POMPEIU’S THEOREM IN AN INNER PRODUCT SPACES

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ABSTRACT. The aim of this paper is to present a version for an inner product space of the classical theorem of Pompeiu.

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1. Introduction

Pompeiu’s theorem is considered one of the most "elegant triangle theorem"[3]. It is said that the segments $MA$, $MB$ and $MC$ could be the sides length of a triangle, for any point $M$ not situated on the circumcircle of the equilateral triangle $ABC$. Today, we find more references about this theorem. (e.g. [2], [5])

Recently [1], it was presented and proved a $n$-dimensional generalization of Pompeiu’s theorem.

**Proposition 1.1.** ([1], Theorem 4) Let $n \geq 2$, and let $S = [A_1, A_2, ..., A_{n+1}]$ be a regular $n$-simplex of edge length $a$. Let $B$ be a point in the affine hull of $S$, and let $b_1, b_2, ..., b_{n+1}$ be the distances from $B$ to the vertices of $S$.

If $n = 2$, i.e., if $S$ is a triangle, then $b_1, b_2, b_3$ can serve as the side lengths of a triangle if and only if $B$ does not lie on the circumcircle of $S$.

If $n \geq 3$, then $b_1, b_2, ..., b_{n+1}$ can serve as the facet contents of an $n$-simplex if and only if $B$ is not a vertex of $S$.

Motivated by this result and the relation from [6], we intend to present a new viewpoint related to Pompeiu’s Theorem. The main result are included in third section and contains a generalization for an inner product space of this theorem. For the proof we need some useful results. These are some relations that hold for any inner product space, some of these being generalizations of similar relations from [1].

To the end of this section, we recall a tool that helps us to complete our proof.
Proposition 1.2. ([4], Theorem 3.1) Let $X$ a real inner product space. Then
\[
\left\| x - \sum_{k=1}^{n} a_k x_k \right\|^2 = \sum_{k=1}^{n} a_k \| x - x_k \|^2 - \sum_{1 \leq k < l \leq n} a_k a_l \| x_l - x_k \|^2,
\]
for all integer $n$, $n \geq 2$, for all $a_1, a_2, \ldots, a_n \in \mathbb{R}$ with $\sum_{k=1}^{n} a_k = 1$, and for any $x, x_1, x_2, \ldots, x_n \in X$.

2. SOME USEFUL RESULTS

In this section $X$ represents a real inner product spaces. Let $n \in \mathbb{N}$, $n \geq 2$ and $x_1, x_2, \ldots, x_n \in X$, distinguies, and $t_0 \in (0, \infty)$ such that $\| x_i - x_j \| = t_0$, for any $i, j \in \{1, 2, \ldots, n\}$, $i \neq j$. Let affin$[x_1, x_2, \ldots, x_n]$ the set $\left\{ \sum_{k=1}^{n} a_k x_k \mid a_1, a_2, \ldots, a_n \in \mathbb{R}, \sum_{k=1}^{n} a_k = 1 \right\}$.

First result is a generalization of Theorem 2 from [1].

**Proposition 2.1.** For any $x \in$ affin$[x_1, x_2, \ldots, x_n]$ we denote $t_k = \| x - x_k \|$, $k \in \{1, 2, \ldots, n\}$. Then
\[
\left( \sum_{k=0}^{n} t_k^2 \right)^2 = n \left( \sum_{k=0}^{n} t_k^4 \right).
\]

**Proof.** If $x \in$ affin$[x_1, x_2, \ldots, x_n]$ there exists $a_1, a_2, \ldots, a_n \in \mathbb{R}$ with $\sum_{k=1}^{n} a_k = 1$ such that $x = \sum_{i=1}^{n} a_i x_i$. Then, for any $k \in \{1, 2, \ldots, n\}$, we have
\[
t_k^2 = \| x_k - x \|^2 = \left\| x_k - \sum_{i=1}^{n} a_i x_i \right\|^2
\]
\[
= \sum_{i=1}^{n} a_i \| x_k - x_i \|^2 - \sum_{1 \leq i < l \leq n} a_i a_l \| x_l - x_i \|^2
\]
\[
= (1 - a_k) t_0^2 - \frac{(\sum_{i=1}^{n} a_i)^2 - \sum_{i=1}^{n} a_i^2 t_0^2}{2}
\]
\[
= \left( 1 - a_k - \frac{1 - \sum_{i=1}^{n} a_i^2}{2} \right) t_0^2 = \frac{1 + \sum_{i=1}^{n} a_i^2 - 2a_k t_0^2}{2}.
\]

With the notation $A = \frac{1 + \sum_{i=1}^{n} a_i^2}{2}$, we obtain $t_k^2 = (A - a_k) t_0^2$. Then
\[
\left( \sum_{k=0}^{n} t_k^2 \right)^2 = t_0^4 \left( 1 + \sum_{k=1}^{n} (A - a_k) \right)^2 = t_0^4 \left( 1 + nA - \sum_{k=1}^{n} a_k \right)^2 = n^2 A^2 t_0^4
\]

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and
\[ n \left( \sum_{k=0}^{n} t_k^4 \right) = n \left( t_0^4 + \sum_{k=1}^{n} (A - a_k)^2 t_k^0 \right) = nt_0^4 \left( 1 + nA^2 - 2A \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} a_k^2 \right) = n(1 + nA^2 - 2A + 2A - 1) = n^2 A^2 t_0^k, \]
that concludes the proof.

Proposition 2.2. For any \( x \in \text{affin}[x_1, x_2, ..., x_n] \) we denote \( t_k = \|x - x_k\|, k \in \{1, 2, ..., n\} \). Then
\[ \left( \sum_{k=1}^{n} t_k^2 \right)^2 - (n-1) \left( \sum_{k=1}^{n} t_k^4 \right) = \frac{n-1}{n} \left( t_0^2 - \frac{1}{n-1} \sum_{k=1}^{n} t_k^2 \right)^2. \] (3)

Proof. Let \( U = \sum_{k=1}^{n} t_k^2 \) and \( V = \sum_{k=1}^{n} t_k^4 \). From previous proposition we have \( (t_0^2 + U)^2 = n (t_0^4 + V) \). Then \( t_0^4 + 2Ut_0^2 + U^2 = nt_0^4 - nV, \) also \( U^2 - nV = (n-1) t_0^4 - 2Ut_0^2 \).

Further
\[ U^2 + \frac{1}{n-1} U^2 - nV = (n-1) t_0^4 - 2Ut_0^2 + \frac{1}{n-1} U^2 - nV, \]
so
\[ \frac{n}{n-1} U^2 - nV = (n-1) \left( t_0^2 - \frac{1}{n-1} U \right)^2. \]
Now, we obtain the conclusion if we multiply with \( \frac{n-1}{n} \).

Further, let \( g = \frac{1}{n} \sum_{k=1}^{n} x_k \) and \( R = t_0 \sqrt{\frac{n-1}{2n}} \). The next two propositions present some properties of \( g \).

Proposition 2.3. For any \( k \in \{1, 2, ..., n\} \) we have \( \|g - x_k\| = R \).

Proof. We apply Proposition 1.2 for \( a_1 = a_2 = ... = a_n = \frac{1}{n} \). We obtain
\[ \|g - x_k\|^2 = \left\| x_k - \frac{1}{n} \sum_{i=1}^{n} x_i \right\|^2 = \frac{1}{n} \sum_{i=1}^{n} \|x_k - x_i\|^2 - \frac{1}{n^2} \sum_{1 \leq i < l \leq n} a_i a_l \|x_l - x_i\|^2 = \frac{n-1}{n} t_0^2 - \frac{n(n-1)}{2n^2} t_0^2 = \frac{n-1}{2n} t_0^2. \]
We obtain \( \|g - x_k\|^2 = R^2 \) and the conclusion follows now.
Proposition 2.4. For any \( x \in X \) we denote \( t_k = \|x - x_k\|, k \in \{1, 2, ..., n\} \). Then \( \|x - g\| = R \) if and only if
\[
(n - 1) t_0^2 = t_1^2 + t_2^2 + ... + t_n^2.
\]

Proof. The equality \( \|x - g\| = R \) is equivalent with \( \|x - g\|^2 = \frac{n-1}{2n} t_0^2 \). By other hand, (1) lead us to
\[
\|x - g\|^2 = \left\| x - \frac{1}{n} \sum_{k=1}^{n} x_k \right\|^2 = \frac{1}{n} \sum_{k=1}^{n} \|x - x_k\|^2 - \frac{1}{n^2} \sum_{1 \leq i < l \leq n} a_i a_l \|x_l - x_i\|^2
\]
\[
= \frac{1}{n} \sum_{k=1}^{n} t_k^2 - \frac{n-1}{2n} t_0^2.
\]
We obtain \( \frac{n-1}{2n} t_0^2 = \frac{1}{n} \sum_{k=1}^{n} t_k^2 - \frac{n-1}{2n} t_0^2 \) and the conclusion follows now.

3. The Pompeiu’s theorem

This section is reserved to the generalization of Pompeiu’s theorem. Let \( x_1, x_2, x_3 \in X \) distingues such that \( \|x_1 - x_2\| = \|x_2 - x_3\| = \|x_3 - x_1\| = t_0 \). Let \( g = \frac{x_1 + x_2 + x_3}{3} \). From Proposition 2.3 we have \( \|x_k - g\| = \frac{1}{\sqrt{3}} t_0 = R \). We denote \( C(g, R) \) the set \( \{ y \in X \mid \|y - g\| = R \} \). This set represent the analogues of the circumcircle from triangle geometry.

Theorem 3.1. For any \( x \in \text{affin}[x_1, x_2, x_3] \), denote \( t_1 = \|x - x_1\|, t_2 = \|x - x_2\| \) and \( t_3 = \|x - x_3\| \). Then \( t_1, t_2, t_3 \) could be the side lengths of a triangle if only if \( x \notin C(g, R) \).

Proof. We will use the following identity that holds for any \( a, b, c \in \mathbb{R} \):
\[
(a + b + c) (-a + b + c) (a - b + c) (a + b - c) = (a^2 + b^2 + c^2)^2 - 2(a^4 + b^4 + c^4).
\]

If \( t_1, t_2, t_3 \) are the side lengths of a triangle then
\[
(t_1 + t_2 + t_3) (-t_1 + t_2 + t_3) (t_1 - t_2 + t_3) (t_1 + t_2 - t_3) > 0.
\]

From (4) we obtain \( (t_1^2 + t_2^2 + t_3^2)^2 > 2(t_1^4 + t_2^4 + t_3^4) \). For \( n = 3 \), the identity (3) led us to conclusion \( t_0^2 \neq \frac{1}{2} (t_1^2 + t_2^2 + t_3^2) \) and, with Proposition 2.4, we obtain \( x \notin C(g, R) \).
For the only if part, we consider \( x \notin C(g, R) \). By reversing the previous calculus we obtain (5), also

\[
(-t_1 + t_2 + t_3)(t_1 - t_2 + t_3)(t_1 + t_2 - t_3) > 0.
\]

We assume that two factors are negative. Let \( t_1 - t_2 + t_3 < 0 \) and \( t_1 + t_2 - t_3 < 0 \). Then \( t_1 < t_2 - t_3 \) and \( t_1 < t_3 - t_2 \) that is false. Then all three factors are positive and \( t_1, t_2, t_3 \) could be the side lengths of a triangle.

We conclude this paper with the next proposition.

**Proposition 3.2.** Let \( n \in \mathbb{N}, n \geq 4 \) and \( x_1, x_2, \ldots, x_n \in X \), distinguies, and \( t_0 \in (0, \infty) \) such that \( \|x_i - x_j\| = t_0 \), for any \( i, j \in \{1, 2, \ldots, n\}, i \neq j \). For any \( x \in \text{affin}[x_1, x_2, \ldots, x_n] \) we denote \( t_k = \|x - x_k\| \), \( k \in \{1, 2, \ldots, n\} \). Then \( t_1, t_2, \ldots, t_n \) could be the side lengths of a \( n \)-side polygon if and only if \( x \notin \{x_1, x_2, \ldots, x_n\} \).

**Proof.** If \( t_1, t_2, \ldots, t_n \) are the side lengths of a \( n \)-side polygon, then \( t_k > 0 \), for any \( k \in \{1, 2, \ldots, n\} \). Then \( \|x - x_k\| > 0 \), so \( x \neq x_k \). We obtain \( x \notin \{x_1, x_2, \ldots, x_n\} \).

Now, if \( x \notin \{x_1, x_2, \ldots, x_n\} \) then \( \left(\sum_{k=1}^{n} t_k^2\right)^2 - (n-1) \left(\sum_{k=1}^{n} t_k^4\right) \geq 0 \) from (2). Now the conclusion is consequence of Theorem 2 from [1].

**References**


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