COEFFICIENT BOUNDS FOR NEW SUBCLASSES OF BI-UNIVALENT FUNCTIONS USING HADAMARD PRODUCT

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ABSTRACT. The aim of the present paper is to introduce a new subclass of bi-univalent functions defined in the open unit disc using Hadamard product. We obtain estimates on the coefficients $|a_2|$ and $|a_3|$ for functions of this class. Some results related to this work will also be pointed out.

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1. Introduction

Let $A$ denote the class of the functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disc $U = \{ z \in C : |z| < 1 \}$ and satisfy the normalization condition $f(0) = f'(0) = 0$. Let $S$ be the subclass of $A$ consisting of functions of the form (1) which are also univalent in $U$. For $n \in \mathbb{N}_0$, we introduce the subclass $Q(n, \delta, \beta, \lambda)$ of $S$ of functions $f$ of the form (1), satisfying the condition

$$\text{Re} \left\{ \frac{(1-\lambda)D_{n,\delta}^k f(z) + \lambda D_{n,\delta}^{k+1} f(z)}{z} \right\} > \beta, \ z \in U,$$

where $D_{n,\delta}^k$ is the differential operator given by Hadamard product between Salagean and Ruscheweyh operators, such as

$$D_{n,\delta}^k f(z) = z + \sum_{n=2}^{\infty} C(\delta, n) n^k a_n z^n.$$
For \( k = \delta = 0 \), it reduces to the class \( Q_\lambda (\beta) \) studied by Ding et al. [3], (see also [4-7]).

Now by having

\[ f^{-1}f(z) = z, \quad (z \in U), \]

and

\[ f^{-1}f(w) = w, \quad (|w| < r_0, f(z) \geq \frac{1}{4}) \]

where \( f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^2 - 5a_3a_4 + a_4)w^4 + \ldots \), we say that a function \( f(z) \in A \) is bi-univalent in \( U \) if both \( f(z) \) and \( f^{-1}(z) \) are univalent in \( U \).

Let \( \Sigma \) denote the class of bi-univalent functions in \( U \) given by (1). For a brief history and interesting examples in the class \( \Sigma \), see [8]. In fact, Brannan and Taha [9] (see also [11]) introduced certain subclasses of the bi-univalent functions similar to the familiar subclasses \( S^*(\alpha) \) and \( K(\alpha) \) of starlike and convex functions of order \( \alpha (0 \leq \alpha < 1) \), respectively (see [10]). Following the same manner of Brannan and Taha [9] (see also [11]), a function \( f \in A \) is in the class of strongly bi-Starlike functions of order \( \alpha (0 < \alpha \leq 1) \) if each of the following conditions is satisfied: For \( f \in \Sigma \),

\[ \left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi \alpha}{2}, \quad \alpha (0 < \alpha \leq 1, z \in U), \]

and

\[ \left| \arg \left\{ \frac{wg'(w)}{g(w)} \right\} \right| < \frac{\pi \alpha}{2}, \quad \alpha (0 < \alpha \leq 1, w \in U), \]

where \( g \) is the extension of \( f^{-1}(z) \) to \( U \). Similarly, a function \( f \in A \) is in the class \( K_{\Sigma}(\alpha) \) of strongly bi-convex functions of order \( \alpha \) if each of the following conditions are satisfied: For \( f \in \Sigma \),

\[ \left| \arg \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} \right| < \frac{\pi \alpha}{2}, \quad \alpha (0 < \alpha \leq 1, z \in U), \]

and

\[ \left| \arg \left\{ 1 + \frac{wg''(w)}{g'(w)} \right\} \right| < \frac{\pi \alpha}{2}, \quad \alpha (0 < \alpha \leq 1, w \in U), \]
where \( g \) is the extension of to \( U \). The classes \( S^*_\Sigma(\alpha) \) and \( K_\Sigma(\alpha) \) of bi-starlike functions of order \( \alpha \) and bi-convex functions of order \( \alpha \), corresponding (respectively) to the classes of \( S^*(\alpha) \) and \( K(\alpha) \) were also introduced analogously. For each of the classes \( S^*_\Sigma(\alpha) \) and \( K_\Sigma(\alpha) \), it was noted that the estimates obtained for the first two coefficients \( |a_2| \) and \( |a_3| \) are not sharp (for details, see [9,11]).

The object of the paper is to introduce two new subclasses of the function class \( \Sigma \) and to find estimates on the coefficients \( |a_2| \) and \( |a_3| \) using the same techniques given earlier by Srivastava et al. [8], Frasin and Aouf [12], and Porwal and Darus [2]. In order to prove our main results, we need the following lemma due to [15].

**Lemma 1.** If \( h \in p \) then \( |c_k| < 1 \), for each \( k \), where \( p \) is the family of all functions \( h \) analytic in \( U \) for which \( \text{Re}\{h(z)\} > 0 \), then

\[
h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots, z \in U.
\]

2. **Coefficient bounds for the function class \( Q_\Sigma(n, \delta, \alpha, \lambda) \)

**Definition 1.** A function \( f(z) \) given by (1) is said to be in the class \( Q_\Sigma(n, \delta, \alpha, \lambda) \) if the following conditions are satisfied: For \( f \in \Sigma \),

\[
\left| \arg \left( 1 - \lambda \right) D_{n,\delta}^k f(z) + \lambda D_{n,\delta}^{k+1} f(z) \over z \right| < \frac{\pi \alpha}{2}, \alpha(0 < \alpha \leq 1, \lambda \geq 1, z \in U),
\]

and

\[
\left| \arg \left( 1 - \lambda \right) D_{n,\delta}^k g(w) + \lambda D_{n,\delta}^{k+1} g(w) \over w \right| < \frac{\pi \alpha}{2}, \alpha(0 < \alpha \leq 1, \lambda \geq 1, w \in U),
\]

where the function \( g \) is given by

\[
g(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^2 - 5a_2 a_3 + a_4) w^4 + \ldots.
\]

We note that for \( k = \delta = 0, \lambda = 1 \), the class \( Q_\Sigma(n, \delta, \alpha, \lambda) \) reduces to the class \( H^*_\Sigma \) introduced and studied by Srivastava et al [8], for \( k = \delta = 0 \), the class reduces to \( Q_\Sigma(\alpha, \lambda) \) introduced and studied by Frasin and Aouf [12]. Also for \( \delta = 0 \), the class \( Q_\Sigma(n, \delta, \alpha, \lambda) \) reduces to \( Q_\Sigma(n, \alpha, \lambda) \) studied by Porwal and Darus [2]. We begin by finding the estimates of the coefficients for functions in the class \( Q_\Sigma(n, \delta, \alpha, \lambda) \).
Theorem 2. Let the function \( f(z) \) given by (1) be in the class \( Q_{\Sigma}(n, \delta, \alpha, \lambda) \), \( n \in \mathbb{N}_0, 0 \leq \beta < 1, \lambda \geq 1 \). Then

\[
|a_2| \leq 4\alpha \frac{|\Gamma(\delta + 1)|}{\Gamma(\delta + 2)} \left[ \frac{1}{\sqrt{4k(1 + \lambda)^2 + \alpha[2.3^k(1 + \lambda) - 4^k(1 + \lambda)^2]}} \right] \tag{6}
\]

and

\[
|a_3| \leq 12\alpha \frac{\Gamma(\delta + 1)}{\Gamma(\delta + 3)} \left[ \frac{1}{(1 - \lambda)3^k + \lambda 3^k(1 + \lambda) + \frac{2\alpha}{(1 - \lambda)2^k + \lambda 2^{k+1}}^2} \right] \tag{7}
\]

Proof. From (3) and (4), we can write

\[
(1 - \lambda)D_{n, \delta}^k f(z) + \lambda D_{n, \delta}^{k+1} f(z) = [p(z)]^\alpha, \tag{8}
\]

and

\[
(1 - \lambda)D_{n, \delta}^k g(w) + \lambda D_{n, \delta}^{k+1} g(w) = [q(w)]^\alpha, \tag{9}
\]

respectively, where \( p(z) \) and \( q(w) \) are in \( \mathbb{C} \) and have the form

\[
p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \ldots \tag{10}
\]

and

\[
q(w) = 1 + p_1 w + q_2 w^2 + q_3 w^3 + \ldots \tag{11}
\]

Now, equating the coefficients in (8) and (9), we obtain

\[
[(1 - \lambda)2^k + \lambda 2^{k+1}]C(\delta, 2) a_2 = \alpha p_1, \tag{12}
\]

\[
[(1 - \lambda)3^k + \lambda 3^{k+1}]C(\delta, 3) a_3 = \frac{1}{2}[2\alpha p_2 + \alpha(\alpha - 1)p_1^2], \tag{13}
\]

\[-[(1 - \lambda)2^k + \lambda 2^{k+1}]C(\delta, 2) a_2 = \alpha q_1, \tag{14}
\]

\[
[(1 - \lambda)3^k + \lambda 3^{k+1}][2C(\delta, 2)]^2 a_2^2 - C(\delta, 3) a_3 = \frac{1}{2}[2\alpha q_2 + \alpha(\alpha - 1)q_1^2]. \tag{15}
\]

From (12) and (14), we obtain

\[
p_1 = -q_1 \tag{16}
\]

and

\[
2[(1 - \lambda)2^k + \lambda 2^{k+1}]^2 [C(\delta, 2)]^2 a_2^2 = \alpha^2(p_1^2 + q_1^2). \tag{17}
\]

Now from (13), (15) and (17), we obtain
Next, in order to find the bound on $a_2$, we subtract (13) from (15) and obtain
\[
2[(1-\lambda)3^k + \lambda 3^{k+1}] [C(\delta, 2)]^2 a_2^2 = \alpha(p_2 + q_2) + \frac{1}{2}[\alpha(\alpha - 1)(p_1^2 + q_1^2)]
\]
\[
= \alpha(p_2 + q_2) + \frac{\alpha(\alpha - 1)}{2} \cdot \frac{2[(1-\lambda)2^k + \lambda 2^{k+1}]^2 [C(\delta, 2)]^2 a_2^2}{\alpha^2}.
\]
Therefore we have
\[
a_2^2 = \frac{\alpha^2(p_2 + q_2)}{[C(\delta, 2)]^2 [\Gamma((1+\lambda)^2 + \alpha(2.3^2 + (1+\lambda))) - 4^2(1+\lambda)^2]^2}.
\]
Applying Lemma 1 for the coefficients $p_2$ and $q_2$, we immediately have
\[
|a_2| \leq 4\alpha \frac{\Gamma(\delta+1)}{\Gamma(\delta+2)} \frac{1}{\sqrt{4^2(1+\lambda)^2 + \alpha(2.3^2 + (1+\lambda)) - 4^2(1+\lambda)^2}}.
\]
This gives the bound as asserted in (6).

Next, in order to find the bound on $|a_3|$, we subtract (13) from (15) and obtain
\[
2[(1-\lambda)3^k + \lambda 3^{k+1}] [C(\delta, 3)a_3 - C(\delta, 2)a_2^2]
\]
\[
= \frac{1}{2} \left( 2\alpha(p_2 - q_2) + \alpha(\alpha - 1)(p_1^2 - q_1^2) \right),
\]
\[
a_3 = \frac{\alpha(p_2 - q_2)}{2[(1-\lambda)3^k + \lambda 3^{k+1}] C(\delta, 3)} + \frac{\alpha^2(p_1^2 - q_1^2)}{2[(1-\lambda)2^k + \lambda 2^{k+1}]^2 C(\delta, 3)}.
\]
Applying Lemma 1 for the coefficients $p_2$ and $q_2$, we immediately have
\[
|a_3| \leq \frac{12\alpha \Gamma(\delta+1)}{[(1-\lambda)3^k + \lambda 3^{k+1}] \Gamma(\delta+3)} + \frac{24\alpha \Gamma(\delta+1) \alpha^2}{[(1-\lambda)2^k + \lambda 2^{k+1}]^2 \Gamma(\delta+3)},
\]
i.e.
\[
|a_3| \leq 12\alpha \frac{\Gamma(\delta+1)}{\Gamma(\delta+3)} \left( \frac{1}{(1-\lambda)3^k + \lambda 3^{k+1}} + \frac{2\alpha}{[(1-\lambda)2^k + \lambda 2^{k+1}]^2} \right).
\]
This completes the proof of Theorem 2.

Putting $\lambda = 1, k = \delta = 0$, in Theorem 2, we have

**Corollary 3.** Let $f(z)$ given by (1) be in the class $H_{\Sigma}^0(0 < \alpha \leq 1)$. Then
\[
|a_2| \leq \alpha \sqrt{\frac{2}{2+\alpha}}
\]
and
\[
|a_3| \leq \frac{\alpha(2+3\alpha)}{3}.
\]
3. Coefficient bounds for the function class $H_{\Sigma}(n, \delta, \beta, \lambda)$

**Definition 2.** A function $f(z)$ given by (1) is said to be in the class $H_{\Sigma}(n, \delta, \beta, \lambda)$ if the following conditions are satisfied:

$$\text{Re} \left\{ \frac{(1-\lambda)D_{n,\delta}^k f(z) + \lambda D_{n,\delta}^{k+1} f(z)}{z} \right\} > \beta, z \in U, n \in \mathbb{N}_0, 0 \leq \beta < 1, \lambda \geq 1.$$  \hspace{1cm} (18)

and

$$\text{Re} \left\{ \frac{(1-\lambda)D_{n,\delta}^k g(w) + \lambda D_{n,\delta}^{k+1} g(w)}{w} \right\} > \beta, w \in U, n \in \mathbb{N}_0, 0 \leq \beta < 1, \lambda \geq 1$$ \hspace{1cm} (19)

where the function $g$ is defined by (5).

We note that for $k = \delta = 0$, and $\lambda = 1$, $H_{\Sigma}(n, \delta, \beta, \lambda)$ the class reduced to the classes $H_{\Sigma}(\beta)$ studied by Srivastava et al.[8], and for $k = \delta = 0$, the class reduced to the classes $H_{\Sigma}(\beta, \lambda)$ studied by Frasin and Aouf [12].

**Theorem 4.** Let the function $f(z)$ given by (1) be in the class $H_{\Sigma}(n, \delta, \beta, \lambda)$, $n \in \mathbb{N}_0, 0 \leq \beta < 1, \lambda \geq 1$. Then

$$|a_2| \leq 2 \left| \frac{\Gamma(\delta + 1)}{\Gamma(\delta + 2)} \right| \sqrt{\frac{2(1-\beta)}{(1-\lambda)3^k + \lambda 3^{k+1}}}$$ \hspace{1cm} (20)

and

$$|a_3| \leq \frac{12(1-\beta)\Gamma(\delta + 1)}{\Gamma(\delta + 3)} \left[ \frac{2(1-\beta)}{((1-\lambda)2^k + \lambda 2^{k+1})^2} + \frac{1}{(1-\lambda)3^k + \lambda 3^{k+1}} \right].$$ \hspace{1cm} (21)

**Proof.** It follows from (18) and (19) that there exists $p, q \in P$ such that

$$\frac{(1-\lambda)D_{n,\delta}^k f(z) + \lambda D_{n,\delta}^{k+1} f(z)}{z} = \beta + (1-\beta)p(z),$$ \hspace{1cm} (22)

and

$$\frac{(1-\lambda)D_{n,\delta}^k g(w) + \lambda D_{n,\delta}^{k+1} g(w)}{w} = \beta + (1-\beta)q(w),$$ \hspace{1cm} (23)

where $p(z)$ and $q(w)$ have the forms (10) and (11), respectively. Equating coefficients in (22) and (23) yields

$$[(1-\lambda)2^k + \lambda 2^{k+1}]C(\delta, 2)a_2 = (1-\beta)p_1,$$ \hspace{1cm} (24)

$$[(1-\lambda)3^k + \lambda 3^{k+1}]C(\delta, 3)a_3 = (1-\beta)p_2,$$ \hspace{1cm} (25)
\[-[(1 - \lambda)2^k + \lambda 2^{k+1}]C(\delta, 2)a_2 = (1 - \beta)q_1. \tag{26}\]

and
\[[(1 - \lambda)3^k + \lambda 3^{k+1}][2C(\delta, 2)]^2a_2^2 - C(\delta, 3)a_3 = (1 - \beta)q_2. \tag{27}\]

From (24) and (26), we have
\[-p_1 = q_1 \tag{28}\]

and
\[2[(1 - \lambda)2^k + \lambda 2^{k+1}]C(\delta, 2)]^2a_2^2 = (1 - \beta)^2(p_1^2 + q_1^2). \tag{29}\]

Also, from (25) and (27), we find that
\[2[(1 - \lambda)3^k + \lambda 3^{k+1}]C(\delta, 2)]^2a_2^2 = (1 - \beta)(p_2 + q_2), \tag{30}\]

\[|a_2|^2 \leq \frac{(1 - \beta)(|p_2| + |q_2|)}{2[(1 - \lambda)3^k + \lambda 3^{k+1}]C(\delta, 2)]^2} \tag{31}\]

i.e.
\[|a_2| \leq 2 \frac{\Gamma(\delta + 1)}{\Gamma(\delta + 2)} \sqrt{\frac{2(1 - \beta)}{1 - \lambda} \frac{3(1 - \beta)(p_2 - q_2)}{2[(1 - \lambda)3^k + \lambda 3^{k+1}]C(\delta, 2)]^2}}. \tag{32}\]

which is the bound on $|a_2|$ as given in (20).

Next, in order to find the bound on $|a_3|$ by subtracting (27) from (25), we obtain
\[2C(\delta, 3)[(1 - \lambda)3^k + \lambda 3^{k+1}]a_3 = 2[(1 - \lambda)3^k + \lambda 3^{k+1}][C(\delta, 2)]^2a_2^2 + (1 - \beta)(p_2 - q_2)\]

or, equivalently
\[a_3 = \frac{2[(1 - \lambda)3^k + \lambda 3^{k+1}][C(\delta, 2)]^2a_2^2}{2C(\delta, 3)[(1 - \lambda)3^k + \lambda 3^{k+1}]} + \frac{(1 - \beta)(p_2 - q_2)}{2C(\delta, 3)[(1 - \lambda)3^k + \lambda 3^{k+1}]} \tag{33}\]

Upon substituting the value of $a_2^2$ from (29), we obtain
\[a_3 = \frac{3(1 - \beta)^2(p_1^2 + q_1^2)\Gamma(\delta + 1)}{[(1 - \lambda)2^k + \lambda 2^{k+1}]^2\Gamma(\delta + 3)} + \frac{3(1 - \beta)(p_2 - q_2)\Gamma(\delta + 1)}{[(1 - \lambda)3^k + \lambda 3^{k+1}][\Gamma(\delta + 3)]}. \tag{33}\]

Applying Lemma 1 for the coefficients $p_1, p_2, q_1$ and $q_2$ we obtain
\[|a_3| \leq \frac{12(1 - \beta)\Gamma(\delta + 1)}{\Gamma(\delta + 3)} \left[ \frac{2(1 - \beta)}{[(1 - \lambda)2^k + \lambda 2^{k+1}]^2} + \frac{1}{(1 - \lambda)3^k + \lambda 3^{k+1}} \right] \tag{34}\]

which is the bound on $|a_3|$ as asserted in (21).
Putting \( \lambda = 1, \ k = \delta = 0 \), in Theorem 4, we have the following corollary.

**Corollary 5.** Let \( f(z) \) given by (1) be in the class \( H_\Sigma(n, \delta, \beta, \lambda), (0 \leq \beta < 1) \). Then

\[
|a_2| \leq \sqrt{\frac{2(1-\beta)}{3}} \tag{35}
\]

and

\[
|a_3| \leq \frac{(1-\beta)(5-3\beta)}{3}. \tag{36}
\]

**Remark 1.** If we put \( \delta = k = 0 \), in Theorems 2 and 3, we obtain the corresponding results due to Frasin and Aouf [12].

**Remark 2.** If we put \( \delta = 0 \), in Theorems 2 and 3, we obtain the corresponding results due to Porwal and Darus [2].

**Remark 3.** If we put \( \delta = k = 0, \lambda = 1 \), in Theorems 2 and 3, we obtain the corresponding results due to Srivastava et al [8].

**Remark 4.** Similarly, just as stated in [2], it would be nice to find estimates for \( |a_n|, n \geq 4 \) (not necessarily sharp) for the class of functions defined in this work.

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**References**


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