PROPERTIES OF AN INTEGRAL OPERATOR

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Abstract. In this paper we define an integral operator for analytic functions in the open unit disk and we determine some properties of this integral operator.

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1. Introduction

Let \( A \) be the class of functions of the form
\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\]
normalized by \( f(0) = f'(0) - 1 = 0 \) which are analytic in the open unit disk \( \mathcal{U} = \{ z \in \mathbb{C} : |z| < 1 \} \).

We consider \( S \) the subclass of \( A \) consisting of functions \( f \in A \), which are univalent in \( \mathcal{U} \).

We denote by \( P \) the class of functions \( p \) of the form
\[
p(z) = 1 + \sum_{k=1}^{\infty} b_k z^k,
\]
which are analytic in open unit disk \( \mathcal{U} \), with \( \text{Re} \ p(z) > 0 \), for all \( z \in \mathcal{U} \).

Let \( \mathcal{H}(\mathcal{U}) \) be the space of holomorphic functions in \( \mathcal{U} \) and let
\[
A_n = \{ f \in \mathcal{H}(\mathcal{U}), f(z) = z + a_{n+1} z^{n+1} + \cdots, z \in \mathcal{U} \}
\]
with \( A_1 = A \).

In this work we introduce a new integral operator \( J_{\alpha,\beta} : \mathcal{H}(\mathcal{U}) \rightarrow \mathcal{H}(\mathcal{U}) \) defined by
\[
J_{\alpha,\beta}(z) = \frac{z^{1-\frac{1}{\alpha}}}{\alpha} \int_0^z t^{\frac{1}{\alpha}-2} (g(t))^{\beta} dt, \ z \in \mathcal{U}, g \in \mathcal{H}(\mathcal{U}),
\]

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for \( \alpha, \beta \) be complex numbers, \( \alpha \neq 0, \beta \neq 0 \).

We have the next remarks:

\( i_1 \) For \( \beta = 1, \alpha = 1 \),
we have the integral operator Alexander [1], \( A : \mathcal{A} \to \mathcal{A} \),
\[
A(z) = \int_0^z \frac{g(t)}{t} dt, \quad z \in \mathcal{U}.
\] (2)

\( i_2 \) For \( \beta = 1, \alpha = \frac{1}{2} \),
we obtain the integral operator Libera [4], \( L : \mathcal{H}(\mathcal{U}) \to \mathcal{H}(\mathcal{U}) \),
\[
L(z) = 2 \int_0^z g(t) dt, \quad z \in \mathcal{U}.
\] (3)

\( i_3 \) If \( \beta = 1, \alpha = \frac{1}{n}, n \in \mathbb{N}^* \),
we get the integral operator Bernardi [3], \( L_n : \mathcal{H}(\mathcal{U}) \to \mathcal{H}(\mathcal{U}) \),
\[
L_n(z) = \frac{n}{z^{n-1}} \int_0^z t^{n-2} g(t) dt, \quad z \in \mathcal{U}.
\] (4)

\( i_4 \) For \( \beta = 1, \alpha \in \mathbb{R}, 0 < \alpha \leq 1 \),
we obtain the integral operator Pascu [6], \( L_\alpha : \mathcal{H}(\mathcal{U}) \to \mathcal{H}(\mathcal{U}) \),
\[
L_\alpha(z) = \frac{z^{1-\frac{1}{\alpha}}}{\alpha} \int_0^z t^{\frac{1}{\alpha}-2} g(t) dt, \quad z \in \mathcal{U}.
\] (5)

These integral operators are integral operators of type Libera.
To discuss our problem for integral operator \( J_{\alpha,\beta} \), we need the following theorem.

**Theorem 1.** (Becker [2]). If \( f(z) = z + a_2 z^2 + \cdots \) is analytic in \( \mathcal{U} \) and

\[
(1 - |z|^2) \left| \frac{zf''(z)}{f'(z)} \right| \leq 1,
\] (6)

for all \( z \in \mathcal{U} \), then the function \( f(z) \) is univalent in \( \mathcal{U} \).

2. **Main results**

**Theorem 2.** Let \( \alpha, \beta \) be complex numbers, \( \alpha \neq 0, \beta \neq 0 \), and the function \( g \in \mathcal{A} \),
\( g(z) = z + a_2 z^2 + \cdots \).

If
\[
\left| \frac{1}{\alpha} + \beta - 2 \right| < 1
\] (7)
and

\[ \left| \frac{zg'(z)}{g(z)} - 1 \right| \leq \frac{1 - \left| \frac{1}{\alpha} + \beta - 2 \right|}{|\beta|}, \quad z \in \mathcal{U}, \quad (8) \]

then \( z^{\frac{1}{\alpha} - 1} J_{\alpha, \beta}(z) \in \mathcal{S} \) and \( J_{\alpha, \beta}(z) \) has the form

\[ J_{\alpha, \beta}(z) = z^{2 - \frac{1}{\alpha} + b_2 z^{3 - \frac{1}{\alpha}} + \cdots}, \quad z \in \mathcal{U}. \quad (9) \]

Proof. We have

\[ J_{\alpha, \beta}(z) = \frac{z^{1 - \frac{1}{\alpha}}}{{\alpha}} \int_0^z t^{\frac{1}{\alpha} + \beta - 2} \left( \frac{g(t)}{t} \right)^\beta dt, \quad (10) \]

for all \( z \in \mathcal{U} \).

We consider the function

\[ G_{\alpha, \beta}(z) = \frac{1}{{\alpha}} \int_0^z t^{\frac{1}{\alpha} + \beta - 2} \left( \frac{g(t)}{t} \right)^\beta dt, \quad z \in \mathcal{U}. \quad (11) \]

From (11) we obtain

\[ G'_{\alpha, \beta}(z) = \frac{1}{{\alpha}} z^{\frac{1}{\alpha} + \beta - 2} \left( \frac{g(z)}{z} \right)^\beta \]

and hence

\[ G''_{\alpha, \beta}(z) = \frac{1}{{\alpha}} \left( \frac{1}{{\alpha}} + \beta - 2 \right) z^{\frac{1}{\alpha} + \beta - 3} \left( \frac{g(z)}{z} \right)^\beta + \]

\[ + \frac{1}{{\alpha}} z^{\frac{1}{\alpha} + \beta - 2} \beta \left( \frac{g(z)}{z} \right)^{\beta - 1} \frac{g'(z) - g(z)}{z^2}. \]

We obtain

\[ (1 - |z|^2) \left| \frac{zG''_{\alpha, \beta}(z)}{G'_{\alpha, \beta}(z)} \right| \leq \frac{1}{|\alpha| + \beta - 2} + |\beta| \left| \frac{zg'(z)}{g(z)} - 1 \right|, \quad (12) \]

for all \( z \in \mathcal{U} \).

From (8) and (12) we have

\[ (1 - |z|^2) \left| \frac{zG''_{\alpha, \beta}(z)}{G'_{\alpha, \beta}(z)} \right| \leq 1, \quad z \in \mathcal{U}. \quad (13) \]

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From (13), using the Theorem 1 we obtain $G_{\alpha,\beta}(z) \in S$ and hence $z^{\alpha - 1}J_{\alpha,\beta}(z) \in S$.

We have

$$z^{\alpha - 1}J_{\alpha,\beta}(z) = z + b_2 z^2 + \cdots$$

(14)

and we obtain $J_{\alpha,\beta}(z)$ is of the form

$$J_{\alpha,\beta}(z) = z^{2 - \frac{1}{\alpha}} + b_2 z^{3 - \frac{1}{\alpha}} + \cdots, \quad z \in U.$$  

(15)

From the Theorem 2 we have the next corollaries.

**Corollary 3.** Let the function $g \in A$, $g(z) = z + a_2 z^2 + \cdots$.

If

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| \leq 1, \quad z \in U,$$

(16)

then $A(z) \in S$ and $A$ is of the form

$$A(z) = z + b_2 z^2 + \cdots, \quad z \in U.$$  

(17)

**Proof.** For $\alpha = 1$, $\beta = 1$, from Theorem 2 we obtain the Corollary 3.

**Corollary 4.** Let the function $g \in A$, $g(z) = z + a_2 z^2 + \cdots$ and $\beta$ be a complex number, $|\beta| < 1$.

If

$$\left| \frac{zg'(z)}{g(z)} - 1 \right| \leq \frac{1 - |\beta|}{|\beta|}, \quad z \in U,$$

(18)

then $J_{\frac{1}{2},\beta}(z)$ is of the form

$$J_{\frac{1}{2},\beta}(z) = 1 + b_2 z + \cdots, \quad z \in U.$$  

(19)

If $\text{Re} J_{\frac{1}{2},\beta}(z) > 0$, then $J_{\frac{1}{2},\beta}(z) \in P$.

**Proof.** For $\alpha = \frac{1}{2}$, from Theorem 2 we have the Corollary 4.
**Corollary 5.** Let \( \alpha \) be a real number, \( \alpha \in \left( \frac{1}{2}, 1 \right] \) and the function 
\( g \in \mathcal{A}, \ g(z) = z + a_2 z^2 + \cdots \).

If

\[
\left| \frac{z g'(z)}{g(z)} - 1 \right| \leq 2 - \frac{1}{\alpha}, \quad z \in \mathcal{U}, \tag{20}
\]

then the integral operator Pascu satisfies the properties:

\[
z^{\frac{1}{\alpha}-1} J_{\alpha,1}(z) \in \mathcal{S}, \quad z \in \mathcal{U}. \tag{21}
\]

and

\[
J_{\alpha,1}(z) = z^{2-\frac{1}{\alpha}} + b_2 z^{3-\frac{1}{\alpha}} + \cdots, \quad z \in \mathcal{U}. \tag{22}
\]

**Proof.** For \( \beta = 1 \), from Theorem 2 we obtain the Corollary 5.

**References**


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