FIXED COEFFICIENTS FOR A NEW SUBCLASS OF UNIFORMLY SPIRALLIKE FUNCTIONS

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Abstract. The main objective of this paper is to give several properties of the new subclass with negative coefficients and with fixed second coefficients.

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1. Introduction and Definitions

Let $S$ denote the class of functions of the form $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ which are analytic and univalent in the open unit disc $U = \{ z \in \mathbb{C} : |z| \leq 1 \}$. Also let $S^*$ and $C$ denote the subclasses of $S$ that are respectively, starlike and convex. Motivated by certain geometric conditions, Goodman [1, 2] introduced an interesting subclass of starlike functions called uniformly starlike functions denoted by UST and an analogous subclass of convex functions called uniformly convex functions, denoted by UCV. From [5, 7] we have

$$f \in UCV \iff \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \geq \left| \frac{zf''(z)}{f'(z)} \right|, z \in U.$$ 

In [7], Ronning introduced a new class $S_p$ of starlike functions which has more manageable properties. The classes UCV and $S_p$ were further extended by Kanas and Wisniowska in [3, 4] as $k - UCV(\alpha)$ and $k - ST(\alpha)$. The classes of uniformly spirallike and uniformly convex spirallike were introduced by Ravichandran et al [6]. This was further generalized in [10] as $UCSP(\alpha, \beta)$. In [11], Herb Silverman introduced the subclass $T$ of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n,$$ 

(1)
which are analytic and univalent in the unit disc $U$. Motivated by [12], new sub-classes with negative coefficients $UCSPT(\alpha, \beta)$ and $SP_T(\alpha, \beta)$ were introduced and studied in [9]. A function $f(z)$ defined by (1) is in $UCSPT(\alpha, \beta)$ if
\[
\Re \left\{ e^{-ia} \left( 1 + \frac{zf''(z)}{f'(z)} \right) \right\} \geq \left| \frac{zf''(z)}{f'(z)} \right| + \beta,
\]
(2)

$|\alpha| < \frac{\pi}{2}$, $0 \leq \beta < 1$. For the class $UCSPT(\alpha, \beta)$, [9] proved the following lemma.

**Lemma 1.** A function $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$ is in $UCSPT(\alpha, \beta)$ if and only if
\[
\sum_{n=2}^{\infty} (2n - \cos \alpha - \beta) n a_n \leq \cos \alpha - \beta.
\]
(3)

Using (1), the functions $f(z) \in UCSPT(\alpha, \beta)$ will satisfy
\[
a_2 \leq \frac{(\cos \alpha - \beta)}{2(4 - \cos \alpha - \beta)}.
\]
(4)

Let $UCSPT_c(\alpha, \beta)$ be the class of functions in $UCSPT(\alpha, \beta)$ of the form
\[
f(z) = z - \frac{c(\cos \alpha - \beta)z^2}{2(4 - \cos \alpha - \beta)} - \sum_{n=3}^{\infty} a_n z^n,
\]
(5)

$(a_n \geq 0)$, where $0 \leq c \leq 1$. When $c = 1$ we get $UCSPT_1(\alpha, \beta) = UCSPT(\alpha, \beta)$.

2. Coefficient Estimat

**Theorem 2.** The function $f(z)$ defined by (5) belongs to $UCSPT_c(\alpha, \beta)$ if and only if
\[
\sum_{n=3}^{\infty} (2n - \cos \alpha - \beta) n a_n \leq (1 - c)(\cos \alpha - \beta).
\]
(6)

The result is sharp.

**Proof.** Taking
\[
a_2 = \frac{c(\cos \alpha - \beta)}{2(4 - \cos \alpha - \beta)}, 0 \leq c \leq 1,
\]
(7)
in (3) we get the required result. Also the result is sharp for the function
\[
f(z) = z - \frac{c(\cos \alpha - \beta)z^2}{2(4 - \cos \alpha - \beta)} - \frac{(1 - c)(\cos \alpha - \beta)z^n}{n(2n - \cos \alpha - \beta)}, (n \geq 3).
\]
(8)
Corollary 3. If $f(z)$ defined by (5) is in the class $UCSPT_c(\alpha, \beta)$ then,

$$ a_n \leq \frac{(1 - c)(\cos \alpha - \beta)}{n(2n - \cos \alpha - \beta)}, (n \geq 3). \quad (9) $$

The result is sharp for the function $f(z)$ given in (8).

3. Closure Theorems

Theorem 4. The class $UCSPT_c(\alpha, \beta)$ is closed under convex linear combination.

Proof. Let $f(z)$ defined by (5) be in $UCSPT_c(\alpha, \beta)$. Now define $g(z)$ by

$$ g(z) = z - \frac{c(\cos \alpha - \beta)z^2}{2(4 - \cos \alpha - \beta)} - \sum_{n=3}^{\infty} b_n z^n, (b_n \geq 0). \quad (10) $$

If $f(z)$ and $g(z)$ belong to $UCSPT_c(\alpha, \beta)$ then it is enough to prove that the function $H(z)$ defined by

$$ H(z) = \lambda f(z) + (1 - \lambda)g(z), (0 \leq \lambda \leq 1) \quad (11) $$

is also in $UCSPT_c(\alpha, \beta)$.

$$ H(z) = z - \frac{c(\cos \alpha - \beta)z^2}{2(4 - \cos \alpha - \beta)} - \sum_{n=3}^{\infty} (\lambda a_n + (1 - \lambda)b_n) z^n. \quad (12) $$

Using theorem (2.1) we get

$$ \sum_{n=3}^{\infty} (2n - \cos \alpha - \beta)n(\lambda a_n + (1 - \lambda)b_n) \leq (1 - c)(\cos \alpha - \beta). \quad (13) $$

Hence $H(z)$ is in $UCSPT_c(\alpha, \beta)$. Thus $UCSPT_c(\alpha, \beta)$ is closed under convex linear combination.

Theorem 5. Let the functions

$$ f_j(z) = z - \frac{c(\cos \alpha - \beta)z^2}{2(4 - \cos \alpha - \beta)} - \sum_{n=3}^{\infty} a_{n,j} z^n, (a_{n,j} \geq 0), \quad (14) $$

be in the class $UCSPT_c(\alpha, \beta)$ for every $j = 1, 2, \ldots m$. Then the function $F(z)$ defined by

$$ F(z) = \sum_{j=1}^{m} d_j f_j(z), (d_j \geq 0), \quad (15) $$

be in the class $UCSPT_c(\alpha, \beta)$. Thus $UCSPT_c(\alpha, \beta)$ is closed under convex linear combination.
is also in the same class $UCSPT_c(\alpha, \beta)$ where
\[ \sum_{j=1}^{m} d_j = 1. \] (16)

Proof. Using (14) and (16) in (15) we have
\[ F(z) = z - \frac{c(\cos \alpha - \beta)z^2}{2(4 - \cos \alpha - \beta)} - \sum_{n=3}^{\infty} \left[ \sum_{j=1}^{m} d_j a_{n,j} \right] z^n. \] (17)

Each $f_j(z) \in UCSPT_c(\alpha, \beta)$ for $j = 1, 2, \ldots, m$, theorem (2.1) gives
\[ \sum_{n=3}^{\infty} (2n - \cos \alpha - \beta) a_{n,j} \leq (1 - c)(\cos \alpha - \beta), \] (18)
for $j = 1, 2, \ldots, m$. Hence we get
\[ \sum_{n=3}^{\infty} n(2n - \cos \alpha - \beta) a_{n,j} \leq (1 - c)(\cos \alpha - \beta). \]

This implies $F(z) \in UCSPT_c(\alpha, \beta)$, by theorem (2.1).

**Theorem 6.** Let
\[ f_2(z) = z - \frac{c(\cos \alpha - \beta)z^2}{2(4 - \cos \alpha - \beta)} \] (19)
and
\[ f_n(z) = z - \frac{c(\cos \alpha - \beta)z^2}{2(4 - \cos \alpha - \beta)} - \frac{(1 - c)(\cos \alpha - \beta)z^n}{n(2n - \cos \alpha - \beta)}, \] (20)
for $n = 3, 4, \ldots$. Then $f(z)$ is in $UCSPT_c(\alpha, \beta)$ if and only if it can be expressed in the form
\[ f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z) \] (21)
where $\lambda_n \geq 0$ and $\sum_{n=2}^{\infty} \lambda_n = 1$.

Proof. First assume that $f(z)$ can be expressed in the form (3.12). Then we have
\[ f(z) = z - \frac{c(\cos \alpha - \beta)z^2}{2(4 - \cos \alpha - \beta)} - \sum_{n=3}^{\infty} \frac{(1 - c)(\cos \alpha - \beta)\lambda_n z^n}{n(2n - \cos \alpha - \beta)}. \] (22)
But
\[ \sum_{n=3}^{\infty} \frac{(1 - c)(\cos \alpha - \beta)}{n(2n - \cos \alpha - \beta)} \lambda_n(2n - \cos \alpha - \beta) = (1 - c)(\cos \alpha - \beta)(1 - \lambda_2) \leq (1 - c)(\cos \alpha - \beta). \] (23)

Hence from (2.1) it follows that \( f(z) \in UCSPT_c(\alpha, \beta) \). Conversely, we assume that \( f(z) \) defined by (1.5) is in the class \( UCSPT_c(\alpha, \beta) \). Then by using (2.4), we get
\[ a_n \leq \frac{(1 - c)(\cos \alpha - \beta)}{n(2n - \cos \alpha - \beta)}, \quad (n = 3, 4, \ldots). \]

Taking \( \lambda_n = \frac{n(2n - \cos \alpha - \beta)\bar{a}_n}{(1 - c)(\cos \alpha - \beta)} \), \((n = 3, 4, \ldots)\) and \( \lambda_2 = 1 - \sum_{n=3}^{\infty} \lambda_n \), we have (21). Hence the proof of theorem (6) is complete.

**Corollary 7.** The extreme points of the class \( UCSPT_c(\alpha, \beta) \) are the functions \( f_n(z), (n \geq 2) \) given by theorem (6).

### 4. Distortion Theorems

In order to obtain the distortion bounds for the function \( f(z) \in UCSPT_c(\alpha, \beta) \), we need the following lemmas.

**Lemma 8.** Let the function \( f_3(z) \) be defined by
\[ f_3(z) = z - \frac{c(\cos \alpha - \beta)z^2}{2(4 - \cos \alpha - \beta)} - \frac{(1 - c)(\cos \alpha - \beta)z^3}{3(6 - \cos \alpha - \beta)}. \] (24)

Then, for \( 0 \leq r < 1 \) and \( 0 \leq c < 1 \),
\[ |f_3(re^{i\theta})| \geq r - \frac{c(\cos \alpha - \beta)r^2}{2(4 - \cos \alpha - \beta)} - \frac{(1 - c)(\cos \alpha - \beta)r^3}{3(6 - \cos \alpha - \beta)}, \] (25)
with equality for \( \theta = 0 \). For either \( 0 \leq c < c_0 \) and \( 0 \leq r \leq r_0 \) or \( c_0 \leq c \leq 1 \),
\[ |f_3(re^{i\theta})| \leq r + \frac{c(\cos \alpha - \beta)r^2}{2(4 - \cos \alpha - \beta)} - \frac{(1 - c)(\cos \alpha - \beta)r^3}{3(6 - \cos \alpha - \beta)}, \] (26)
with equality for \( \theta = \pi \). Further, for \( 0 \leq c < c_0 \) and \( r_0 \leq r < 1 \),
\[ |f_3(re^{i\theta})| \leq r \left[ 1 + \frac{9c^2(\cos \alpha - \beta)(6 - \cos \alpha - \beta)}{16(1 - c)(4 - \cos \alpha - \beta)^2} \right] \]
\[ + r^2(\cos \alpha - \beta) \left[ \frac{2(1 - c)}{3(6 - \cos \alpha - \beta)} - \frac{c^2(\cos \alpha - \beta)}{8(4 - \cos \alpha - \beta)^2} \right] \]
\[ + r^4(1 - c)(\cos \alpha - \beta)^2 \left[ \frac{(1 - c)}{9(6 - \cos \alpha - \beta)} + \frac{c^2(\cos \alpha - \beta)}{16(4 - \cos \alpha - \beta)^2} \right]^{1/2}, \]
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Proof. We employ the techniques used by Silverman and Silvia\[12\]. Since

\[ c_0 = \frac{1}{2(\cos \alpha - \beta)} \left[ (11 \cos \alpha + 11\beta - 49) \right. \\
\left. + \sqrt{(49 - 11 \cos \alpha - 11 \cos \beta)^2 - 32(\cos \alpha - \beta)(4 - \cos \alpha - \beta)} \right] \]

(27)

and

\[ r_0 = \frac{1}{c(1 - c)(\cos \alpha - \beta)} \left[ -4(1 - c)(4 - \cos \alpha - \beta) \right. \\
\left. + \sqrt{16(1 - c)^2(4 - \cos \alpha - \beta)^2 + 3c^2(1 - c)(6 - \cos \alpha - \beta)(\cos \alpha - \beta)} \right]. \]

(28)

we see that \( \frac{\partial |f_3(re^{i\theta})|^2}{\partial \theta} = 0 \), for \( \theta_1=0, \theta_2=\pi \) and

\[ \theta_3 = \cos^{-1} \left[ \frac{(\cos \alpha - \beta)c(1 - c)r^2 - 3c(6 - \cos \alpha - \beta)}{8(1 - c)(4 - \cos \alpha - \beta)r} \right], \]

(30)

since \( \theta_3 \) is a valid root only when \(-1 \leq \cos \theta_3 \leq 1\). Hence there is a third root if and only if \( r_0 \leq r < 1 \) and \( 0 \leq c \leq c_0 \). Thus the results of the theorem follow by comparing the extremal values \( |f_3(re^{i\theta_k})|, (k=1,2,3) \) on the appropriate intervals.

**Lemma 9.** Let the function \( f_n(z) \) be defined by (20) and \( n \geq 4 \). Then

\[ |f_n(re^{i\theta})| \leq |f_n(-r)|. \]

(31)

Proof. Since \( f_n(z) = z - \frac{c(\cos \alpha - \beta)z^2}{2(4 - \cos \alpha - \beta)} \left[ -\frac{(1-c)(\cos \alpha - \beta)z^n}{n(2n-\cos \alpha - \beta)} \right] \) and \( \frac{r^n}{n} \) is a decreasing function of \( n \), we have

\[ |f_n(re^{i\theta})| \leq r + \frac{c(\cos \alpha - \beta)r^2}{2(4 - \cos \alpha - \beta)} + \frac{(1-c)(\cos \alpha - \beta)r^n}{n(2n-\cos \alpha - \beta)} \]

\[ \leq r + \frac{c(\cos \alpha - \beta)r^2}{2(4 - \cos \alpha - \beta)} + \frac{(1-c)(\cos \alpha - \beta)r^4}{4(8 - \cos \alpha - \beta)} = -f_4(-r), \]

which gives (31).
**Theorem 10.** Let the function $f(z)$ defined by (5) belong to the class $UCSPT_c(\alpha, \beta)$. Then for $0 \leq r < 1$,
\[
|f(re^{i\theta})| \geq r - \frac{c(\cos \alpha - \beta)r^2}{2(4 - \cos \alpha - \beta)} - \frac{(1 - c)(\cos \alpha - \beta)r^3}{3(6 - \cos \alpha - \beta)},
\]
with equality for $f_3(z)$ at $z=r$ and
\[
|f(re^{i\theta})| \leq \max\{\max_{\theta}|f_3(re^{i\theta})|, -f_4(-r)\},
\]
where $\max_{\theta}|f_3(re^{i\theta})|$ is given by lemma 4.1.

The proof is obtained by comparing the bounds of lemma 4.1 and lemma 4.2.

**Corollary 11.** Let the function $f(z)$ be defined by (1) be in the class $UCSPT(\alpha, \beta)$. Then for $|z| = r < 1$, we have
\[
r - \frac{(\cos \alpha - \beta)r^2}{2(4 - \cos \alpha - \beta)} \leq |f(z)| \leq r + \frac{(\cos \alpha - \beta)r^2}{2(4 - \cos \alpha - \beta)}.
\]
The result is sharp.

**Corollary 12.** Let the function $f(z)$ be defined by (5) be in the class $UCSPT_c(\alpha, \beta)$. Then the disk $|z| < 1$ is mapped onto a domain that contains the disk
\[
|w| < \frac{6(6 - \cos \alpha - \beta)(4 - \cos \alpha - \beta) - (\cos \alpha - \beta)(8 + 10c + (c + 2)(\cos \alpha - \beta))}{6(4 - \cos \alpha - \beta)(6 - \cos \alpha - \beta)}.
\]
The result is sharp with the extremal function
\[
f_3(z) = z - \frac{c(\cos \alpha - \beta)z^2}{2(4 - \cos \alpha - \beta)} - \frac{(1 - c)(\cos \alpha - \beta)z^3}{3(6 - \cos \alpha - \beta)}.
\]

**Proof.** The result follows by letting $r \to 1$ in theorem 4.3.

**Lemma 13.** Let the function $f_3(z)$ be defined by (24). Then for $0 \leq r < 1$ and $0 \leq c \leq 1$,
\[
|f_3'(re^{i\theta})| \geq 1 - \frac{c(\cos \alpha - \beta)r}{(4 - \cos \alpha - \beta)} - \frac{(1 - c)(\cos \alpha - \beta)r^2}{(6 - \cos \alpha - \beta)},
\]
with equality for $\theta = 0$. For either $0 \leq c < c_1$ and $0 \leq r \leq r_1$ or $c_1 \leq c \leq 1$,
\[
|f_3'(re^{i\theta})| \leq 1 + \frac{c(\cos \alpha - \beta)r}{(4 - \cos \alpha - \beta)} - \frac{(1 - c)(\cos \alpha - \beta)r^2}{(6 - \cos \alpha - \beta)},
\]

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with equality for $\theta = \pi$. Further, $0 \leq c < c_1$ and $r_1 \leq r < 1$,

$$|f'_3(re^{i\theta})| \leq \left\{ 1 + \frac{c^2(\cos \alpha - \beta)(6 - \cos \alpha - \beta)}{4(1 - c)(4 - \cos \alpha - \beta)^2} \right\}$$

$$+ (\cos \alpha - \beta) \left[ \frac{2(1 - c)}{6 - \cos \alpha - \beta} \right] r^2$$

$$+ \left\{ \frac{1 - c}{6 - \cos \alpha - \beta} \left[ \frac{(1 - c)}{6 - \cos \alpha - \beta} + \frac{c^2(\cos \alpha - \beta)}{4(4 - \cos \alpha - \beta)^2} \right] r^4 \right\}^{1/2},$$

with equality for

$$\theta = \cos^{-1} \left[ \frac{(1 - c)(\cos \alpha - \beta)r^2 - c(6 - \cos \alpha - \beta)}{4(1 - c)r(4 - \cos \alpha - \beta)} \right],$$

where

$$c_1 = -(22 - 6 \cos \alpha - 4\beta) + \sqrt{(22 - 6 \cos \alpha - 4\beta)^2 + 16(4 - \cos \alpha - \beta)(\cos \alpha - \beta)}$$

and

$$r_1 = \frac{1}{c(1 - c)(\cos \alpha - \beta)} \left\{ -2(1 - c)(4 - \cos \alpha - \beta) $$

$$+ \sqrt{4(1 - c)^2(4 - \cos \alpha - \beta)^2 - c^2(1 - c)(\cos \alpha - \beta)(6 - \cos \alpha - \beta)} \right\}.$$

The proof of lemma(4.4) is given in the same way as lemma(4.1).

**Theorem 14.** Let the function $f(z)$ defined by (1.5) be in the class $UCSPT_c(\alpha, \beta)$. Then for $0 \leq r < 1$,

$$|f'(re^{i\theta})| \geq 1 - \frac{c(\cos \alpha - \beta)r}{4 - \cos \alpha - \beta} - \frac{(1 - c)(\cos \alpha - \beta)r^2}{6 - \cos \alpha - \beta},$$

with equality for $f'_3(z)$ at $z=r$ and

$$\left| f'(re^{i\theta}) \right| \leq \max \{ \max_\theta \left| f'_3(re^{i\theta}) \right|, f'_4(-r) \},$$

where $\max_\theta \left| f'_3(re^{i\theta}) \right|$ is given by lemma (4.4).

Remark: For $c=1$ in theorem 6 we obtain:

**Corollary 15.** Let the function $f(z)$ defined by (1.1) be in the class $UCSPT(\alpha, \beta)$. Then for $|z|=r < 1$, we have

$$1 - \frac{(\cos \alpha - \beta)r}{4 - \cos \alpha - \beta} \leq |f'(z)| \leq 1 + \frac{(\cos \alpha - \beta)r}{4 - \cos \alpha - \beta},$$

the result is sharp.
5. Radii of starlikeness and convexity

**Theorem 16.** Let the function $f(z)$ defined by (5) be in the class $UCSPT_c(\alpha, \beta)$. Then $f(z)$ is starlike of order $\rho (0 \leq \rho < 1)$ in the disc $|z| < r_1(\alpha, \beta, c, \rho)$ where $r_1(\alpha, \beta, c, \rho)$ is the largest value for which

$$\frac{c(\cos \alpha - \beta)(2 - \rho)r}{2(4 - \cos \alpha - \beta)} + \frac{(1 - c)(\cos \alpha - \beta)(n - \rho)r^{n-1}}{n(2n - \cos \alpha - \beta)} \leq 1 - \rho,$$

for $n \geq 3$. The result is sharp with the extremal function

$$f_n(z) = z - c(\cos \alpha - \beta)\frac{z^2}{2(4 - \cos \alpha - \beta)} - \frac{(1 - c)(\cos \alpha - \beta)z^n}{n(2n - \cos \alpha - \beta)},$$

for some $n$.

**Proof.** It suffices to show that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho, \quad (o \leq \rho < 1),$$

for $|z| < r_1(\alpha, \beta, c, \rho)$. Note that

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \leq \frac{c(\cos \alpha - \beta)r}{2(4 - \cos \alpha - \beta)} + \sum_{n=3}^{\infty} (n - 1)a_nr^{n-1} - \frac{1 - c(\cos \alpha - \beta)r}{2(4 - \cos \alpha - \beta)} - \sum_{n=3}^{\infty} a_nr^{n-1} \leq 1 - \rho,$$

for $|z| \leq r$ if and only if

$$\frac{c(\cos \alpha - \beta)(2 - \rho)r}{2(4 - \cos \alpha - \beta)} + \sum_{n=3}^{\infty} (n - \rho)a_nr^{n-1} \leq 1 - \rho.$$

Since $f(z)$ is in $UCSPT_c(\alpha, \beta)$ from (2.1) we may take

$$a_n = \frac{(1 - c)(\cos \alpha - \beta)\lambda_n}{n(2n - \cos \alpha - \beta)}, \quad (n \geq 3),$$

where $\lambda_n \geq 0 (n \geq 3)$ and $\sum_{n=3}^{\infty} \lambda_n \leq 1$. For each fixed $r$, we choose the positive integer $n_0 = n_0(r)$ for which $\frac{(n-\rho)r^{n-1}}{n}$ is maximal. Then it follows that

$$\sum_{n=3}^{\infty} (n - \rho)a_nr^{n-1} \leq \frac{(1 - c)(\cos \alpha - \beta)(n_0 - \rho)r^{n_0-1}}{n_0(2n_0 - \cos \alpha - \beta)}.$$
Hence \( f(z) \) is starlike of order \( \rho \) in \( |z| < r_1(\alpha, \beta, c, \rho) \) provided that
\[
\frac{c(\cos \alpha - \beta)(2 - \rho)r}{2(4 - \cos \alpha - \beta)} + \frac{(1 - c)(\cos \alpha - \beta)(n_0 - \rho)r^{n_0-1}}{n_0(2n_0 - \cos \alpha - \beta)} \leq 1 - \rho.
\]
We find the value \( r_0 = r_0(\alpha, \beta, c, \rho) \) and the corresponding integer \( n_0(\rho) \) so that
\[
\frac{c(\cos \alpha - \beta)(2 - \rho)r}{2(4 - \cos \alpha - \beta)} + \frac{(1 - c)(\cos \alpha - \beta)(n_0 - \rho)r^{n_0-1}}{n_0(2n_0 - \cos \alpha - \beta)} = 1 - \rho.
\]
Then this value \( r_0 \) is the radius of starlikeness of order \( \rho \) for functions \( f(z) \) belonging to the class \( UCSP_{c}(\alpha, \beta) \).

We prove the following theorem concerning the radius of convexity of order \( \rho \) for functions in the class \( UCSP_{c}(\alpha, \beta) \).

**Theorem 17.** Let the function \( f(z) \) be defined by (5) be in the class \( UCSP_{c}(\alpha, \beta) \). Then \( f(z) \) is convex of order \( \rho (0 \leq \rho < 1) \) in the disc \( |z| < r_2(\alpha, \beta, c, \rho) \), where \( r_2(\alpha, \beta, c, \rho) \) is the largest value for which
\[
\frac{c(\cos \alpha - \beta)(2 - \rho)r}{(4 - \cos \alpha - \beta)} + \frac{(1 - c)(\cos \alpha - \beta)(n - \rho)r^{n-1}}{(2n - \cos \alpha - \beta)} \leq 1 - \rho,
\]
for \( n \geq 3 \). The result is sharp for the function \( f(z) \) given by (33).

6. The class \( UCSP_{c_n,N}(\alpha, \beta) \)

We now fix finitely many coefficients instead of fixing just the second coefficients. Let \( UCSP_{c_n,N}(\alpha, \beta) \) denote the class of functions in \( UCSP_{c}(\alpha, \beta) \) of the form
\[
f(z) = z - \sum_{n=2}^{N} \frac{c_n(\cos \alpha - \beta)z^n}{n(2n - \cos \alpha - \beta)} - \sum_{n=N+1}^{\infty} a_n z^n,
\]
where \( 0 \leq \sum_{n=2}^{N} c_n = c \leq 1 \). Note that \( UCSP_{c_n,2}(\alpha, \beta) = UCSP_{c}(\alpha, \beta) \).

**Theorem 18.** The extreme points of the class \( UCSP_{c_n,N}(\alpha, \beta) \) are
\[
z - \sum_{n=2}^{N} \frac{c_n(\cos \alpha - \beta)z^n}{n(2n - \cos \alpha - \beta)}
\]
and
\[
z - \sum_{n=2}^{N} \frac{c_n(\cos \alpha - \beta)z^n}{n(2n - \cos \alpha - \beta)} - \frac{(1 - c)(\cos \alpha - \beta)z^n}{n(2n - \cos \alpha - \beta)},
\]
for \( n=N+1, N+2, \ldots \).
The characterization of the extreme points enables us to solve the standard extremal problems in the same manner as was done in $UCSPT_e(\alpha, \beta)$. The details are omitted.

References


