CONVEXITY PROPERTIES FOR A NEW INTEGRAL OPERATOR

V. T. NGUYEN, A. OPREA, D. BREAZ

Abstract. For some classes of analytic functions \( f, g \) and \( h \) in the open unit disk \( U \), we define a new integral operator \( H_{n, \alpha}(z) = \int_0^z \prod_{i=1}^n \left( \frac{f_i(t) g_i(t)}{h_i(t)} \right)^{\alpha_i} dt \) and we study convexity properties of this general integral operator.

2010 Mathematics Subject Classification: 30C45.

Keywords: analytic functions, integral operator, convex, order

1. Introduction

Let \( U = \{ z : |z| < 1 \} \) be the unit disk and \( \mathcal{A} \) be the class of all functions of the form
\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in U
\]
which are analytic in \( U \) and satisfy the conditions \( f(0) = f'(0) - 1 = 0 \).

We denote by \( S \) the subclass of \( \mathcal{A} \) consisting of univalent functions on \( U \).

A function \( f \in \mathcal{A} \) is a convex function of complex order \( b \), \( (b \in \mathbb{C} \setminus \{0\}) \) and type \( \lambda \) \((0 \leq \lambda < 1)\), if it verifies one of these conditions
\[
\text{Re} \left\{ 1 + \frac{1}{b} \left( \frac{zf''(z)}{f'(z)} \right) \right\} > \lambda, \quad \left| \frac{1}{b} \frac{zf''(z)}{f'(z)} \right| < 1 - \lambda, \quad z \in U. \quad (2)
\]

We denote by \( C^{*}_\lambda(b) \) the class of these functions.

A function \( f \in \mathcal{A} \) is a starlike function of order \( \beta \), \( 0 \leq \beta < 1 \) if it satisfies one of the conditions
\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \beta, \quad \left| \frac{zf'(z)}{f(z)} \right| < \beta, \quad z \in U. \quad (3)
\]

We denote this class by \( S^*(\beta) \).
We done by $K(\beta)$ the class of convex functions of order $\beta$, $0 \leq \beta < 1$ that satisfies the inequality

$$\Re \left( \frac{zf''(z)}{f'(z)} + 1 \right) > \beta, \ z \in U.$$  \hspace{1cm} (4)

A function $f \in \mathcal{A}$ belongs to class $R(\beta)$, $0 \leq \beta < 1$, if

$$\Re \left( \frac{f'(z)}{f(z)} \right) > \beta, \ z \in U.$$  \hspace{1cm} (5)

A function $f \in \mathcal{A}$ is a starlike function of the complex order $b$, $b \in \mathbb{C} \setminus \{0\}$ and type $\lambda$, $(0 \leq \lambda < 1)$, if and only if

$$\Re \left\{ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > \lambda \text{ or } \left| \frac{1}{b} \frac{zf'(z)}{f(z)} \right| \leq 1 - \lambda, \ z \in U.$$  \hspace{1cm} (6)

We denote by $S_{\lambda}^{*}(b)$ the class of these functions.

F. Ronning introduced in [6] the class of univalent functions $\mathcal{SP}(\alpha, \beta)$, $\alpha > 0$, $\beta \in [0, 1)$. So, we denote by $\mathcal{SP}(\alpha, \beta)$ the class of all functions $f \in \mathcal{S}$ which satisfies the inequality:

$$\left| \frac{zf'(z)}{f(z)} - (\alpha + \beta) \right| \leq \Re \frac{zf''(z)}{f(z)} + \alpha - \beta, \ z \in U.$$  \hspace{1cm} (7)

Silverman defined in [7] the class $G_{b}$. So, a function $f \in \mathcal{A}$ is in the class $G_{b}$, $0 < b \leq 1$ if and only if

$$\left| 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right| < b \left| \frac{zf''(z)}{f(z)} \right|, \ z \in U.$$  \hspace{1cm} (8)

Uralegaddi in [8], Owa and Srivastava in [3] defined the class $\mathcal{N}(\beta)$. So, a function $f \in \mathcal{A}$ is in the class $\mathcal{N}(\beta)$ if it verifies the inequality

$$\Re \left( \frac{zf''(z)}{f'(z)} + 1 \right) < \beta, \ z \in U, \ \beta > 1.$$  \hspace{1cm} (9)

2. Main results

In this paper, we study new properties for a general integral operator defined by

$$H_{n,\alpha}(z) = \int_{0}^{z} \prod_{i=1}^{n} \left( \frac{f_{i}(t)}{h_{i}(t)} \right)^{\alpha_{i}} \, dt.$$  \hspace{1cm} (10)
Remark 1. If we consider $h_i(z) = z$, for $i = 1, 2, ..., n$, in relation (10), we obtain the integral operator:

$$G_n(z) = \int_0^z \prod_{i=1}^n \left( \frac{f_i(t)}{g_i(t)} \right)^{\alpha_i} dt$$

introduced and studied by Adriana Oprea and Daniel Breaz in [2].

Remark 2. If $f_i(z) = z, h_i(z) = z$, for $i = 1, 2, ..., n$ from (10), we obtain the integral operator:

$$F_{\alpha_1, \alpha_2, ..., \alpha_n}(z) = \int_0^z (g_1(t))^{\alpha_1} (g_2(t))^{\alpha_2} ... (g_n(t))^{\alpha_n} dt,$$

introduced and studied by D.Breaz et all in [1].

Remark 3. For $n = 1$, $f(z) = z, h(z) = z, g_1 = g, \alpha_1 = \gamma_1 = \gamma$ from (10), we obtain the integral operator:

$$G(z) = \int_0^z (g'(t))^{\gamma} dt$$

studied in [4] and [5].

Theorem 1. Let $f_i, g_i, h_i \in A$, where $g_i \in G_{b_i}, 0 < b_i \leq 1$, for $i = 1, 2, ..., n$. For any $M_i, N_i \geq 1$, which verify

$$\left| \frac{zf_i'(z)}{f_i(z)} \right| \leq M_i, \left| \frac{zh_i'(z)}{h_i(z)} \right| \leq N_i \text{ and } \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| < 1,$$

for all $z \in U$, there are $\alpha_i$ real numbers, with $\alpha_i > 0$, $i = 1, 2, ... n$, so that

$$\lambda = 1 - \sum_{i=1}^n \alpha_i (M_i + N_i + 2b_i + 1) > 0.$$

In these conditions, the integral operator

$$H_{n, \alpha}(z) = \int_0^z \prod_{i=1}^n \left( \frac{f_i(t)}{h_i(t) g_i(t)} \right)^{\alpha_i} dt$$

is in the class $K(\lambda)$. 

77
Proof. We calculate the first and second order derivatives for \( H_{n,\alpha} \) and we obtain:

\[
H'_{n,\alpha}(z) = \prod_{i=1}^{n} \left( \frac{f_i(z)}{h_i(z)g_i(z)} \right)^{\alpha_i}
\]

and

\[
H''_{n,\alpha}(z) = \sum_{i=1}^{n} \alpha_i \left( \frac{f_i(z)}{h_i(z)g_i(z)} \right)^{\alpha_i-1} \left[ \frac{f_i'(z)h_i(z)}{h_i^2(z)}g_i(z) + \frac{f_i(z)h_i'(z)g_i'(z)}{h_i(z)g_i''(z)} \right]
\]

\[
\times \prod_{k=1, k \neq i}^{n} \left( \frac{f_k(z)}{h_k(z)g_k(z)} \right)^{\alpha_k}
\]

Further, we have:

\[
\frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} = \sum_{i=1}^{n} \alpha_i \left[ \frac{zf_i'(z)}{f_i(z)} - \frac{zh_i'(z)}{h_i(z)} \right] + \sum_{i=1}^{n} \alpha_i \frac{zg_i''(z)}{g_i(z)}
\]

\[
= \sum_{i=1}^{n} \alpha_i \left[ \frac{zf_i'(z)}{f_i(z)} - \frac{zh_i'(z)}{h_i(z)} \right] + \sum_{i=1}^{n} \alpha_i \left( \frac{zg_i''(z)}{g_i(z)} - \frac{zg_i'(z)}{g_i(z)} + 1 \right)
\]

\[
+ \sum_{i=1}^{n} \alpha_i \left( \frac{zg_i'(z)}{g_i(z)} - 1 \right). \tag{16}
\]

So, we have:

\[
\frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} \leq \sum_{i=1}^{n} \alpha_i \left| zf_i'(z) \right| + \sum_{i=1}^{n} \alpha_i \left| zh_i'(z) \right|
\]

\[
+ \sum_{i=1}^{n} \alpha_i \left| zg_i''(z) \right| - \sum_{i=1}^{n} \alpha_i \left( \frac{zg_i'(z)}{g_i(z)} - 1 \right). \tag{17}
\]

Since functions \( g_i \in G_{b_i}, \ 0 < b_i \leq 1, \) for \( i = 1, 2, ..., n, \) using inequality (8), we get:

\[
\frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} \leq \sum_{i=1}^{n} \alpha_i (M_i + N_i) + \sum_{i=1}^{n} \alpha_i b_i \left| \frac{zg_i'(z)}{g_i(z)} \right| + \sum_{i=1}^{n} \alpha_i \left( \frac{zg_i'(z)}{g_i(z)} - 1 \right)
\]

\[
\leq \sum_{i=1}^{n} \alpha_i (M_i + N_i) + \sum_{i=1}^{n} \alpha_i b_i \left( \left| \frac{zg_i'(z)}{g_i(z)} - 1 \right| + 1 \right) + \sum_{i=1}^{n} \alpha_i \left( \frac{zg_i'(z)}{g_i(z)} - 1 \right).
\]

78
then the integral operator \( H_{n,\alpha} \) is in the class \( K(\lambda) \).

If we consider \( n = 1 \) in Theorem 1, we get the following corollary:

**Corollary 2.** Let \( f, g, h \in \mathcal{A} \), where \( g \in G_b, 0 < b \leq 1 \). For any \( M, N \geq 1 \), which verify the conditions

\[
\frac{|zf'(z)|}{f(z)} \leq M, \quad \frac{|zh'(z)|}{h(z)} \leq N, \quad \left| \frac{zg'(z)}{g(z)} - 1 \right| < 1,
\]

for all \( z \in U \), with \( \alpha > 0 \) is a real number, so that \( \lambda = 1 - \alpha(M + N + 2b + 1) > 0 \). In these conditions, the integral operator \( H_{1,\alpha}(z) = \int_0^z \left( \frac{f(t)}{h(t)} g'(t) \right)^\alpha dt \) is in the class \( K(\lambda) \).

**Theorem 3.** Let \( f_i \in S^*(\beta_i) \) and \( h_i \in S^*(\delta_i) \), with \( 0 \leq \beta_i, \delta_i < 1 \) and \( g_i \in K(\lambda_i), 0 \leq \lambda_i < 1 \), for \( i = 1, 2, ..., n \). If \( \alpha_i \) are real numbers with \( \alpha_i > 0 \), for \( i = 1, 2, ..., n \) so that

\[
\sum_{i=1}^{n} \alpha_i (\beta_i + \delta_i - \lambda_i + 3) < 1,
\]

then the integral operator

\[
H_{n,\alpha}(z) = \int_0^z \prod_{i=1}^{n} \left( \frac{f_i(t)}{h_i(t)} g_i'(t) \right)^{\alpha_i} dt
\]

is convex of order \( \rho = 1 + \sum_{i=1}^{n} \alpha_i (\lambda_i - \beta_i - \delta_i - 3) \), for all \( i = 1, 2, ..., n \).

**Proof.** After the same steps as in the proof of Theorem 1, we get

\[
\frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} = \sum_{i=1}^{n} \alpha_i \frac{zf''_i(z)}{f_i(z)} - \sum_{i=1}^{n} \alpha_i \frac{zh''_i(z)}{h_i(z)} + \sum_{i=1}^{n} \alpha_i \frac{zg''_i(z)}{g_i(z)}.
\]

Further, we obtain

\[
\frac{|zH''_{n,\alpha}(z)|}{|H'_{n,\alpha}(z)|} \leq \sum_{i=1}^{n} \alpha_i \left| \frac{zf''_i(z)}{f_i(z)} \right| + \sum_{i=1}^{n} \alpha_i \left| \frac{zh''_i(z)}{h_i(z)} \right| + \sum_{i=1}^{n} \alpha_i \left| \frac{zg''_i(z)}{g_i(z)} \right|.
\]
\[
\leq \sum_{i=1}^{n} \alpha_i \left( \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + 1 \right) + \sum_{i=1}^{n} \alpha_i \left( \left| \frac{zh_i(z)}{h_i(z)} - 1 \right| + 1 \right) + \sum_{i=1}^{n} \alpha_i \left| \frac{zg''_i(z)}{g'_i(z)} \right|
\]

\[
\leq \sum_{i=1}^{n} \alpha_i [\beta_i + 1 + \delta_i + 1 + 1 - \lambda_i] = \sum_{i=1}^{n} \alpha_i (\beta_i + \delta_i - \lambda_i + 3). \quad (21)
\]

From (21), we get:
\[
\left| \frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} \right| \leq \sum_{i=1}^{n} \alpha_i (\beta_i + \delta_i - \lambda_i + 3) = 1 - \rho. \quad (22)
\]

So, the integral operator \(H_{n,\alpha}\) is convex of order \(\rho = 1 + \sum_{i=1}^{n} \alpha_i (\lambda_i - \beta_i - \delta_i - 3)\), for \(i = 1, 2, ..., n\).

If we consider \(n = 1\) in Theorem 3, we get the following corollary:

**Corollary 4.** Let \(f \in S^*(\beta), h \in S^*(\delta), 0 \leq \beta < 1, 0 \leq \delta < 1\) and \(g \in K(\lambda), 0 \leq \lambda < 1\). If \(\alpha\) is a real number so that \(\alpha > 0\) and \(\alpha(\beta + \delta - \lambda + 3) < 1\), then the integral operator
\[
H_{1,\alpha}(z) = \int_0^z \left( \frac{f(t)}{h(t)} g_i'(t) \right) \alpha \, dt
\]
is convex of order \(1 + \alpha(\lambda - \beta - \delta - 3)\).

**Theorem 5.** Let functions \(f_i \in \mathcal{SP}(\alpha, \beta), h_i \in \mathcal{SP}(\delta, \eta), \) with \(\alpha > 0\) and \(\delta > 0, \beta \in [0, 1), \eta \in [0, 1)\) and \(g_i \in \mathcal{N}(\lambda_i), \lambda_i > 1\) for \(i = 1, 2, ..., n\). For any \(M_i, N_i \geq 1, i = 1, 2, ..., n,\) which verify
\[
\left| \frac{zf_i'(z)}{f_i(z)} \right| \leq M_i, \quad \left| \frac{zh_i'(z)}{h_i(z)} \right| \leq N_i \quad \text{for all } z \in U, \quad (23)
\]

there are \(\alpha_i > 0\) real numbers with \(\alpha_i > 0, i = 1, 2, ..., n,\) so that
\[
\rho = 1 + \sum_{i=1}^{n} \alpha_i (M_i + N_i + 2\alpha + 4\delta - 2\eta + \lambda_i - 1) > 1. \quad (24)
\]

In these conditions, the integral operator
\[
H_{n,\alpha}(z) = \int_0^z \prod_{i=1}^{n} \left( \frac{f_i(t)}{h_i(t)} g_i'(t) \right)^{\alpha_i} \, dt
\]
is in the class \(\mathcal{N}(\rho)\).
Proof. From Theorem 3, we get:

\[
\frac{zH''_{n,0}(z)}{H'_{n,0}(z)} + 1 = \sum_{i=1}^{n} \alpha_i \frac{zf'_{i}(z)}{f_{i}(z)} - \sum_{i=1}^{n} \alpha_i \frac{h'_{i}(z)}{h_{i}(z)} + \sum_{i=1}^{n} \alpha_i \frac{zg''_{i}(z)}{g_{i}(z)} + 1
\]

\[
= \sum_{i=1}^{n} \alpha_i \left( \frac{zf'_{i}(z)}{f_{i}(z)} + \alpha - \beta \right) - \sum_{i=1}^{n} \alpha_i \left( \frac{zh'_{i}(z)}{h_{i}(z)} + \delta - \eta \right)
\]

\[+ \sum_{i=1}^{n} \alpha_i (\delta - \eta - \alpha + \beta) + \sum_{i=1}^{n} \alpha_i \left( \frac{zg''_{i}(z)}{g_{i}(z)} + 1 \right) - \sum_{i=1}^{n} \alpha_i + 1. \tag{25}\]

We calculate the real part of both terms in the above expression and obtain:

\[
\text{Re} \left( \frac{zH''_{n,0}(z)}{H'_{n,0}(z)} + 1 \right) = \sum_{i=1}^{n} \alpha_i \text{Re} \left( \frac{zf'_{i}(z)}{f_{i}(z)} + (\alpha - \beta) \right) - \sum_{i=1}^{n} \alpha_i \text{Re} \left( \frac{zh'_{i}(z)}{h_{i}(z)} + (\delta - \eta) \right)
\]

\[+ \sum_{i=1}^{n} \alpha_i (\delta - \eta - \alpha + \beta) + \sum_{i=1}^{n} \alpha_i \text{Re} \left( \frac{zg''_{i}(z)}{g_{i}(z)} + 1 \right) - \sum_{i=1}^{n} \alpha_i + 1 \tag{26}\]

Since \(\text{Re} \omega \leq |\omega|\), we have:

\[
\text{Re} \left( \frac{zH''_{n,0}(z)}{H'_{n,0}(z)} + 1 \right) \leq \sum_{i=1}^{n} \alpha_i \left| \frac{zf'_{i}(z)}{f_{i}(z)} + (\alpha - \beta) \right| - \sum_{i=1}^{n} \alpha_i \left| \frac{zh'_{i}(z)}{h_{i}(z)} + (\delta - \eta) \right|
\]

\[+ \sum_{i=1}^{n} \alpha_i (\delta - \eta - \alpha + \beta) + \sum_{i=1}^{n} \alpha_i \text{Re} \left( \frac{zg''_{i}(z)}{g_{i}(z)} + 1 \right) - \sum_{i=1}^{n} \alpha_i + 1 \]

\[\leq \sum_{i=1}^{n} \alpha_i \left| \frac{zf'_{i}(z)}{f_{i}(z)} + (\alpha - \beta) \right| + \sum_{i=1}^{n} \alpha_i \left| \frac{zh'_{i}(z)}{h_{i}(z)} + (\delta - \eta) \right|
\]

\[+ \sum_{i=1}^{n} \alpha_i (\delta - \eta - \alpha + \beta) + \sum_{i=1}^{n} \alpha_i \text{Re} \left( \frac{zg''_{i}(z)}{g_{i}(z)} + 1 \right) - \sum_{i=1}^{n} \alpha_i + 1. \tag{27}\]
Since \( f_i \in \mathcal{SP}(\alpha, \beta), \alpha > 0, \beta \in [0, 1) \) and \( h_i \in \mathcal{SP}(\delta, \eta), \delta > 0, \eta \in [0, 1) \), for \( i = 1, 2, \ldots, n \) and \( g_i \in \mathcal{N}(\lambda_i), \lambda_i > 1, i = 1, 2, \ldots, n \), we have:

\[
\left| \frac{zf_i'(z)}{f_i(z)} - (\alpha + \beta) \right| \leq \text{Re} \left( \frac{zf_i'(z)}{f_i(z)} \right) + \alpha - \beta,
\]

\[
\left| \frac{zh_i'(z)}{h_i(z)} - (\delta + \eta) \right| \leq \text{Re} \left( \frac{zh_i'(z)}{h_i(z)} \right) + \delta - \eta,
\]

and

\[
\text{Re} \left( \frac{zg_i''(z)}{g_i(z)} \right) \leq \lambda_i, \lambda_i > 1, i = 1, 2, \ldots, n, \quad z \in U.
\]

Using above inequalities, we get:

\[
\text{Re} \left( \frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} + 1 \right) \leq \sum_{i=1}^{n} \alpha_i \left( \text{Re} \left( \frac{zf_i'(z)}{f_i(z)} \right) + \alpha - \beta \right) - \sum_{i=1}^{n} \alpha_i \left( \text{Re} \left( \frac{zh_i'(z)}{h_i(z)} \right) + \delta - \eta \right)
\]

\[
+ \sum_{i=1}^{n} \alpha_i(2\alpha + 2\delta) + \sum_{i=1}^{n} \alpha_i(\delta - \eta - \alpha - \beta) + \sum_{i=1}^{n} \alpha_i\lambda_i - \sum_{i=1}^{n} \alpha_i + 1. \tag{28}
\]

From (28), we obtain:

\[
\text{Re} \left( \frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} + 1 \right) \leq \sum_{i=1}^{n} \alpha_i \left| \frac{zf_i'(z)}{f_i(z)} \right| + \sum_{i=1}^{n} \alpha_i \left| \frac{zh_i'(z)}{h_i(z)} \right| + \sum_{i=1}^{n} \alpha_i(\alpha - \beta + 2\alpha)
\]

\[
+ \sum_{i=1}^{n} \alpha_i(\delta - \eta + 2\delta) + \sum_{i=1}^{n} \alpha_i(\delta - \eta + \alpha + \beta) + \sum_{i=1}^{n} \alpha_i\lambda_i - \sum_{i=1}^{n} \alpha_i + 1
\]

\[
= \sum_{i=1}^{n} \alpha_i(M_i + N_i + 2\alpha + 4\delta - 2\eta + \lambda_i - 1) + 1 = \rho \tag{29}
\]

So, the integral operator \( H_{n,\alpha} \) is in the class \( \mathcal{N}(\rho) \).

If we consider \( n = 1 \) in Theorem 5, we obtain the following corollary:

**Corollary 6.** Let functions \( f \in \mathcal{SP}(\alpha, \beta), h \in \mathcal{SP}(\delta, \eta) \) with \( \alpha > 0, \delta > 0, \beta \in [0, 1), \eta \in [0, 1) \) and \( g \in \mathcal{N}(\lambda), \lambda > 1 \). For any \( M, N \geq 1 \), which verify

\[
\left| \frac{zf'(z)}{f(z)} \right| \leq M, \quad \left| \frac{zh'(z)}{h(z)} \right| \leq N, \quad \text{for all} \quad z \in U,
\]

82
there is $\alpha$ real number with $\alpha > 0$, so that

$$\rho = 1 + \alpha(M + N + 2\alpha + 4\delta - 2\eta + \lambda - 1) > 1.$$ 

In these conditions, the integral operator

$$H_{1,\alpha}(z) = \int_0^z \prod_{i=1}^n \left( \frac{f(t)}{h(t)} g_i(t) \right) dt$$

is in the class $N(\rho)$.

**Theorem 7.** Let $f_i \in S^{*}_{\lambda_i}(b)$, $h_i \in S^{*}_{\delta_i}(b)$, $g_i \in C_{\lambda_i}(b)$, with $0 \leq \lambda_i < 1$, $0 \leq \delta_i < 1$ for $i = 1, 2, ..., n$ and $b \in \mathbb{C} - \{0\}$. Also, let $\alpha_i$ be real numbers, with $\alpha_i > 0$ for $i = 1, 2, ..., n$. If

$$0 \leq 1 + \sum_{i=1}^n \alpha_i(2\lambda_i + \delta_i - 5) < 1,$$

then the integral operator

$$H_{n,\alpha}(z) = \int_0^z \prod_{i=1}^n \left( \frac{f_i(t)}{h_i(t)} g_i(t) \right)^{\alpha_i} dt$$

is in the class $C_{\mu}(b)$, with $\mu = 1 + \sum_{i=1}^n \alpha_i(2\lambda_i + \delta_i - 5)$, for $i = 1, 2, ..., n$.

**Proof.** After the same steps from previous Theorems, we obtain:

$$\frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} = \sum_{i=1}^n \alpha_i \left[ \frac{zf_i'(z)}{f_i(z)} - \frac{zh_i'(z)}{h_i(z)} + \frac{zg_i''(z)}{g_i(z)} \right].$$

Multiplying relation with $1/b$, we get:

$$\frac{1}{b} \frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} = \sum_{i=1}^n \alpha_i \left[ \frac{1}{b} \left( \frac{zf_i'(z)}{f_i(z)} - \frac{zh_i'(z)}{h_i(z)} \right) + \frac{1}{b} \frac{zg_i''(z)}{g_i(z)} \right].$$

Further, we have
\[ \left| \frac{1}{b} \frac{z H''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} \right| = \left| \sum_{i=1}^{n} \frac{1}{\alpha_i} \left( z f_i'(z) - \sum_{i=1}^{n} \frac{1}{\alpha_i} z h_i'(z) \right) + \sum_{i=1}^{n} \frac{1}{\alpha_i} \frac{1}{b} g_i''(z) \right| \]

\[ \leq \sum_{i=1}^{n} \alpha_i \left| \frac{1}{b} \frac{z f_i'(z)}{f_i(z)} \right| + \sum_{i=1}^{n} \alpha_i \left| \frac{1}{b} \frac{z h_i'(z)}{h_i(z)} \right| + \sum_{i=1}^{n} \alpha_i \left| \frac{1}{b} g_i''(z) \right| \]

\[ \leq \sum_{i=1}^{n} \alpha_i \left( \left| \frac{1}{b} \left( \frac{z f_i'(z)}{f_i(z)} - 1 \right) \right| + 1 \right) + \sum_{i=1}^{n} \alpha_i \left( \left| \frac{1}{b} \left( \frac{z h_i'(z)}{h_i(z)} - 1 \right) \right| + 1 \right) \]

\[ + \sum_{i=1}^{n} \alpha_i \left| \frac{1}{b} g_i''(z) \right| . \] (30)

Since \( f_i \in S^*_\lambda(b), \ h_i \in S^*_\delta(b) \) and \( g_i \in C_\mu(b) \) for \( i = 1, 2, ..., n \), we have

\[ \left| \frac{1}{b} \left( \frac{z f_i'(z)}{f_i(z)} - 1 \right) \right| \leq 1 - \lambda_i, \ \left| \frac{1}{b} \left( \frac{z h_i'(z)}{h_i(z)} - 1 \right) \right| \leq 1 - \delta_i \] and \( \left| \frac{1}{b} g_i''(z) \right| \leq 1 - \lambda_i. \)

So, we get:

\[ \left| \frac{1}{b} \frac{z H''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} \right| \leq \sum_{i=1}^{n} \alpha_i ((1 - \lambda_i) + 1) + \sum_{i=1}^{n} \alpha_i ((1 - \delta_i) + 1) + \sum_{i=1}^{n} \alpha_i (1 - \lambda_i) \]

\[ = \sum_{i=1}^{n} (2 - \lambda_i) + \sum_{i=1}^{n} \alpha_i (2 - \delta_i) + \sum_{i=1}^{n} \alpha_i (1 - \lambda_i) = \sum_{i=1}^{n} \alpha_i (5 - 2\lambda_i - \delta_i). \]

Since \( 0 \leq 1 + \sum_{i=1}^{n} \alpha_i (2\lambda_i + \delta_i) < 1 \), we get, \( H_{n,\alpha} \) is in the class \( C_\mu(b) \), with \( \mu = 1 + \sum_{i=1}^{n} \alpha_i (2\lambda_i + \delta_i - 5) \).

If we consider \( n = 1 \) in Theorem 7, we get the following corollary:

**Corollary 8.** Let \( f \in S^*_\lambda \) and \( h \in S^* \delta, \ g \in C_\lambda(b) \) with \( 0 \leq \lambda < 1, \ 0 \leq \delta < 1 \) and \( b \in \mathbb{C} \setminus \{0\} \). Also, let \( \alpha \) be a real number, with \( \alpha > 0 \). If \( 0 \leq 1 + \alpha (2\lambda + \delta - 5) < 1 \), then the integral operator

\[ H_{1,\alpha}(z) = \int_{0}^{z} \left( \frac{f(t)}{h(t)} g'(t) \right)^{\alpha} dt \]

84
is in the class $C_{\mu}(b)$, with $\mu = 1 + \alpha(2\lambda + \delta - 5)$.

**Theorem 9.** Let $f_i, g_i, h_i \in \mathcal{A}$, where $g_i \in \mathcal{N}(\lambda_i)$, with $\lambda_i > 1$ for $i = 1, 2, ..., n$. For any $\lambda_i > 1$, and $f_i, h_i$ verifying conditions

$$\left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| \leq 1, \quad \left| \frac{zh_i'(z)}{h_i(z)} - 1 \right| \leq 1, \quad z \in U$$

there are numbers $\alpha_i \in \mathbb{R}$ with $\alpha_i > 0$ so that $\mu = \sum_{i=1}^{n} \alpha_i(\lambda_i+1)+1$ for $i = 1, 2, ..., n$. In these conditions, the integral operator

$$H_{n,\alpha}(z) = \int_{0}^{z} \prod_{i=1}^{n} \left( \frac{f_i(t)}{h_i(t)} g_i(t) \right)^{\alpha_i} dt$$

is in the class $\mathcal{N}(\mu)$.

**Proof.** From the previous Theorems, we obtain

$$\frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} = \sum_{i=1}^{n} \alpha_i \left( \frac{zf_i'(z)}{f_i(z)} - 1 \right) - \sum_{i=1}^{n} \alpha_i \left( \frac{zh_i'(z)}{h_i(z)} - 1 \right)$$

$$+ \sum_{i=1}^{n} \alpha_i \left( \frac{zg_i''(z)}{g_i'(z)} + 1 \right) - \sum_{i=1}^{n} \alpha_i + 1 \quad (31)$$

Further, we get:

$$\Re \left( \frac{zH''_{n}(z)}{H'_{n}(z)} + 1 \right) = \sum_{i=1}^{n} \alpha_i \Re \left( \frac{zf_i'(z)}{f_i(z)} - 1 \right) - \sum_{i=1}^{n} \alpha_i \Re \left( \frac{zh_i'(z)}{h_i(z)} - 1 \right)$$

$$+ \sum_{i=1}^{n} \alpha_i \Re \left( \frac{zg_i''(z)}{g_i'(z)} + 1 \right) - \sum_{i=1}^{n} \alpha_i + 1. \quad (32)$$

Since $g_i \in \mathcal{N}(\lambda_i), i = 1, 2, ..., n$ and $\Re(\omega) \leq |\omega|$ and applying the conditions from the hypothesis of Theorem 9, (31) and (32), we get:

$$\Re \left( \frac{zH''_{n,\alpha}(z)}{H'_{n,\alpha}(z)} + 1 \right) \leq \sum_{i=1}^{n} \alpha_i \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| + \sum_{i=1}^{n} \alpha_i \left| \frac{zh_i'(z)}{h_i(z)} - 1 \right|$$

$$+ \sum_{i=1}^{n} \alpha_i\lambda_i - \sum_{i=1}^{n} \alpha_i + 1$$
\[
\leq 2 \sum_{i=1}^{n} \alpha_i + \sum_{i=1}^{n} \alpha_i \lambda_i - \sum_{i=1}^{n} \alpha_i + 1 = \sum_{i=1}^{n} \alpha_i (\lambda_i + 1) + 1. \tag{33}
\]

So, \( H_{n, \alpha} \) is in the class \( \mathcal{N}(\mu) \), where \( \mu = 1 + \sum_{i=1}^{n} \alpha_i (\lambda_i + 1), i = 1, 2, \ldots, n. \)

If consider \( n = 1 \) and in Theorem 9, we get the following corollary:

**Corollary 10.** Let \( f, h \in \mathcal{A} \), where \( g \in \mathcal{N}(\lambda), \lambda > 1 \) and \( f, h \) verify conditions

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1, \quad \left| \frac{zh'(z)}{h(z)} - 1 \right| \leq 1, \quad z \in U,
\]

there is number \( \alpha \in \mathbb{R} \) with \( \alpha > 0 \) so that \( \mu = \alpha(\lambda + 1) \).

In these conditions, the integral operator

\[
H_{1, \alpha}(z) = \int_{0}^{z} \left( \frac{f(t)}{h(t)} \right)^{\alpha} dt
\]

is in the class \( \mathcal{N}(\mu) \).

**Acknowledgement** This work was supported by a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS/CCCDI - UEFISCDI, project number PN-III-P2-2.1-PED-2016-1835, within PNCDI III.

**References**


Van Tuan Nguyen  
Department of Mathematics, University of Pitesti  
Targul din Vale Str., No.1, 110040, Pitesti, Arges, Romania  
email: vataninguyenedu@gmail.com

Adriana Oprea  
Department of Mathematics, University of Pitesti  
Targul din Vale Str., No.1, 110040, Pitesti, Arges, Romania  
email: adriana_oprea@yahoo.com

Daniel Breaz  
”1 Decembrie” University of Alba Iulia, Romania  
N. Iorga Str., No. 11-13,510009, Alba Iulia, Romania,  
email: dbreaz@uab.ro