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ON $(\in, \in \vee Q_K)$-FUZZY KU-IDEALS OF KU-ALGEBRAS

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Abstract. We define $(\in, \in \vee q_k)$-fuzzy KU-ideals of KU-algebras and then some related results have been provided.

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1. Introduction

Fuzzy set theory was first introduced by Zadeh [9] in 1965. The concept of KU-algebras was given by Prabpayak and Leerawat [6, 7] in 2009. The study of fuzzy KU-algebras was first initiated by Mostafa et al. [4]. They also studied KU-algebras in terms of interval-valued fuzzy sets in [5]. Akram et al. [2] introduced the concept of interval valued $(\widetilde{\theta}, \widetilde{\delta})$-fuzzy KU-ideals of KU-algebras and Yaqoob et al. [8] introduced the concept of cubic KU-ideals of KU-algebras.

In this article, we study the concept of $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-subalgebra and $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of KU-algebras.

2. Review of literature

Now we recall some known concepts related to KU-algebra from the literature which will be helpful in further study of this article.

Definition 1. *[6] By a KU-algebra we mean an algebra with a binary operation "$*$", satisfying the following conditions:*

$(i) : (l * m) * [(m * n) * (l * n)] = 0,$
$(ii) : l * 0 = 0, \forall\, l \in \mathbf{X},$
$(iii) : 0 * l = l, \forall\, l \in \mathbf{X},$
$(iv) : l * m = 0 = m * l$ *implies* $l = m, \forall\, l, m, n \in \mathbf{X}.$

We call it an algebra $(\mathbf{X}, *, 0)$ of type $(2, 0)$. In further study of this article we denote a KU-algebra by $\mathbf{X}$. We define " $\leq$ " in $\mathbf{X}$ as if $l \leq m$ if and only if $m * l = 0$.

Definition 2. *[7] A subset S of KU-algebra $\mathbf{X}$ is called KU-subalgebra of $\mathbf{X}$ if $l * m \in S$, whenever $l, m \in S$.*

Definition 3. *[7] A non-empty subset A of a KU-algebra $\mathbf{X}$ is called a KU-ideal of $\mathbf{X}$ if it satisfies the following conditions:*
(1) $0 \in A$,
(2) $l * (m * n) \in A$, $m \in A$ *implies* $l * n \in A$, *for all* $l, m, n \in \mathbf{X}$.

Definition 4. *Fuzzy point in a KU-algebra $\mathbf{X}$ is defined as*

$$\psi(z) = \begin{cases} t & \text{if } z = x \\ 0 & \text{otherwise} \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by x_t. The notation $x_t \alpha \psi$ means that $\psi(x) \geq t$ and $x_t q \psi$ means that $\psi(x) + t > 1$ and $x_t q_k \psi \Rightarrow \psi(x) + t + k > 1$, while the notation $x_t \overline{\alpha} \psi \Rightarrow x_t \alpha \psi$ does not hold.

3. $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideals in KU-algebras

In this section we study the properties of $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideals.

Definition 5. *A fuzzy subset $\psi : \mathbf{X} \to [0, 1]$ is said to be $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-subalgebra of $\mathbf{X}$ if it satisfy the following conditions:*
(i) $[x,\ t] \in \psi \Rightarrow [0,\ t] \in \vee \mathbf{q}_k \psi$,
(ii) $[x * y,\ t_1] \in \psi, [y,\ t_2] \in \psi \Rightarrow [x,\ t_1 \wedge t_2] \in \vee \mathbf{q}_k \psi$.

Example 1. *Let us consider the KU-algebra $(X, *, 0)$ in which $*$ is defined as follows:*

$*$	0	l	m	n	p
0	0	l	m	n	p
l	0	0	m	n	p
m	0	l	0	n	n
n	0	0	m	0	m
p	0	0	0	0	0

Let us define $\psi(0) = 0.9$, $\psi(l) = 0.8$, $\psi(m) = 0.7$, $\psi(n) = 0.6$, $\psi(p) = 0.5$. Let $t = 0.49$ and $k = 0.48$ then by routine calculation it is clear that ψ is an $(\in, \in \vee \mathbf{q}_{0.48})$-fuzzy KU-subalgebra of $\mathbf{X}$.

Definition 6. *A fuzzy subset $\psi : \mathbf{X} \to [0, 1]$ is said to be an $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of $\mathbf{X}$ if it satisfy the following conditions:*
(i) $[x,\ t] \in \psi \Rightarrow [0,\ t] \in \vee \mathbf{q}_k \psi$,
(ii) $[x * (y * z),\ t_1] \in \psi, [y,\ t_2] \in \psi \Rightarrow [x * z,\ t_1 \wedge t_2] \in \vee \mathbf{q}_k \psi$.

Theorem 1. *A fuzzy subset* ψ *of* $\mathbf{X}$ *is said to be an* $(\in, \in \vee \mathbf{q}_k)$*-fuzzy KU-ideal of* X *if and only if it satisfy:*

(*i*) $\psi(0) \geq \min\left\{\psi(x), \frac{1-k}{2}\right\}$,

(*ii*) $\psi(x * z) \geq \min\left\{\psi(x * (y * z)),\ \psi(y),\ \frac{1-k}{2}\right\}\ \ \forall x, y, z \in \mathbf{X}$.

Proof. Let ψ of $\mathbf{X}$ is an $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of $\mathbf{X}$. Let there exist some x, y, z in $\mathbf{X}$ such that

(*i*) $\psi(0) < \min\left\{\psi(x), \frac{1-k}{2}\right\}$,

(*ii*) $\psi(x * z) < \min\left\{\psi(x * (y * z)),\ \psi(y),\ \frac{1-k}{2}\right\}$.

Now consider (*i*) and if $\psi(x) < \frac{1-k}{2} \Rightarrow \psi(0) < \psi(x)$ and $\psi(0) < t \leq \psi(x)$ for some $t \in (0,1) \Rightarrow [x,\ t] \in \psi$ but $[0,\ t]\,\overline{\in}\psi$. Moreover $\psi(0) + t < 2t < 1 - k$ which implies that $[0,\ t]\,\overline{q}_k\psi$. Hence $[0,\ t]\,\overline{\in \vee q_k}\psi$, which contradicts the given hypothesis. Now if $\psi(x) \geq \frac{1-k}{2}$ then it will imply that $\left[x,\ \frac{1-k}{2}\right] \in \psi$ and then $\psi(0) < \frac{1-k}{2} \Rightarrow \left[0,\ \frac{1-k}{2}\right]\overline{\in}\psi$. Moreover if $\psi(0) + \frac{1-k}{2} < 1 - k \Rightarrow \left[0,\ \frac{1-k}{2}\right]\overline{q}_k\psi$ and consequently $\left[0,\ \frac{1-k}{2}\right]\overline{\in \vee q_k}\psi$, which contradicts the given hypothesis and thus $\psi(0) \geq \min\left\{\psi(x), \frac{1-k}{2}\right\}$. Now consider (*ii*) and if

$$\min\left\{\psi(x * (y * z)),\ \psi(y)\right\} < \frac{1-k}{2} \Rightarrow \psi(x * z) < \min\left\{\psi(x * (y * z)),\ \psi(y)\right\}$$

and for some $t \in (0,1)$ we have

$$\psi(x * z) < t \leq \min\left\{\psi(x * (y * z)),\ \psi(y)\right\}.$$

Which implies that $[x * (y * z), t] \in \psi$ and $[y, t] \in \psi$ but $[x * z, t]\,\overline{\in}\psi$. And if $\psi(x * z) + t < 2t < 1 - k$ and thus $[x * z, t]\,\overline{\mathbf{q}_k}\psi$. Consequently $[x * z, t]\,\overline{\in \vee \mathbf{q}_k}\psi$ which is contradiction and if $\min\left\{\psi(x * (y * z)),\ \psi(y)\right\} \geq \frac{1-k}{2}$ we get again $[x * z, t]\,\overline{\in \vee \mathbf{q}_k}\psi$, which again contradicts the given hypothesis and thus

$$\psi(x * z) \geq \min\left\{\psi(x * (y * z)),\ \psi(y),\ \frac{1-k}{2}\right\}.$$

Conversely assume that (*i*) and (*ii*) are valid and we have to prove that ψ of $\mathbf{X}$ is $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of $\mathbf{X}$. For this let $[x, t] \in \psi$ for $x \in \mathbf{X}$ and $t \in [0, 1]$. Which implies that $\psi(x) \geq t$. But $\psi(0) \geq \min\left\{\psi(x), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\}$. Now if $t > \frac{1-k}{2}$ then $\psi(0) \geq \frac{1-k}{2} \Rightarrow \psi(0) + t > 1 - k \Rightarrow [0,\ t]\,q_k\psi$ and if $t > \frac{1-k}{2}$ then it is obvious that $[0,\ t] \in \psi$, thus $[0,\ t] \in \vee \mathbf{q}_k\psi$. Hence $[x,\ t] \in \psi \Rightarrow [0,\ t] \in \vee \mathbf{q}_k\psi$. Similarly we can show that

$$[x * (y * z), t_1] \in \psi, [y,\ t_2] \in \psi \Rightarrow [x * z,\ t_1 \wedge t_2] \in \vee \mathbf{q}_k\psi.$$

This completes the proof.

Corollary 2. *A fuzzy subset ψ of $\mathbf{X}$ is said to be an $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-subalgebra of X if and only if it satisfies:*

(*i*) $\psi(0) \geq \min\left\{\psi(x), \frac{1-k}{2}\right\}$,

(*ii*) $\psi(x) \geq \min\left\{\psi(x * y),\ \psi(y),\ \frac{1-k}{2}\right\}\ \forall\ x, y \in \mathbf{X}$.

Proof. By putting $z = 0$ in the proof of the above theorem we can easily prove it.

Next we characterize $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of $\mathbf{X}$ in terms of level sets.

Theorem 3. *A fuzzy subset ψ of $\mathbf{X}$ is said to be $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of $\mathbf{X}$ if and only if the following set $U[\psi, t] = \{x \in \mathbf{X} |\ \psi(x) \geq t\}$ is a KU-ideal of $\mathbf{X}$ where $t \in (0, \frac{1-k}{2}]$.*

Proof. Assume that ψ of $\mathbf{X}$ is an $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of $\mathbf{X}$ and let $x \in U[\psi, t]$ which implies by definition that $\psi(x) \geq t$ for some $t \in (0, \frac{1-k}{2}]$. But $\psi(0) \geq \min\left\{\psi(x), \frac{1-k}{2}\right\} \geq \min\left\{t, \frac{1-k}{2}\right\} = t$, which implies that $0 \in U[\psi, t]$. Now again let $(x * (y * z)) \in U[\psi, t]$ and $y \in U[\psi, t]$ then by definition we get $\psi(x * (y * z)) \geq t$ and $\psi(y) \geq t$ but

$$\psi(x * z) \geq \min\left\{\psi(x * (y * z)),\ \psi(y),\ \frac{1-k}{2}\right\} \geq \min\left\{t,\ t,\ \frac{1-k}{2}\right\} = t,$$

which implies that $x * z \in U[\psi, t]$. Hence $U[\psi, t]$ is a KU-ideal of $\mathbf{X}$ where $t \in (0, \frac{1-k}{2}]$.

Conversely let $U[\psi, t]$ is a KU-ideal of $\mathbf{X}$ where $t \in (0, \frac{1-k}{2}]$ and we show that ψ of $\mathbf{X}$ is an $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of $\mathbf{X}$. For this let there exist some $t \in (0, \frac{1-k}{2}]$ such that $\psi(0) < t \leq \min\left\{\psi(x), \frac{1-k}{2}\right\}$ which implies that $x \in U[\psi, t]$ but $0 \overline{\in} U[\psi, t]$ which is contradiction and hence $\psi(0) \geq \min\left\{\psi(x), \frac{1-k}{2}\right\}$. Similarly we can prove that $\psi(x * z) \geq \min\left\{\psi(x * (y * z)),\ \psi(y),\ \frac{1-k}{2}\right\}$. This completes the proof.

Example 2. *Let us consider the KU-algebra $(\mathbf{X}, *, 0)$ in which $*$ is defined as follows*

$*$	0	l	m	n	p	q
0	0	l	m	n	p	q
l	0	0	m	m	p	q
m	0	0	0	l	p	q
n	0	0	0	0	p	q
p	0	0	0	l	0	q
q	0	0	0	0	0	0

Define a fuzzy subset ψ of $\mathbf{X}$ as $\psi(0) = 0.9$, $\psi(l) = 0.8$, $\psi(m) = 0.75$, $\psi(n) = 0.7$, $\psi(p) = 0.65$, $\psi(q) = 0.3$. *Then*

$$U[\psi, t] = \begin{cases} \mathbf{X} & \text{if } t \in (0, 0.3] \text{ for } k = 0.4 \\ \{0, l, m, n, p\} & \text{if } t \in (0.3, 0.4] \text{ for } k = 0.2 \end{cases}$$

As $\mathbf{X}$ *and* $\{0, l, m, n, p\}$ *are KU-ideals of* $\mathbf{X}$*, so by Theorem 3,* ψ *of* $\mathbf{X}$ *is* $(\in, \in \vee \mathbf{q}_k)$*-fuzzy KU-ideal of* $\mathbf{X}$.

Corollary 4. *A fuzzy subset* ψ *of* $\mathbf{X}$ *is said to be* $(\in, \in \vee \mathbf{q}_k)$*-fuzzy KU-subalgebra of* $\mathbf{X}$ *if and only if* $U[\psi, t] = \{x \in \mathbf{X} | \ \psi(x) \geq t\}$ *is a KU-subalgebra of* $\mathbf{X}$ *where* $t \in (0, \frac{1-k}{2}]$.

Proof. By putting $z = 0$ in the proof of the above theorem we can easily prove it.

Theorem 5. *Every* $(\in, \in)$*-fuzzy KU-subalgebra (resp., KU-ideal) implies* $(\in, \in \vee \mathbf{q}_k)$*-fuzzy KU-subalgebra (resp., KU-ideal) of* $\mathbf{X}$.

Proof. The proof is straightforward.

Definition 7. *A fuzzy subset* $\psi : \mathbf{X} \to [0, 1]$ *is said to be* $(\in, \mathbf{q}_k)$*-fuzzy KU-algebra of* $\mathbf{X}$ *if it satisfy the following conditions:*
(*i*) $[x, \ t] \in \psi \Rightarrow [0, \ t] \in \mathbf{q}_k \psi$,
(*ii*) $[x * y, \ t_1] \in \psi, [y, \ t_2] \in \psi \Rightarrow [x, \ t_1 \wedge t_2] \in \mathbf{q}_k \psi$.

Definition 8. *A fuzzy subset* $\psi : \mathbf{X} \to [0, 1]$ *is said to be* $(\in, \mathbf{q}_k)$*-fuzzy KU-ideal of* $\mathbf{X}$ *if it satisfy the following conditions:*
(*i*) $[x, \ t] \in \psi \Rightarrow [0, \ t] \in \mathbf{q}_k \psi$,
(*ii*) $[x * (y * z), \ t_1] \in \psi, [y, \ t_2] \in \psi \Rightarrow [x * z, \ t_1 \wedge t_2] \in \mathbf{q}_k \psi$.

Theorem 6. *Every* $(\in, \mathbf{q}_k)$*-fuzzy KU-subalgebra (resp., KU-ideal) implies* $(\in, \in \vee \mathbf{q}_k)$*-fuzzy KU-subalgebra (resp., KU-ideal) of* $\mathbf{X}$.

Proof. The proof is straightforward.

Example 3. *Let us consider the KU-algebra* $(\mathbf{X}, *, 0)$ *in which* $*$ *is defined as follows*

$*$	0	l	m	n	p
0	0	l	m	n	p
l	0	0	m	n	p
m	0	l	0	n	n
n	0	0	m	0	m
p	0	0	0	0	0

Define a fuzzy subset ψ *as*

$$\psi(x) = \begin{cases} 0.65 & \text{if } x = 0 \\ 0.74 & \text{if } x = l \\ 0.55 & \text{if } x \in \{m, n\} \\ 0.35 & \text{if } x = p \end{cases}$$

then ψ *is an* $(\in, \in \vee \mathbf{q}_k)$*-fuzzy KU-ideal of* $\mathbf{X}$ *for* $k = 0.2$ *but* ψ *is not an* $(\in, q_k)$ $-$*fuzzy KU-ideal of* $\mathbf{X}$ *because* $l_{0.7} \in \psi$ *but* $0_{0.7} \notin \psi$.

Theorem 7. *Let $\emptyset \neq A \subset \mathbf{X}$, the characteristic function ψ_A of A is an $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-subalgebra (resp., KU-ideal) of $\mathbf{X}$ if and only if A is a KU-subalgebra (resp., KU-ideal) of $\mathbf{X}$.*

Proof. Let A be a KU-ideal of $\mathbf{X}$, then it is obviously an $(\in, \in)$-fuzzy KU-ideal of $\mathbf{X}$ which then implies that A is an $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of $\mathbf{X}$.

Conversely assume that A is an $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of $\mathbf{X}$ and we show that A is a KU-ideal of $\mathbf{X}$. For this let $x * (y * z) \in A$, $y \in A$ then by definition $\psi_A(x * (y * z)) = 1$ and $\psi_A(y) = 1 \Rightarrow [(x * (y * z)), 1] \in \psi_A$ and $[y, 1] \in \psi_A$. But by hypothesis

$$\psi_A(x * z) \geq \min\left\{\psi_A(x * (y * z)),\ \psi_A(y),\ \frac{1-k}{2}\right\} = \min\left\{1, 1, \frac{1-k}{2}\right\} = \frac{1-k}{2}$$

and as $k \in [0, 1)$ so $\frac{1-k}{2} \neq 0$ and hence $\psi_A(x * z) \geq 1 \Rightarrow x * z \in A$. Moreover in the same way $\psi_A(0) \geq \min\left\{\psi_A(x),\ \frac{1-k}{2}\right\} = 1 \Rightarrow 0 \in A$. Hence A is a KU-ideal of $\mathbf{X}$. The other case can be seen in a similar way.

Theorem 8. *If $\{\psi_i : i \in \wedge\}$ be a family of $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-subalgebra (resp., KU-ideal) of $\mathbf{X}$ then so is their intersection $\psi = \underset{i\in\wedge}{\cap} \psi_i$.*

Proof. Let $\{\psi_i : i \in \wedge\}$ be a family of $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of $\mathbf{X}$ and we have to show that $\psi = \underset{i\in\wedge}{\cap} \psi_i$ is an $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of $\mathbf{X}$. For this let $[x,\ t] \in \psi$ and we have to show that $[0,\ t] \in \vee \mathbf{q}_k \psi$. Assume that $[0,\ t]\, \overline{\in \vee \mathbf{q}_k} \psi \Rightarrow \psi(0) < t$ and $\psi(0) + t < 1 - k$. Which implies that $\psi(0) < \frac{1-k}{2}$. Now let

$$\Delta_1 = \{i \in \wedge |\ \ [0,\ t] \in \vee \psi_i\}$$

and

$$\Delta_2 = \{i \in \wedge |\ \ [0,\ t]\, \mathbf{q}_k \psi_i\} \cap \{i \in \wedge\ |\ \ [0,\ t]\, \overline{\in} \psi_i\}$$

then we have $\wedge = \Delta_1 \cup \Delta_2$ and $\Delta_1 \cap \Delta_2 = \emptyset$. Let us suppose that if $\Delta_2 = \emptyset$, then

$$[0,\ t] \in \vee \psi_i \forall i \in \wedge \Rightarrow \psi_i(0) \geq t,\ \forall i \in \wedge \Rightarrow \psi(0) = \underset{i\in\wedge}{\cap} \psi_i(0) \geq t,$$

which contradicts the assumption and so $\Delta_2 \neq \emptyset$. Thus for each $i \in \Delta_2$ we have $[0,\ t] + t \geq 1 - k$ and $[0,\ t] < t$, it implies that $t > \frac{1-k}{2}$. Now since $[x,\ t] \in \psi \Rightarrow \psi(x) \geq t$ and we can write it as $\psi(x) \geq t > \frac{1-k}{2}\ \forall\ i \in \wedge$. Next assume that $\psi_i(0) < \frac{1-k}{2} = t_1$ and let $t_1 < r < \frac{1-k}{2}$ which implies $[x, r] \in \psi_i$, but $[0,\ t]\, \overline{\in \vee \mathbf{q}_k} \psi_i$, which contradicts the fact that ψ_i is given to be an $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of $\mathbf{X}$. Thus $\psi_i(0) \geq \frac{1-k}{2}$ $\forall\ i \in \wedge$ and hence $\psi(0) \geq \frac{1-k}{2} \Rightarrow [0,\ t] \in \vee \mathbf{q}_k \psi$. Similarly we can show that if $[x * (y * z),\ t_1] \in \psi$, $[y,\ t_2] \in \psi$, then it implies that $[x * z,\ t_1 \wedge t_2] \in \vee \mathbf{q}_k \psi$. Which shows that $\psi = \underset{i\in\wedge}{\cap} \psi_i$ is $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of $\mathbf{X}$. The other case can be seen in a similar way.

For any fuzzy subset ψ in $\mathbf{X}$ and $t \in (0,1]$, we denote $\psi_t = \{x \in \mathbf{X} | \ [x,t]\, \mathbf{q}_k \psi\}$ and $[\psi]_t = \{x \in \mathbf{X} | \ [x,t] \in \vee \mathbf{q}_k \psi\}$ then it is clear that $[\psi]_t = U[x,t] \cup \psi_t$.

Theorem 9. *Let $\psi : \mathbf{X} \to [0,1]$ be a fuzzy subset of $\mathbf{X}$ then ψ is an $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-subalgebra (resp., KU-ideal) of $\mathbf{X}$ if and only if $[\psi]_t$ is a KU-subalgebra (resp., KU-ideal) of $\mathbf{X}$ for all $t \in (0,1]$.*

Proof. Let us assume that ψ is an $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of $\mathbf{X}$ and we aim to prove that $[\psi]_t$ is a KU-ideal of $\mathbf{X}$ for all $t \in (0,1]$. For this let $x \in [\psi]_t = U[x,t] \cup \psi_t$, which then implies that $[x,t] \in \vee \mathbf{q}_k \psi \Rightarrow \psi(x) \geq t$ or $\psi(x) + t > 1-k$. As $\psi(0) \geq \min\left\{\psi(x), \frac{1-k}{2}\right\}$, so we have the following cases.

(i) If $\psi(x) \geq t$ and $t > \frac{1-k}{2}$ then $\psi(0) \geq \frac{1-k}{2} \Rightarrow \psi(0) + t > \frac{1-k}{2} + \frac{1-k}{2} = 1-k$, which implies that $[0,t]\, \mathbf{q}_k \psi$ and if $t \leq \frac{1-k}{2}$ then $\psi(0) \geq t \Rightarrow [0,t] \in \psi$. Hence $[0,t] \in \vee \mathbf{q}_k \psi$.

(ii) If $\psi(x) + t > 1-k$ and $t > \frac{1-k}{2}$ then $\psi(0) \geq (1-k-t) \wedge \frac{1-k}{2} \Rightarrow \psi(0) \geq 1-k-t$, which implies that $[0,t]\, \mathbf{q}_k \psi$ and if $t \leq \frac{1-k}{2}$ then $\psi(0) \geq (1-k-t) \wedge \frac{1-k}{2} = \frac{1-k}{2} = t \Rightarrow [0,t] \in \psi$. Hence $[0,t] \in \vee \mathbf{q}_k \psi$. Thus form both cases we get $0 \in [\psi]_t$.

Again let $(x*(y*z)) \in [\psi]_t$ and $y \in [\psi]_t \Rightarrow [x*(y*z),\ t] \in \vee \mathbf{q}_k \psi$ and $[y,\ t] \in \vee \mathbf{q}_k \psi \Rightarrow [x*(y*z),\ t] \in \psi$ or $[x*(y*z),\ t]\, \mathbf{q}_k \psi$ and $[y,\ t] \in \psi$ or $[y,\ t]\, \mathbf{q}_k \psi \Rightarrow \psi(x*(y*z)) \geq t$ or $\psi(x*(y*z)) + t + k > 1$ and $\psi(y) \geq t$ or $\psi(y) + t + k > 1$. So we discuss the following cases.

(i) If $\psi(x*(y*z)) \geq t$ and $\psi(y) \geq t$. So $\psi(x*z) \geq \min\left\{t,\ t,\ \frac{1-k}{2}\right\}$ and if $t > \frac{1-k}{2} \Rightarrow \psi(x) \geq \frac{1-k}{2}$ and hence $\psi(x*z) + t > 1-k \Rightarrow [x*z,t]\, q_k \psi$ and if $t \leq \frac{1-k}{2}$ then $\psi(x*z) \geq t \Rightarrow [x*z,t] \in \psi$. Hence $[x*z,t] \in \vee \mathbf{q}_k \psi$.

Similarly from all other cases we get $[x*z,t] \in \vee \mathbf{q}_k \psi$. Which shows that $[\psi]_t$ is a KU-ideal of $\mathbf{X}$ for all $t \in (0,1]$.

Conversely assume that $[\psi]_t$ is a KU-ideal of $\mathbf{X}$ for all $t \in (0,1]$ and we have to show that ψ is an $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of $\mathbf{X}$. Suppose there exist some $t \in (0,1]$ such that

$$\begin{aligned} \psi(0) &< t \leq \min\left\{\psi(x), \frac{1-k}{2}\right\}, \psi(x*z) < t \\ &\leq \min\left\{\psi(x*(y*z)),\ \psi(y),\ \frac{1-k}{2}\right\} \Rightarrow x \in U[\psi,t] \subseteq [\psi]_t \Rightarrow 0 \in [\psi]_t \end{aligned}$$

by hypothesis. Which then implies that $\psi(0) \geq t$ or $\psi(0) + t + k > 1$, this is a contradiction. Similarly

$$\psi(x*z) < t \leq \min\left\{\psi(x*(y*z)),\ \psi(y),\ \frac{1-k}{2}\right\}$$

leads to a contradiction. Thus $\forall\ x, y, z \in \mathbf{X}$ we have

$$\psi(0) \geq \min\left\{\psi(x), \frac{1-k}{2}\right\} \text{ and } \psi(x*z) \geq \min\left\{\psi(x*y)*z,\ \psi(y),\ \frac{1-k}{2}\right\},$$

which shows that ψ is an $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of $\mathbf{X}$. The other case can be seen in a similar way.

Theorem 10. *Let there is an $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-subalgebra (resp., KU-ideal) of X such that $\left\{\psi(x) \mid \psi(x) < \frac{1-k}{2}\right\} \geq 2$ then ψ can be expressed as the union of two proper non-equivalent $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-subalgebra (resp., KU-ideal) of* $\mathbf{X}$.

Proof. Let us define the fuzzy sets as

$$\mu(x) = \begin{cases} t_1 & \text{if } x \in [\psi]_{t_1}, \\ t_2 & \text{if } x \in [\psi]_{t_2} \setminus [\psi]_{t_1}, \\ \vdots & \vdots \\ t_r & \text{if } x \in [\psi]_{t_r} \setminus [\psi]_{t_{r-1}}, \end{cases}$$

and

$$\psi(x) = \begin{cases} \psi(x) & \text{if } x \in [\psi]_{\frac{1-k}{2}}, \\ t_2 & \text{if } x \in [\psi]_{t_2} \setminus [\psi]_{\frac{1-k}{2}}, \\ \vdots & \vdots \\ t_r & \text{if } x \in [\psi]_{t_r} \setminus [\psi]_{t_{r-1}}. \end{cases}$$

for $[\psi]_{\frac{1-k}{2}} \subseteq [\psi]_{t_1} \subseteq ... \subseteq [\Psi]_{t_r} = \mathbf{X}$ and $\left\{\psi(x) \mid \psi(x) < \frac{1-k}{2}\right\} = \{t_1, t_2, ..., t_r\}$ for $t_1 > t_2 > ... > t_r$ with $r \geq 2$. Then by level cut theorem μ and λ are $(\in, \in \vee \mathbf{q}_k)$-fuzzy KU-ideal of $\mathbf{X}$ and the chain of $(\in, \in \vee \mathbf{q}_k)$-level KU-ideals μ and λ are given by respectively as $[\psi]_{t_1} \subseteq [\psi]_{t_2} \subseteq ... \subseteq [\psi]_{t_r}$ and $[\psi]_{\frac{1-k}{2}} \subseteq [\psi]_{t_2} \subseteq ... \subseteq [\psi]_{t_r}$. They are non-equivalent and $\psi = \mu \cup \lambda$. This completes the proof. The other case can be seen in a similar way.

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