

A NEW CO-FILTER IN IMPLICATIVE SEMIGROUPS WITH APARTNESS

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ABSTRACT. The concept of implicative semigroups with apartness in Bishop's constructive algebra was introduced and analyzed by the author in several of his papers. In some of them, the concepts of co-ideals and co-filters in such semigroups were introduced and analyzed. This paper continues investigation of implicative semigroups with tight apartness and of their co-filters. In particular, the notion of an implicative co-filters was introduced and analyzed.

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1. INTRODUCTION

The notions of implicative semigroup and ordered filter were introduced on [5] by M. W. Chan and K. P. Shum. The idea of implicative ordered filters in such semigroups was developed by Y. B. Jun [9].

In the setting of Bishops constructive mathematics, the notion of implicative semigroups with tight apartness was introduced in [11], following the ideas of above mentioned authors, and some fundamental characterizations of these semigroups were given. Then the author continued his research on implicative semigroups with tight apartness using co-order relations instead of partial order. In particular, strongly extensional homomorphisms between implicative semigroups with tight apartness are discussed in [12]; co-filters and co-ideals in such semigroups were considered in [14, 15, 16]. An interested reader can find in paper [13] more information on the relations of co-quasiorder and co-order and their applications in algebraic structures having sets with apartness as carriers.

This paper continues investigation of implicative semigroups with tight apartness and of their co-filters. In particular, the notion of an implicative co-filters was introduced and analyzed. In addition, in proving the stated claims, some specifics of the applied techniques in constructive algebra are demonstrated.

2. PRELIMINARIES

In this section, we recall from [11, 12, 14, 15, 16] some concepts and processes necessary in the sequel of this paper and the reader is referred to [13] for undefined notions and notations. This investigation is in Bishop's constructive algebra in the sense of paper [6, 7, 8, 13] and books [1, 2, 3, 4, 10] and Chapter 8: Algebra of [17].

2.1. Set with apartness

Let $(S, =, \neq)$ be a constructive set (i.e. it is a relational system with the relation " \neq "). A diversity relation " \neq " satisfying conditions

$$\neg(x \neq x), x \neq y \implies y \neq x, x \neq y \wedge y = z \implies x \neq z$$

for any $x, y, z \in S$ is called apartness. In this paper, we assume that the apartness is tight, i.e. it satisfies the following

$$(\forall x, y \in S)(\neg(x \neq y) \implies x = y).$$

A subset X of S is called a strongly extensional subset of S if and only if

$$(\forall x \in X)(\forall y \in S)(x \neq y \vee y \in S).$$

Let X, Y be subsets of S . According with Bridge and Vita definition (see for instance [4]), we say that X is set-set apartned from Y (denoted $X \bowtie Y$) if and only if $(\forall x \in X)(\forall y \in Y)(x \neq y)$.

We set $x \triangleleft Y$ and $x \neq y$, instead of $\{x\} \bowtie Y$ and $\{x\} \bowtie \{y\}$ respectively. With $X^\triangleleft = \{x \in S : x \triangleleft X\}$ we denote the apartness complement of X .

We say that a relation $\alpha \subseteq S \times S$ is a co-order relation on the semigroup S , if it fulfills the following properties

$$\begin{aligned} &(\forall x, y \in S)((x, y) \in \alpha \implies x \neq y) \quad (\text{consistency}) \\ &(\forall x, y, z \in S)((x, z) \in \alpha \implies ((x, y) \in \alpha \vee (y, z) \in \alpha)) \quad (\text{co-transitivity}) \\ &(\forall x, y \in S)(x \neq y \implies ((x, y) \in \alpha \vee (y, z) \in \alpha)) \quad (\text{linearity}) \text{ and} \end{aligned}$$

compatibility with the product of S in the following sense

$$(\forall x, y, z \in S)((xz, yz) \in \alpha \vee (zx, zy) \in \alpha) \implies (x, y) \in \alpha.$$

2.2. Implicative semigroups with apartness

Let $((S, =, \neq), \cdot)$ be a semigroup with apartness. We recall that its binary operation ' \cdot ' has to be extensional and strongly extensional, i.e. ' \cdot ' is a function from $S \times S$ into S such that

$$\begin{aligned} &(\forall a, b, u, v \in S)((a, b) = (u, v) \implies ab = uv), \\ &(\forall a, b, u, v \in S)(ab \neq uv \implies (a, b) \neq (u, v)) \end{aligned}$$

where we wrote ab instead of $a \cdot b$.

By a *negatively co-ordered* semigroup (briefly, n.a-o. semigroup) we mean a semigroup with apartness S with a co-order α such that for all $x, y, z \in S$ the following hold:

- (1) $(xy)z = x(yz)$,
- (2) $(xz, yz) \in \alpha$ or $(zx, zy) \in \alpha$ implies $(x, y) \in \alpha$, and
- (3) $(xy, x) \triangleleft \alpha$ and $(xy, y) \triangleleft \alpha$.

In such case *alpha* will be called a negative co-order relation on S .

A n.a-o. semigroup $(S, =, \neq, \cdot, \alpha)$ is said to be *implicative* if there is an additional binary operation $\otimes : S \times S \rightarrow S$ such that the following is true

- (4) $(z, x \otimes y) \in \alpha \iff (zx, y) \in \alpha$

for any elements x, y, z of S .

In addition, let us recall that the internal binary operation ' \otimes ' must satisfy the following implications:

$$\begin{aligned} &(\forall a, b, u, v \in S)((a, b) = (u, v) \implies a \otimes b = u \otimes v), \\ &(\forall a, b, u, v \in S)(a \otimes b \neq u \otimes v \implies (a, b) \neq (u, v)). \end{aligned}$$

The operation \otimes is called *implication*. From now on, an implicative n.a-o. semigroup is simply called an *implicative semigroup*.

In any implicative semigroup S there exists a special element of S , the biggest element in $(S, \alpha^{\triangleleft})$, which is the left neutral element in (S, \cdot) .

Example 1. Let $S = \{1, 2, 3, 4, 5, 0\}$ and operations ' \cdot ' and ' \otimes ' defined on S as follows:

\cdot	1	2	3	4	5	0		\otimes	1	2	3	4	5	0
1	1	2	3	4	5	0		1	1	2	3	4	5	0
2	2	3	3	5	0	0	and	2	1	1	2	4	4	5
3	3	3	3	0	0	0		3	1	1	1	4	4	4
4	4	5	0	4	5	0		4	1	2	3	1	2	3
5	5	0	0	5	0	0		5	1	1	2	1	1	2
0	0	0	0	0	0	0		0	1	1	1	1	1	1

Then $S = ((S, =, \neq), \cdot, \not\leq, \otimes)$ is an implicative semigroup where the co-order relation ' $\not\leq$ ' is defined as follows $\not\leq = \{(1, 2), (1, 3), (1, 4), (1, 5), (1, 0), (2, 3), (2, 4), (2, 5), (2, 0), (3, 4), (3, 5), (3, 0), (4, 2), (4, 3), (4, 5), (4, 0), (5, 3), (5, 0)\}$.

2.3. Co-filters

In this subsection the author reminds readers on the concept of co-filters of an implicative semigroup with apartness ([11, 14]): An inhabited subset G of S is called *ordered co-filter* if the following holds:

- (G1) $(\forall x, y \in S)(xy \in G \implies x \in G \vee y \in G)$ and
- (G2) $(\forall x, y \in S)(y \in G \implies (x, y) \in \alpha \vee x \in G)$.

It is easy to check that a co-filter is a strongly extensional subset of S . Moreover, strong complement G^\triangleleft of an co-filter G is a filter in S ([11], pp. 162-163).

The following theorem gives equivalent conditions of ordered co-filters.

Theorem 1 ([11], Theorem 3.7). *An inhabited proper subset G of an implicative semigroup S is an ordered co-filter of S if and only if it satisfies the following conditions:*

- (G3) $1 \triangleleft G$;
- (G4) $(\forall x \in S)(\forall y \in S)(y \in G \implies x \otimes y \in G \vee x \in G)$.

Example 2. *Let S be the implicative semigroup as in Example 1. Then the subsets $\{4, 5, 0\}$, $\{2, 3, 5, 0\}$ and $\{2, 3, 4, 5, 0\}$ are ordered co-filters of S .*

3. CONCEPT OF IMPLICATIVE CO-FILTERS

Definition 1. *Let $((S, =, \neq), \cdot, \alpha, \otimes)$ be an implicative semigroup. An inhabited subset G of S is called an *implicative co-filter* of S if it satisfies (G3) and*

- (GI) $(\forall x, y, z \in S)(x \otimes z \in G \implies (x \otimes (y \otimes z) \in G \vee x \otimes y \in G))$.

Example 3. *Let S be the implicative semigroup as in Example 1. Then the subset $\{4, 5, 0\}$ is an implicative co-filter of S but the ordered co-filter $\{2, 3, 4, 5, 0\}$ is not an implicative co-filter of S .*

Proposition 1. *Every implicative co-filter is a co-filter.*

Proof. Let G be an implicative co-filter of an implicative semigroup S . Let $y, z \in S$ be such that $y \in G$. Then $1 \otimes y \in G$ by Corollary 3.3. in [11]. Thus $1 \otimes y \in G \implies (1 \otimes (y \otimes z) \vee 1 \otimes y \in G)$ by (GI). Hence $y \in G \implies (y \otimes z \in G \vee z \in G)$ in accordance with Corollary 3.3. in [11]. We have proved that G satisfies the conditions of Theorem 1. So, G is a co-filter in S .

In what follows, the following proposition will be useful to us.

Proposition 2. *Let S be an implicative semigroup with apartness. If G is an implicative co-filter of S , then the following holds:*

- (G5) $(\forall x, y \in S)(x \otimes y \in G \implies x \otimes (x \otimes y) \in G)$

Proof. Let G be an implicative co-filter of S . If we put $z = y$ and $y = x$ in (GI), in accordance of Theorem 3.3 in [11] we have get

$$x \otimes y \in G \implies (x \otimes (x \otimes y)) \in G \vee 1 = x \otimes x \in G).$$

As the second option is impossible due to (G3), we get (G5) from here.

Let G be an ordered co-filter of an implicative semigroup with apartness S and let $a \in S$. Define $G_a := \{x \in S : a \otimes x \in G\}$. Note that $G_1 = G$ according Corollary 3.3 in [11].

By using the set G_a ($a \in G$), we can design a condition for an ordered co-filter to be an implicative co-filter.

Theorem 2. *Let G be an ordered co-filter of an implicative semigroup with apartness S . Then G is an implicative co-filter of S if and only if for any $a \in S$, the set G_a is an ordered co-filter of S .*

Proof. Suppose that G is an implicative co-filter of S . Clearly, $1 \triangleleft G_a$. Indeed. For $u \in G_a$, by (G1) we have $a \otimes u \neq 1 = a \otimes 1$ by Corollary 3.2 in [11]. Thus $u \neq 1$. Let $x, y \in S$ be such that $y \in G_a$. Then $a \otimes y \in G$. Since G is an implicative co-filter of S , it follows that $a \otimes (x \otimes y) \in G$ or $a \otimes x \in G$ by (GI). This means $x \otimes y \in G_a \vee x \in G_a$ by definition of G_a . So the set G_a satisfies the conditions (G3) and (G4). Hence, G_a is an ordered co-filter of S .

Conversely, suppose G_a is an ordered co-filter for any $a \in S$. Let $x, y, z \in S$ be arbitrary elements such that $x \otimes z \in G$. This means $z \in G_x$. Then $y \otimes z \in G_x \vee y \in G_x$ by (G4) because G_x is an ordered co-filter of S . This means $x \otimes (y \otimes z) \in G \vee x \otimes y \in G$. Hence, G is an implicative co-filter of S since it satisfies conditions (G3) and (GI).

In what follows, we will deal with commutative implicative semigroups with apartness: An implicative semigroup with apartness $((S, =, \neq), \cdot, \alpha, \otimes)$ is commutative if it satisfies $(\forall x, y \in S)(x \cdot y = y \cdot x)$.

Lemma 3. *If S is a commutative implicative semigroup with apartness, then the following holds*

$$(5) (\forall x, y, z \in S)(x \otimes (y \otimes z) = y \otimes (x \otimes z)).$$

Proof. Let $x, y, z \in S$ be arbitrary elements. Than

$$x \otimes (y \otimes z) = (x \cdot y) \otimes z = (y \cdot x) \otimes z = y \otimes (x \otimes z)$$

in accordance with statement (2) of Theorem 3.6 in [11].

Lemma 4. *If S is a commutative implicative semigroup with apartness, then the following holds*

$$(6) (\forall x, y, z \in S)((y \otimes z, (x \otimes y) \otimes (x \otimes z)) \triangleleft \alpha).$$

Proof. For elements $x, y, z \in S$ we have $(z, (x \otimes y) \otimes z) \triangleleft \alpha$ by statement (iii) of Theorem 3.5 in [11]. Then $(x \otimes z, x \otimes ((x \otimes y) \otimes z)) \triangleleft \alpha$ by statement (1) of Theorem 3.6 in [11].

On the other hand, we have $x \otimes ((x \otimes y) \otimes z) = (x \cdot (x \otimes y)) \otimes z$ by Statement (2) of Theorem 3.6 in [11]. Due to the commutativity of the multiplication operation in S , we have $(x \cdot (x \otimes y)) \otimes z = ((x \otimes y) \cdot x) \otimes z$. Hence it follows $((x \otimes y) \cdot x) \otimes z = (x \otimes y) \otimes (x \otimes z)$ according to statement (2) of Theorem 3.6 in [11].

This proves that (6) is a valid formula.

Theorem 5. *Let S be a commutative implicative semigroup with apartness. If an inhabited subset G of S satisfies the condition (G2), (G3) and the condition*

$$(G6) (\forall x, y, z \in S)(y \otimes z \in G \implies (x \otimes (y \otimes (y \otimes z)) \in G \vee x \in G)),$$

then G is an implicative co-filter of S .

Proof. Let $x, y, z \in S$ be arbitrary element such that $\otimes z \in G$. If we put $x = x \otimes y$ and $y = x$ in (G6), we get

$$(x \otimes y) \otimes (x \otimes (x \otimes z)) \in G \vee x \otimes y \in G.$$

The first option it follows

$$(x \otimes (y \otimes z), (x \otimes y) \otimes (x \otimes (x \otimes z))) \in \alpha \vee x \otimes (y \otimes z) \in G$$

according to (G2). If the first option in the previous formula was valid, then according to (5) we would have

$$(x \otimes (y \otimes z), x \otimes ((x \otimes y) \otimes (x \otimes z))) \in \alpha.$$

From here, due to the strictly extensionality of the operation \otimes in S , we would get

$$(y \otimes z, (x \otimes y) \otimes (x \otimes z)) \in \alpha$$

which is contrary to (6). The obtained contradiction confirms the following possibilities

$$x \otimes (y \otimes z) \in G \vee x \otimes y \in G.$$

This proves that G is an implicative co-filter in S .

Theorem 6. *Let S be a commutative implicative semigroup with apartness. If an ordered co-filter G of S satisfies the condition*

$$(G7) \ (\forall x, y, z \in S)((x \otimes y) \otimes (x \otimes z) \in G \implies x \otimes (y \otimes z) \in G),$$

then G is an implicative co-filter of S .

Proof. We will prove that G satisfies the condition (G6).

Let $x, y, z \in S$ be arbitrary elements such that $y \otimes z \in G$. Then $x \otimes (y \otimes z) \in G$ or $x \in G$ by (G4). On the other hand, according to (5) and Theorem 3.3. Corollary 3.3 in [11], we have

$$G \ni x \otimes (y \otimes z) = y \otimes (x \otimes z) = 1 \otimes (y \otimes (x \otimes z)) = (y \otimes y) \otimes (y \otimes (x \otimes z)).$$

From here, according to hypothesis (G7), we get $y \otimes (y \otimes (x \otimes z)) \in G$. This, according to (5), can be transformed into $x \otimes (y \otimes (y \otimes z)) \in G$.

Since the ordered co-filter G satisfies condition (G6), we conclude that G is an implicative co-filter of S according to Theorem 5.

Let us prove that for an ordered co-filter G of an implicative semigroup with apartness S condition (G5) is sufficient for it to be an implicit co-filter of S .

Theorem 7. *Let S be a commutative implicative semigroup with apartness. If an ordered co-filter G of S satisfies the condition (G5), then it is an implicative co-filter of S .*

Proof. Suppose that an ordered co-filter G of S satisfies the condition (G5). To prove that G is an implicative co-filter of S , it suffices to prove that it satisfies the condition (G7).

Let $x, y, z \in S$ be arbitrary elements such that $(x \otimes y) \otimes (x \otimes z) \in G$. Then $x \otimes ((x \otimes y) \otimes z) \in G$ by (5). Thus $x \otimes (x \otimes (x \otimes y) \otimes z) \in G$ by (G5). This, according to (5), can be transformed into $x \otimes ((x \otimes y) \otimes (x \otimes z)) \in G$. Hence it follows

$$(x \otimes (y \otimes z), x \otimes ((x \otimes y) \otimes (x \otimes z))) \in \alpha \vee x \otimes (y \otimes z) \in G$$

in accordance with (G2) since G is an ordered co-filter of S . The first option gives us

$$(y \otimes z, (x \otimes y) \otimes (x \otimes z)) \in \alpha$$

due to the strongly extensionality of the operation x , which is in contrast to (6). The obtained contradiction confirms the possibility of $x \otimes (y \otimes z) \in G$.

4. FINAL COMMENTS

The specificity of Bishop's constructive algebra is, in addition to the intuitionistic logical environment, also the principled-philosophical orientations of Bishop's constructive mathematics. The invalidity of the principle of exclusion of the third it allows the existence of a separation relation ' \neq ' independent of the equality relation '='. A consistent, co-transitive, and symmetric separation relation is called an apartness relation. It is clear that this relation must be extensive with respect to the equality relation. So, a set with apartness $(S, =, \neq)$ is one relational system. In this logical environment, every predicate P and every function f should be strongly extensional with respect to the apartness. Therefore, if $((S, =, \neq), \cdot)$ is an algebraic structure, then the multiplication in S is a strictly extensional function. These orientation allow us design a 'co-substructure' in S as a dual of the observed sub-structure.

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