ON CERTAIN SUBCLASS OF MULTIVALENT ANALYTIC FUNCTIONS ASSOCIATED WITH ERDELYI-KOBER TYPE INTEGRAL OPERATOR

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ABSTRACT. In this paper, we introduce a certain subclasses of multivalent uniformly starlike analytic functions by making use of Erdeyi-Kober type integral operator. Further, we determine coefficient estimates and Holder's inequality results. Also, results for family of class preserving integral operators are obtained for the class $US^*_pT(n,a,c;\alpha,\beta)$.

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1. Introduction

Let $A(p,n)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (n,p \in \mathbb{N} = \{1,2,...\}),$$

which are analytic and $p$-valent in open unit disc $U = \{ z : z \in \mathbb{C}, |z| < 1 \}$. Also, we note that $A(1,1) = A$, that is the class of analytic univalent functions.

A function $f \in A(p,n)$ is said to be in the class $S(p,n,\alpha)$ of $p$-valent starlike functions of order $\alpha$ if it satisfies the condition

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in U; \ 0 \leq \alpha < p).$$

A function $f \in A(p,n)$ is said to be in the class $K(p,n,\alpha)$ of $p$-valent convex functions of order $\alpha$ if it satisfies the condition

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in U; \ 0 \leq \alpha < p).$$

The classes $S(p,n,\alpha)$ and $K(p,n,\alpha)$ were studied by Owa [18]. The class $S^*(p,\alpha) = S(p,1,\alpha)$ was considered by Patil and Thakare [19].
We denote by $T(p,n)$ the subclass of $A(p,n)$ consisting of functions of the form

$$f(z) = z^p - \sum_{k=n}^{\infty} a_{k+p} z^{k+p} \quad (a_{k+p} \geq 0; \ n, \ p \in \mathbb{N} = \{1, 2, \ldots\}),$$

and define two further classes $T^*(p,n,\alpha)$ and $C(p,n,\alpha)$ by

$$T^*(p,n,\alpha) = S(p,n,\alpha) \cap T(p,n), \quad C(p,n,\alpha) := K(p,n,\alpha) \cap T(p,n).$$

Further, the classes $T^*(p,\alpha) = S^*(p,\alpha) \cap T(p,n), \ C(p,\alpha) := K(p,\alpha) \cap T(p,n)$. The function $f(z) \in T(p,n)$ given by (4) is said to be $\beta$–uniformly starlike of order $\alpha (-p \leq \alpha < p)$ and $\beta \geq 0$ denote by $US^*_{p,T}(n,\alpha,\beta)$ if and only if

$$\text{Re} \left( \frac{zf'(z)}{f(z)} - \alpha \right) > \beta \left| \frac{zf''(z)}{f'(z)} - p \right| \quad (z \in U).$$

Also, function $f(z)$ is said to be $\beta$–uniformly convex of order $\alpha$ denoted by $UC_{p,V}(n,\alpha,\beta)$ [10] if and only if

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} - \alpha \right) > \beta \left| \frac{zf''(z)}{f'(z)} - (p-1) \right| \quad (z \in U).$$

Note that, the classes $US^*_{1,T}(1,\alpha,\beta) = US^*_{1,T}(\alpha,\beta)$ and $UC_{1,V}(1,\alpha,\beta) = UC_{V}(\alpha,\beta)$ are introduced and studied by Bharati et al. [4]. In particular, the classes $UC_{V}(0,1)$ and $UC_{V}(0,\beta)$ were introduced by Goodman [7] and Kanas and Wisniowska [9].

**Definition 1.** [2] For $f \in A(p,n), \ p, \ n \in \mathbb{N}, \ \mu > 0, \ a, c \in \mathbb{C}, \ \text{Re} \ (a) \geq -\mu p$ and $\text{Re} \ (c-a) > 0$, El-Ashwah and Drbuk define the differ-integral operator which called Erdelyi-Kober type integral operator $I^{a,c}_{p,\mu} : A(p,n) \to A(p,n)$ as follows

$$I^{a,c}_{p,\mu}f(z) = z^p + \sum_{k=n}^{\infty} \Psi^{a,c}_{p,\mu}(k) a_{k+p} z^{k+p},$$

where

$$\Psi^{a,c}_{p,\mu}(k) = \frac{\Gamma(c+\mu p) \Gamma(a+\mu(k+p))}{\Gamma(a+\mu p) \Gamma(c+\mu(k+p))}.$$
If \( a = c \), then we have \( I_{p,\mu}^{a,c} f (z) = f (z) \). It easily to verify that,

\[
z \left( I_{p,\mu}^{a,c} f (z) \right)' = \frac{a + \mu p}{\mu} I_{p,\mu}^{a+1,c} f (z) - \frac{a}{\mu} I_{p,\mu}^{a,c} f (z).
\]

We also note that the operator \( I_{p,\mu}^{a,c} f (z) \) generalizes several previously studied familiar operators and we will mention some of the interesting particular cases as follows:

(1) For \( p = 1 \), we can obtain the operator \( I_{\mu}^{a,c} f (z) \) defined in [11, ch.5] (see also [20] and [21, with \( m = 0 \));

(2) For \( a = \beta \), \( c = \beta + 1 \) and \( \mu = 1 \), we obtain the familiar integral operator \( I_{\beta,p} f (z) \) \((\beta > -p)\) which studies by Saitoh et al. [23];

(3) For \( a = \beta \), \( c = \alpha + \beta - \gamma + 1 \) and \( \mu = 1 \), we obtain the operator \( R_{\beta,\gamma}^{\alpha} f (z) \) \((\gamma > 0; \alpha \geq \gamma - 1; \beta > -1)\) studied by Aouf et al. [1];

(4) For \( p = 1 \), \( a = \beta \), \( c = \alpha + \beta \) and \( \mu = 1 \), we obtain the operator \( Q_{\beta}^{\alpha} f (z) \) \((\alpha \geq 0, \beta > -1)\) studied by Jung et al. [8];

(5) For \( p = 1 \), \( a = \alpha - 1 \), \( c = \beta - 1 \) and \( \mu = 1 \), we obtain the operator \( l (\alpha, \beta) f (z) \) \((\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0, \mathbb{Z}_0 = \{0, -1, -2, \ldots\}\) studied by Carlson and Shafer [5];

(6) For \( p = 1 \), \( a = \rho - 1 \), \( c = \ell \) and \( \mu = 1 \), we obtain the operator \( I_{\rho,\ell} f (z) \) \((\rho > 0; \ell > -1)\) studied by Choi et al. [6];

(7) For \( p = 1 \), \( a = \alpha \), \( c = 0 \) and \( \mu = 1 \), we obtain the operator \( D^{\alpha} f (z) \) \((\alpha > 1)\) studied by Ruscheweyh [22];

(8) For \( p = 1 \), \( a = 1 \), \( c = n \) and \( \mu = 1 \), we obtain the operator \( I_{n} f (z) \) \((n \in \mathbb{N}_0)\) studied by Noor and Noor [17]; and Noor [16];

(10) For \( p = 1 \), \( a = \beta \), \( c = \beta + 1 \) and \( \mu = 1 \), we obtain the integral operator \( I_{\beta,1} \) which studied by Bernardi [3];

(11) For \( p = 1, a = 1, c = 2 \) and \( \mu = 1 \), we obtain the integral operator \( I_{1,1} = I \) which studied by Libera [12] and Livingston [14].

Now, we introduced a new subclasses of \( p \)-valent functions and discussed some interesting geometric properties of this generalized function class.
Definition 2. A function \( f \in A(p, n) \) is said to be in the class \( US_p^*(n, a, c; \mu, \alpha, \beta) \) if it satisfies the inequality

\[
\text{Re} \left( \frac{z(I_{p, \mu}^{a,c} f(z))'}{I_{p, \mu}^{a,c} f(z)} - \alpha \right) > \beta \left| \frac{z(I_{p, \mu}^{a,c} f(z))'}{I_{p, \mu}^{a,c} f(z)} - p \right| , \quad (z \in U),
\]

which is equivalent to

\[
\text{Re} \left( \frac{I_{p, \mu}^{a+1,c} f(z)}{I_{p, \mu}^{a,c} f(z)} - \frac{a + \alpha \mu}{a + \mu p} \right) > \beta \left| \frac{I_{p, \mu}^{a+1,c} f(z)}{I_{p, \mu}^{a,c} f(z)} - 1 \right| , \quad (z \in U) \quad (8)
\]

for some \(-p \leq \alpha < p, \beta \geq 0, p, n \in \mathbb{N}, \mu > 0, a, c \in \mathbb{C}, \text{Re}(a) \geq -\mu p \) and \( \text{Re}(c - a) > 0 \).

Furthermore, we define the class \( US_p^*T(n, a, c; \mu, \alpha, \beta) \) by \( US_p^* (n, a, c; \mu, \alpha, \beta) \cap T(p, n) \).

The main object of this work is to determine coefficient estimates for the analytic functions class \( US_p^*T(n, a, c; \mu, \alpha, \beta) \). We study some interesting Holder’s inequality for the class \( US_p^*T(n, a, c; \mu, \alpha, \beta) \). Also, the family of class preserving integral operators for functions \( f \) in the class \( US_p^*T(n, a, c; \mu, \alpha, \beta) \) are obtained.

2. COEFFICIENT INEQUALITIES

Unless otherwise mention, we assume in the reminder of this paper that \( \mu > 0, a, c \in \mathbb{R}, a > -\mu p, (a - c) > 0, -p \leq \alpha < p, \beta \geq 0, p, n \in \mathbb{N} \). First, we give a coefficients inequality for the class \( US_p^* (n, a, c; \mu, \alpha, \beta) \).

Theorem 1. A sufficient condition for a function \( f(z) \) of the form (1) to be in \( US_p^* (n, a, c; \mu, \alpha, \beta) \) is

\[
\sum_{k=n}^{\infty} [k(1 + \beta) + (p - \alpha)] \Psi_{p, \mu}^{a,c}(k) |a_{k+p}| \leq (p - \alpha)
\]

where

\[
\Psi_{p, \mu}^{a,c}(k) = \frac{\Gamma(c + \mu p) \Gamma(a + \mu(k + p))}{\Gamma(a + \mu p) \Gamma(c + \mu(k + p))},
\]

Proof. It is sufficient to show that

\[
\beta \left| \frac{z(I_{p, \mu}^{a,c} f(z))'}{I_{p, \mu}^{a,c} f(z)} - p \right| - \text{Re} \left( \frac{z(I_{p, \mu}^{a,c} f(z))'}{I_{p, \mu}^{a,c} f(z)} - p \right) \leq p - \alpha.
\]
We have

\[ \beta \left| \frac{z(I_{p,\mu}^{a,c} f(z))'}{I_{p,\mu}^{a,c} f(z)} - p \right| - \text{Re} \left( \frac{z(I_{p,\mu}^{a,c} f(z))'}{I_{p,\mu}^{a,c} f(z)} - p \right) \leq (1 + \beta) \left| \frac{z(I_{p,\mu}^{a,c} f(z))'}{I_{p,\mu}^{a,c} f(z)} - p \right| \leq (1 + \beta) \left| \frac{p z^p + \sum_{k=n}^{\infty} (k + p) \Psi_{p,\mu}^{a,c} (k) a_{k+p} z^{k+p}}{z^p + \sum_{k=n}^{\infty} \Psi_{p,\mu}^{a,c} (k) a_{k+p} z^{k+p}} - p \right| \leq (1 + \beta) \sum_{k=n}^{\infty} k \Psi_{p,\mu}^{a,c} (k) a_{k+p} z^{k}. \]

The last expression is bounded by \((p - \alpha)\), if

\[ \sum_{k=n}^{\infty} [k (1 + \beta) + (p - \alpha)] \Psi_{p,\mu}^{a,c} (k) a_{k+p} \leq (p - \alpha), \]

and hence the proof is completed.

**Theorem 2.** A necessary and sufficient condition for a function \( f(z) \) of the form (4) to be in \( US_p^{a,c} (n, a, c; \mu, \alpha, \beta) \) is

\[ \sum_{k=n}^{\infty} [k (1 + \beta) + (p - \alpha)] \Psi_{p,\mu}^{a,c} (k) a_{k+p} \leq (p - \alpha). \]

**Proof.** The sufficient condition follows from Theorem 1. To prove the necessity, let \( f \in US_p^{a,c} (n, a, c; \mu, \alpha, \beta) \) and \( z \) is real, then

\[ \frac{p - \sum_{k=n}^{\infty} (k + p) \Psi_{p,\mu}^{a,c} (k) a_{k+p} z^{k}}{1 - \sum_{k=n}^{\infty} \Psi_{p,\mu}^{a,c} (k) a_{k+p} z^{k}} - \alpha \geq \beta \left| \frac{p - \sum_{k=n}^{\infty} (k + p) \Psi_{p,\mu}^{a,c} (k) a_{k+p} z^{k}}{1 - \sum_{k=n}^{\infty} \Psi_{p,\mu}^{a,c} (k) a_{k+p} z^{k}} - p + \sum_{k=n}^{\infty} p \Psi_{p,\mu}^{a,c} (k) a_{k+p} z^{k} \right|. \]

Let \( z \to 1^- \), we obtain

\[ \frac{p - \sum_{k=n}^{\infty} (k + p) \Psi_{p,\mu}^{a,c} (k) a_{k+p}}{1 - \sum_{k=n}^{\infty} \Psi_{p,\mu}^{a,c} (k) a_{k+p}} - \alpha \geq \beta \left| \frac{p - \sum_{k=n}^{\infty} k \Psi_{p,\mu}^{a,c} (k) a_{k+p}}{1 - \sum_{k=n}^{\infty} \Psi_{p,\mu}^{a,c} (k) a_{k+p}} \right|. \]

or, equivalently

\[ p - \sum_{k=n}^{\infty} (k + p) \Psi_{p,\mu}^{a,c} (k) a_{k+p} - \alpha \left( 1 - \sum_{k=n}^{\infty} \Psi_{p,\mu}^{a,c} (k) a_{k+p} \right) \geq \beta \sum_{k=n}^{\infty} k \Psi_{p,\mu}^{a,c} (k) a_{k+p}. \]
Thus, we have
\[ \sum_{k=n}^{\infty} [k (1 + \beta) + (p - \alpha)] \Psi_{p,\mu}^{a,c} (k) |a_{k+p}| \leq (p - \alpha).\]

Then the proof is completed.

**Corollary 3.** If \( f(z) \) of the form (4) is in \( US^*_p T(n, a, c; \mu, \alpha, \beta) \), then
\[ a_{p+k} \leq \frac{(p - \alpha)}{[k (1 + \beta) + (p - \alpha)] \Psi_{p,\mu}^{a,c} (k)}, \quad (k \geq n, n \in \mathbb{N}). \quad (10)\]

with equality only for the function
\[ f(z) = z^p - \frac{(p - \alpha)}{[k (1 + \beta) + p - \alpha] \Psi_{p,\mu}^{a,c} (k)} z^{p+k}, \quad (k \geq n, n \in \mathbb{N}). \quad (11)\]

### 3. Holder’s Inequality

For function \( f_j(z) \in T(p, n) \) are given by
\[ f_j(z) = z^p - \sum_{k=n}^{\infty} a_{k+p,j} z^{k+p} \quad (a_{k+p,j} \geq 0, j = 1, 2, 3, ..., m). \]

Now, we define the modified Hadmard product of \( f_j(z) \) and the generalization of the modified Hadmard product as follows
\[ G_m(z) = z^p - \sum_{k=n}^{\infty} \left( \prod_{j=1}^{m} a_{k+p,j} \right) z^{k+p} \]

and
\[ H_m(z) = z^p - \sum_{k=n}^{\infty} \left( \prod_{j=1}^{m} a_{q_j k+p,j} \right) z^{k+p}, \quad (q_j > 0, j = 1, 2, 3, ..., m). \]

(i) For \( m = 2 \), then \( G_2(z) = (f_1 * f_2)(z) \).

(ii) For \( q_j = 1 \), we have \( G_m(z) = H_m(z) \).
Further, for functions $f_j(z)\ (j = 1, 2, \ldots, m)$, the familiar Holder’s inequality assumes the following form

$$\sum_{k=n}^{\infty} \left( \prod_{j=1}^{m} a_{k+p,j} \right) \leq \prod_{j=1}^{m} \left( \sum_{k=n}^{\infty} (a_{k+p,j})^{q_j} \right)^{\frac{1}{q_j}}, \quad \left( q_j > 1, \sum_{j=1}^{m} \frac{1}{q_j} \geq 1, \ j = 1, 2, 3, \ldots, m \right).$$

Recently, Nishiwaki and Owa [15] have studied some results of Holder’s inequalities for a subclass of $p$-valent starlike and convex function.

**Theorem 4.** Let $f_j(z) \in US_{p}^{s}T(n,a,c;\mu,\alpha_j,\beta)$ $(j = 1, 2, 3, \ldots, m)$, then $H_m(z) \in US_{p}^{s}T(n,a,c;\mu,\eta,\beta)$,

$$[k (1 + \beta)] \prod_{j=1}^{m} (p - \alpha_j)^{s_j} \eta \leq \frac{\prod_{j=1}^{m} [k (1 + \beta) + (p - \alpha_j)]^{s_j} [\Psi_{p,\mu}^{a,c}(k)]^{s_j-1} - \prod_{j=1}^{m} (p - \alpha_j)^{s_j}}{\sum_{j=1}^{m} \frac{1}{q_j}, \sum_{j=1}^{m} \frac{1}{q_j} \geq 1;}$$

where $k \geq n$, $s_j \geq \frac{1}{q_j}$, $q_j > 1$, $\sum_{j=1}^{m} \frac{1}{q_j} \geq 1$; $j = 1, 2, 3, \ldots, m$.

**Proof.** Let $f_j(z) \in US_{p}^{s}T(n,a,c;\mu,\alpha_j,\beta)$, then

$$\sum_{k=n}^{\infty} \frac{[k (1 + \beta) + (p - \alpha_j)] \Psi_{p,\mu}^{a,c}(k)}{(p - \alpha_j)} a_{k+p,j} \leq 1. \quad (12)$$

which implies

$$\left( \sum_{k=n}^{\infty} \frac{[k (1 + \beta) + (p - \alpha_j)] \Psi_{p,\mu}^{a,c}(k)}{(p - \alpha_j)} a_{k+p,j} \right)^{\frac{1}{q_j}} \leq 1, \quad \left( q_j > 1, \sum_{j=1}^{m} \frac{1}{q_j} \geq 1 \right). \quad (13)$$

From (13), we have

$$\prod_{j=1}^{m} \left( \sum_{k=n}^{\infty} \frac{[k (1 + \beta) + (p - \alpha_j)] \Psi_{p,\mu}^{a,c}(k)}{(p - \alpha_j)} a_{k+p,j} \right)^{\frac{1}{q_j}} \leq 1.$$

Applying Holder’s inequality, we find that

$$\sum_{k=n}^{\infty} \left[ \prod_{j=1}^{m} \frac{[k (1 + \beta) + (p - \alpha_j)] \Psi_{p,\mu}^{a,c}(k)}{(p - \alpha_j)} \right]^{\frac{1}{q_j}} \left( a_{k+p,j} \right)^{\frac{1}{q_j}} \leq 1.$$
Thus, we have to determine the largest $\eta$ such that

$$\sum_{k=n}^{\infty} \frac{[k (1 + \beta) + (p - \eta)] \Psi_{p,\mu}^{\alpha, \epsilon}(k)}{(p - \eta)} \left( \prod_{j=1}^{m} a_{k+p,j}^{s_j} \right) \leq 1.$$ 

That is

$$\sum_{k=n}^{\infty} \frac{[k (1 + \beta) + (p - \eta)] \Psi_{p,\mu}^{\alpha, \epsilon}(k)}{(p - \eta)} \left( \prod_{j=1}^{m} a_{k+p,j}^{s_j} \right) \leq \sum_{k=n}^{\infty} \left[ \prod_{j=1}^{m} \left( \frac{[k (1 + \beta) + (p - \alpha_j)] \Psi_{p,\mu}^{\alpha, \epsilon}(k)}{(p - \alpha_j)} \right)^{\frac{1}{\eta_j}} \right] \frac{1}{(a_{k+p,j})^{\eta_j}}.$$ 

Therefore, we need to find the largest $\eta$ such that

$$\frac{[k (1 + \beta) + (p - \alpha_j)] \Psi_{p,\mu}^{\alpha, \epsilon}(k)}{(p - \alpha_j)} \left( \prod_{j=1}^{m} (a_{k+p,j})^{s_j - \frac{1}{\eta_j}} \right) \leq \frac{\prod_{j=1}^{m} \left( [k (1 + \beta) + (p - \alpha_j)] \Psi_{p,\mu}^{\alpha, \epsilon}(k) \right)^{\frac{1}{\eta_j}}}{(a_{k+p,j})^{\eta_j}}, \ (k \geq n).$$

Since

$$\prod_{j=1}^{m} \left( [k (1 + \beta) + (p - \alpha_j)] \Psi_{p,\mu}^{\alpha, \epsilon}(k) \right)^{s_j - \frac{1}{\eta_j}} \leq \prod_{j=1}^{m} \left( [k (1 + \beta) + (p - \alpha_j)] \Psi_{p,\mu}^{\alpha, \epsilon}(k) \right)^{\frac{1}{\eta_j}}$$

we see that,

$$\prod_{j=1}^{m} (a_{k+p,j})^{s_j - \frac{1}{\eta_j}} \leq \frac{1}{\prod_{j=1}^{m} \left( [k (1 + \beta) + (p - \alpha_j)] \Psi_{p,\mu}^{\alpha, \epsilon}(k) \right)^{s_j - \frac{1}{\eta_j}}}.$$ 

This implies that

$$\frac{[k (1 + \beta) + (p - \eta)] \Psi_{p,\mu}^{\alpha, \epsilon}(k)}{(p - \eta)} \leq \prod_{j=1}^{m} \left( [k (1 + \beta) + (p - \alpha_j)] \Psi_{p,\mu}^{\alpha, \epsilon}(k) \right)^{s_j} \prod_{j=1}^{m} (p - \alpha_j)^{s_j}.$$ 

Then

$$\eta \leq p - \frac{k (1 + \beta) \prod_{j=1}^{m} (p - \alpha_j)^{s_j}}{\prod_{j=1}^{m} [k (1 + \beta) + (p - \alpha_j)^{s_j} \left( \Psi_{p,\mu}^{\alpha, \epsilon}(k) \right)^{s_j - \frac{1}{\eta_j}} - \prod_{j=1}^{m} (p - \alpha_j)^{s_j}]}.$$ 

This completes the proof of the theorem.

**Remark 1.** Putting $a = c$, $\mu = 1$, $\beta = 0$ in Theorem 4, we obtain the corresponding result obtained by Nishiwaki and Owa [15];

**Corollary 5.** Let $f_j(z) \in US_{p}^\alpha T (n, a, c; \mu, \alpha_j, \beta)$ $(j = 1, 2, 3, ..., m)$, then $H_m (z) \in US_{p}^\alpha T (n, a, c; \mu, \eta, \beta)$ with

$$\eta \leq p - \frac{n (1 + \beta) \prod_{j=1}^{m} (p - \alpha_j)^{s_j}}{\left( \Psi_{p,\mu}^{\alpha, \epsilon}(n) \right)^{r-1} \prod_{j=1}^{m} \left[ n (1 + \beta) + (p - \alpha_j)^{s_j} - \prod_{j=1}^{m} (p - \alpha_j)^{s_j} \right]}$$

where $r = \sum_{j=1}^{m} s_j > 1 + \frac{p - \alpha}{n (1 + \beta)}$, $s_j \geq \frac{1}{q_j}$, $q_j > 1$, $\sum_{j=1}^{m} \frac{1}{q_j} \geq 1$; $j = 1, 2, 3, ..., m$. 

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Putting $\alpha_j = \alpha$ in Corollary 5 we obtain the following corollary.

**Corollary 6.** Let $f_j(z) \in US^*_p T(n, a, c; \mu, \alpha_j, \beta)$ ($j = 1, 2, 3, \ldots, m$), then $H_m(z) \in US^*_p T(n, a, c; \mu, \eta, \beta)$ with

$$\eta \leq p - \frac{[n(1 + \beta)](p - \alpha)^r}{[n(1 + \beta) + (p - \alpha)]^r [\Psi^{a,c}_{p,\mu}(n)]^{r-1} - (p - \alpha)^r},$$

where $r = \sum_{j=1}^{m} s_j > 1 + \frac{p-\alpha}{n(1+\beta)}$, $s_j \geq \frac{1}{q_j}$, $q_j > 1$, $\sum_{j=1}^{m} \frac{1}{q_j} \geq 1$; $j = 1, 2, 3, \ldots, m$.

**Example 1.** Let $f_j(z)$ ($j = 1, 2, 3, \ldots, m$) define as follows

$$f_j(z) = z^p - \frac{(p - \alpha)}{[n(1 + \beta) + (p - \alpha)] [\Psi^{a,c}_{p,\mu}(n)]} \epsilon z^{n+p} - \frac{(p - \alpha)}{[(n+j)(1 + \beta) + (p - \alpha)] [\Psi^{a,c}_{p,\mu}(n+j)]} \epsilon_j z^{n+p+j},$$

$(\epsilon + \epsilon_j \leq 1)$,

then $H_m(z) \in US^*_p T(n, a, c; \mu, \eta, \beta)$ with

$$\eta \leq p - \frac{[n(1 + \beta)](p - \alpha)^r}{[n(1 + \beta) + (p - \alpha)]^r [\Psi^{a,c}_{p,\mu}(n)]^{r-1} - (p - \alpha)^r}.$$

Since

$$f_j(z) = z^p - \frac{(p - \alpha)}{[n(1 + \beta) + (p - \alpha)] [\Psi^{a,c}_{p,\mu}(n)]} \epsilon z^{n+p} - \frac{(p - \alpha)}{[(n+j)(1 + \beta) + (p - \alpha)] [\Psi^{a,c}_{p,\mu}(n+j)]} \epsilon_j z^{n+p+j},$$

$(\epsilon + \epsilon_j \leq 1, \ j = 1, 2, 3, \ldots, m)$,

we have

$$\sum_{k=n}^{\infty} \frac{[k(1 + \beta) + (p - \alpha)] [\Psi^{a,c}_{p,\mu}(k)]}{(p - \alpha)} a_{k+p} = \frac{[n(1 + \beta) + (p - \alpha)] [\Psi^{a,c}_{p,\mu}(n)]}{(p - \alpha)} \epsilon a_{n+p} + \frac{[(n+j)(1 + \beta) + (p - \alpha)] [\Psi^{a,c}_{p,\mu}(n+j)]}{(p - \alpha)} \epsilon_j a_{n+p+j} = \epsilon + \epsilon_j \leq 1.$$

Then $f_j(z) \in US^*_p T(n, a, c; \mu, \alpha, \beta)$ and we have

$$H_m(z) = z^p - \left( \frac{(p - \alpha)}{[n(1 + \beta) + (p - \alpha)] [\Psi^{a,c}_{p,\mu}(n)]} \right)^r z^{n+p},$$

and $H_m(z) \in US^*_p T(n, a, c; \mu, \eta, \beta)$.  

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4. Modified Hadamard Products

Let the functions $f_i(z) \ (i = 1, 2)$ be defined by

$$f_i(z) = z^p - \sum_{k=n}^{\infty} a_{k+p,i} z^{k+p} \quad (14)$$

The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=n}^{\infty} a_{k+p,1} a_{k+p,2} z^{k+p}.$$

**Corollary 7.** Let the functions $f_i(z) \ (i = 1, 2)$ defined by (14) be in the class $US_p^T(n, a, c; \mu, \alpha, \beta)$ and $US_p^{*T}(n, a, c; \mu, \alpha, \beta)$, then

$$(f_1 * f_2)(z) \in US_p^T(n; a, c; \mu, \delta, \beta) \quad \text{where} \quad \delta \leq p - \frac{n (1 + \beta) (p - \alpha_1) (p - \alpha_2)}{n (1 + \beta) + (p - \alpha_1) \Psi_{p,\mu}^c(n) + (p - \alpha_2)} \quad (z \in U; n \in \mathbb{N}).$$

**Corollary 8.** Let the functions $f_i(z) \ (i = 1, 2)$ defined by (14) be in the class $US_p^{*T}(n, a, c; \mu, \alpha, \beta)$ then $(f_1 * f_2)(z) \in US_p^{*T}(n, a, c; \mu, \delta, \beta)$ where

$$\delta \leq p - \frac{n (1 + \beta) (p - \alpha)^2}{\Psi_{p,\mu}^c(n) [n (1 + \beta) + (p - \alpha)]^2} \quad (z \in U; n \in \mathbb{N}).$$

**Theorem 9.** Let the functions $f_i(z) \ (i = 1, 2)$ defined by (14) be in the class $US_p^{*T}(n, a, c; \mu, \alpha, \beta)$, then $h(z) = z^p - \sum_{k=p+1}^{\infty} \left( a_{k,1}^2 + a_{k,2}^2 \right) z^p$ belongs to the class $US_p^{*T}(n, a, c; \mu, \delta, \beta)$ where

$$\delta = \Omega(n) \leq p - \frac{2n (1 + \beta) (p - \alpha)^2}{n (1 + \beta) + (p - \alpha)]^2 \Psi_{p,\mu}^{a,c}(n) - 2 (p - \alpha)} \quad (z \in U, n \in \mathbb{N}).$$

**Proof.** To prove the theorem, we need to find the largest $\delta$ such that

$$\sum_{k=p+1}^{\infty} \frac{[k (1 + \beta) + (p - \delta)] \Psi_{p,\mu}^{a,c}(k)}{(p - \delta)} (a_{k,1}^2 + a_{k,2}^2) \leq 1. \quad (15)$$

Hence

$$\sum_{k=n}^{\infty} \left\{ \frac{[k (1 + \beta) + (p - \alpha)] \Psi_{p,\mu}^{a,c}(k)}{(p - \alpha)} \right\}^2 a_{k,i}^2 \leq \sum_{k=n}^{\infty} \frac{[k (1 + \beta) + (p - \alpha)] \Psi_{p,\mu}^{a,c}(k)}{(p - \alpha)} a_{k,i} \leq 1, \quad (i = 1, 2). \quad (16)$$
Then
\[ \sum_{k=n}^{\infty} \frac{\left[ k (1 + \beta) + (p - \alpha) \right] \Psi_{p,\mu}^{a,c}(k)}{(p - \alpha)} \left( a_{k,1}^2 + a_{k,2}^2 \right)^2 \leq 1, \]
and (15) is true if
\[ \sum_{k=n}^{\infty} \frac{\left[ k (1 + \beta) + (p - \delta) \right] \Psi_{p,\mu}^{a,c}(k)}{(p - \delta)} \left( a_{k,1}^2 + a_{k,2}^2 \right)^2 \leq \sum_{k=n}^{\infty} \frac{\left[ k (1 + \beta) + (p - \alpha) \right] \Psi_{p,\mu}^{a,c}(k)}{(p - \alpha)} \left( a_{k,1}^2 + a_{k,2}^2 \right)^2. \]

If
\[ \frac{\left[ k (1 + \beta) + (p - \delta) \right]}{(p - \delta)} \leq \frac{\left[ k (1 + \beta) + (p - \alpha) \right]^2}{2 (p - \alpha)^2} \Psi_{p,\mu}^{a,c}(k), \]
then
\[ \delta \leq \Omega(k) = p - \frac{2k (1 + \beta) (p - \alpha)^2}{\left[ k (1 + \beta) + (p - \alpha) \right]^2 \Psi_{p,\mu}^{a,c}(k) - 2 (p - \alpha)^2}, \quad (k \geq n, n \in \mathbb{N}) \]
which is an increasing function of \( k \geq n, \ 0 \leq \alpha < p, \ p \in \mathbb{N}, \ 0 < \beta \leq 1. \)

Then
\[ \delta = \Omega(n) \leq p - \frac{2n (1 + \beta) (p - \alpha)^2}{\left[ n (1 + \beta) + (p - \alpha) \right]^2 \Psi_{p,\mu}^{a,c}(n) - 2 (p - \alpha)^2}. \]
The proof is completed.

5. Closure Properties under Integral Operators

In this section, we discuss some preserving integral operators. We recall here the generalized Komatu integral operator (see [13]) define by
\[
K(z) = \frac{(\gamma + p)^d}{\Gamma(d)} \int_0^z t^{\gamma-1} \left( \log \frac{z}{t} \right)^{d-1} f(t) \, dt \quad (f(z) \in T(n,p))
\]
\[ = z^p - \sum_{k=n}^{\infty} \left( \frac{\gamma + p}{\gamma + k + p} \right)^d a_{k+p} z^{k+p} \quad (d \geq 0; \gamma > -p; z \in U). \quad (17) \]

Also the generalized Jung-Kim-Srivastava operator (see[11]) define by
\[
I(z) = Q_{p}^{a,c} I(z) = \left( \frac{p + d + \gamma - 1}{p + \gamma - 1} \right)^d \int_0^z t^{\gamma-1} \left( \frac{1 - \frac{t}{z}}{1 - \frac{z}{t}} \right)^{d-1} f(t) \, dt \quad (f(z) \in T(n,p))
\]
\[ = z^p - \sum_{k=n}^{\infty} \frac{\Gamma(p + k + \gamma) \Gamma(p + \gamma + d)}{\Gamma(p + k + \gamma + d) \Gamma(p + \gamma)} a_{k+p} z^{k+p} \quad (d \geq 0; \gamma > -p; z \in U). \quad (18) \]
Theorem 10. Let $d > 0$, $\gamma > -p$ and $f(z) \in US_p^*T(n, a, c; \mu, \alpha, \beta)$, then $K(z)$ defined by (17) belongs to $US_p^*T(n, a, c; \mu, \alpha, \beta)$.

Proof. Let $f(z) \in US_p^*T(n, a, c; \mu, \alpha, \beta)$ defined by (4), and $K(z)$ defined by (17). Then $K(z) \in US_p^*T(n, a, c; \mu, \alpha, \beta)$ if

$$
\sum_{k=n}^{\infty} \left[ \frac{k(1+\beta)+(p-\alpha)}{(p-\alpha)} \Psi_{p,\mu}^{a,c}(k) \right] \left( \frac{\gamma + p}{\gamma + k + p} \right)^d \left| a_{k+p} \right| \leq 1.
$$

(19)

Now, from Theorem 2, $f(z) \in US_p^*T(n, a, c; \mu, \alpha, \beta)$ if and only if

$$
\sum_{k=n}^{\infty} \left[ \frac{k(1+\beta)+(p-\alpha)}{(p-\alpha)} \Psi_{p,\mu}^{a,c}(k) \right] \left( \frac{\gamma + p}{\gamma + k + p} \right)^d \left| a_{k+p} \right| \leq 1.
$$

Since $\frac{\gamma + p}{\gamma + k + p} \leq 1$, for $k \geq n$, then (19) holds true. Therefore $K(z) \in US_p^*T(n, a, c; \mu, \alpha, \beta)$.

Theorem 11. Let $d > 0$, $\gamma > -p$ and $f(z) \in US_p^*T(n, a, c; \mu, \alpha, \beta)$, then $K(z)$ defined by (17) is $p-$valent in the disk $|z| < R_1$, where

$$
R_1 = \inf_k \left\{ \frac{p \left[ k(1+\beta)+(p-\alpha) \right] (\gamma + k + p)^d \left( \frac{\gamma + p}{\gamma + k + p} \right)^d \left| \Psi_{p,\mu}^{a,c}(k) \right| }{(k+p)(p-\alpha)(\gamma + p)^d} \right\}^{\frac{1}{p}}
$$

(20)

Proof. In order to prove the assertion, it is enough to show that

$$
\left| \frac{K'(z)}{z^{p-1}} - p \right| \leq p.
$$

(21)

Now, in view of (21), we get

$$
\left| \frac{K'(z)}{z^{p-1}} - p \right| = \left| -\sum_{k=n}^{\infty} (k+p) \left( \frac{\gamma + p}{\gamma + k + p} \right)^d \left| a_{k+p} \right| z^{k+p} \right| \leq \sum_{k=n}^{\infty} (k+p) \left( \frac{\gamma + p}{\gamma + k + p} \right)^d \left| a_{k+p} \right| \left| z^k \right|.
$$

This expression is bounded by $p$ if

$$
\sum_{k=n}^{\infty} \left( \frac{k+p}{p} \right) \left( \frac{\gamma + p}{\gamma + k + p} \right)^d \left| a_{k+p} \right| \left| z^k \right| \leq 1.
$$

(22)
Since $f(z) \in US_p^* T(n, a, c; \mu, \alpha, \beta)$, and from Theorem 2 (22) holds if
\[
(\frac{k+p}{p}) (\frac{\gamma+p}{\gamma+k+p})^d a_{k+p} |z|^k \leq \frac{[k(1+\beta) + (p-\alpha)] \Psi_{p,\mu}^{a,c}(k)}{(p-\alpha)} a_{n+p}, \quad (k \in \mathbb{N}).
\]
That is
\[
|z| \leq \left\{ \frac{p[k(1+\beta) + (p-\alpha)] (\gamma+k+p)^d}{(k+p)(p-\alpha)(\gamma+p)^d} \Psi_{p,\mu}^{a,c}(k) \right\}^{\frac{1}{k}}.
\]
The result follows by setting $|z| = R_1$.

Following similar steps as in the proofs of Theorem 10 and Theorem 11, we have the following results for $I(z)$.

**Theorem 12.** Let $d > 0$, $\gamma > -p$ and $f(z) \in US_p^* T(n, a, c; \mu, \alpha, \beta)$, then $I(z)$ defined by (18) belongs to $US_p^* T(n, a, c; \mu, \alpha, \beta)$.

**Theorem 13.** Let $d > 0$, $c > -p$ and $f(z) \in US_p^* T(n; a, c; \mu, \alpha, \beta)$. Then $I(z)$ defined by (18) is $p-$valent in the disk $|z| < R_2$, where
\[
R_2 = \inf_k \left\{ \frac{p[k(1+\beta) + (p-\alpha)] \Gamma(p+k+\gamma+d) \Gamma(p+\gamma)}{(k+p)(p-\alpha) \Gamma(p+k+\gamma) \Gamma(p+\gamma+d)} \Psi_{p,\mu}^{a,c}(k) \right\}^{\frac{1}{k}}.
\]

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