\infty} \frac{C_{n+1}^1}{(n+1)^2} z^n = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n+1} = \ln 2,\ z \in U.$$

In the other case :

$$\mathrm{Re} f'(z) > 1 + \inf_{z\in U} \mathrm{Re}\sum_{n=1}^{\infty} \frac{(n+1)C_{n+1}^1}{(n+1)^2} z^n = 1 + \inf_{z\in U} \mathrm{Re}\frac{z}{1-z} = \frac{1}{2},\ z \in U.$$

Corollary 3 *Let* $p \in \mathbb{N}$, $p \geq 3$. *If* $f \in \mathcal{A}$ *and* $S(p,k)$, $p \geq k$ *are the numbers of Stirling of the second kind defined by*

$$S(p,k) = \frac{1}{k!}\sum_{l=1}^{k-1}(-1)^l C_k^l (k-l)^p,\ k = \overline{1,p},$$

then the inequality

$$Re\left(\sqrt{\sum_{k=1}^{p} S(p,k) z^{k-1} f^{(k)}(z)}\right) > \frac{1}{2},\ z \in U \tag{3}$$

implies that

$$Re\frac{f(z)}{z} > \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{p-1}},\ z \in U,$$

and the result is sharp.

Proof. According to Theorem 1follows that:

$$\mathrm{Re}\frac{f(z)}{z} > 1 + \inf_{z\in U}\mathrm{Re}\left(\sum_{n=1}^{\infty}\frac{C_{n+1}^1}{P(n+1)}z^n\right) \tag{4}$$

and we have:

$$P(x) = \sum_{k=1}^{p} S(p,k) x(x-1)\ldots(x-k+1) = x^p. \tag{5}$$

We have to determine:

$$\inf_{z\in U}\mathrm{Re}\left(\sum_{n=1}^{\infty}\frac{C_{n+1}^1}{P(n+1)}z^n\right) = \inf_{\theta\in(0,2\pi)}\mathrm{Re}\left(\sum_{n=1}^{\infty}\frac{e^{in\theta}}{(n+1)^{p-1}}\right).$$

We will use the following integral representation:

$$\sum_{n=1}^{\infty}\frac{e^{in\theta}}{(n+1)^{p-1}} = \underbrace{\int_0^1\int_0^1\ldots\int_0^1}_{p-1}(t_1t_2\ldots t_{p-1}e^{i\theta})^n dt_1dt_2\ldots dt_{p-1} =$$

$$= \underbrace{\int_0^1\int_0^1\ldots\int_0^1}_{p-1} t_1t_2\ldots t_{p-1}\frac{e^{i\theta} - t_1t_2\ldots t_{p-1}}{1 + t_1^2t_2^2\ldots t_{p-1}^2 - 2t_1t_2\ldots t_{p-1}\cos\theta}dt_1dt_2\ldots dt_{p-1}$$

If we denote $t_1 t_2 \dots t_{p-1} = u$, then $u \in [0,1]$ and

$$\frac{\cos\theta - u}{1 + u^2 - 2u\cos\theta} \geq \frac{-1}{1+u}, \theta \in [0, 2\pi]. \tag{6}$$

We get from (6) the inequality:

$$\int_0^1 \dots \int_0^1 t_1 \dots t_{p-1} \frac{\cos\theta - t_1 \dots t_{p-1}}{1 + t_1^2 \dots t_{p-1}^2 - 2t_1 \dots t_{p-1}\cos\theta} dt_1 \dots dt_{p-1} \geq$$
$$\geq -\int_0^1 \dots \int_0^1 \frac{t_1 \dots t_{p-1}}{1 + t_1 \dots t_{p-1}} dt_1 \dots dt_{p-1}$$

where in case of $\theta = \pi$ the equality holds. This implies that:

$$\inf_{\theta \in (0,2\pi)} \mathrm{Re} \sum_{n=1}^{\infty} \frac{e^{in\theta}}{(n+1)^{p-1}} =$$
$$= \inf_{\theta \in (0,2\pi)} \int_0^1 \dots \int_0^1 t_1 \dots t_{p-1} \frac{\cos\theta - t_1 \dots t_{p-1}}{1 + t_1^2 \dots t_{p-1}^2 - 2t_1 \dots t_{p-1}\cos\theta} dt_1 \dots dt_{p-1} =$$
$$= -\int_0^1 \int_0^1 \dots \int_0^1 \frac{t_1 \dots t_{p-1}}{1 + t_1 \dots t_{p-1}} dt_1 \dots dt_{p-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(n+1)^{p-1}},$$

and the proof is done. ∎

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