ON $\eta$-EINSTEIN $LP$-SASAKIAN MANIFOLDS

U.C. De and K. De

ABSTRACT. The object of the present paper is to study $\eta$-Einstein $LP$-Sasakian manifolds.

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1. Introduction

In 1989 Matsumoto [7] introduced the notion of Lorentzian para-Sasakian manifolds. Then Mihai and Rosca [5] defined the same notion independently and they obtained several results in this manifold. LP-Sasakian manifolds have also been studied by Matsumoto and Mihai [8], Matsumoto, Mihai and Rosca [9], De and Shaikh [13], Ozgur [4] and many others.

The Ricci tensor $S$ of an LP-Sasakian manifold is said to be $\eta$-Einstein if its Ricci tensor satisfies the following condition

$$S(X,Y) = a g(X,Y) + b \eta(X) \eta(Y),$$  \hspace{1cm} (1)

where $a, b$ are smooth functions. $\eta$-Einstein LP-Sasakian manifolds have been studied by Mihai, Shaikh and De [6]. Also Shaikh, De and Binh [2] studied $K$-contact $\eta$-Einstein manifolds satisfying certain curvature conditions. Example of an $\eta$-Einstein manifold is given by Okumura [10]. Motivated by the above works we study some properties of $\eta$-Einstein LP-Sasakian manifolds. The paper is organized as follows:

In section 2, some preliminary results are recalled. After preliminaries, we find out the significance of the associated scalars in an LP-Sasakian $\eta$-Einstein manifold. In the next Section, we prove that the functions $a$ and $b$ of the defining equation (1) are constants, provided $tr\phi = 0$. We also obtain a necessary and sufficient condition for an LP-Sasakian manifold to be an $\eta$-Einstein manifold. Finally, we cited some examples of $\eta$-Einstein LP-Sasakian manifolds.
2. Preliminaries

Let $M^n$ be an $n$-dimensional differentiable manifold endowed with a $(1, 1)$ tensor field $\phi$, a contravariant vector field $\xi$, a covariant vector field $\eta$ and a Lorentzian metric $g$ of type $(0,2)$ such that for each point $p \in M$, the tensor $g_p: T_pM \times T_pM \to \mathbb{R}$ is a non-degenerate inner product of signature $(-, +, +, \ldots, +)$, where $T_pM$ denotes the tangent space of $M$ at $p$ and $\mathbb{R}$ is the real number space which satisfies

$$\phi^2(X) = X + \eta(X)\xi, \eta(\xi) = -1,$$

and

$$g(X, \xi) = \eta(X), g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y)$$

for all vector fields $X, Y$. Then such a structure $(\phi, \xi, \eta, g)$ is termed as Lorentzian almost paracontact structure and the manifold $M^n$ with the structure $(\phi, \xi, \eta, g)$ is called Lorentzian almost paracontact manifold [7]. In the Lorentzian almost paracontact manifold $M^n$, the following relations hold [7]:

$$\phi \xi = 0, \eta(\phi X) = 0,$$

$$\Omega(X, Y) = \Omega(Y, X),$$

where $\Omega(X, Y) = g(X, \phi Y)$.

Let $\{e_i\}$ be an orthonormal basis such that $e_1 = \xi$. Then the Ricci tensor $S$ and the scalar curvature $r$ are defined by

$$S(X, Y) = \sum_{i=1}^{n} \epsilon_i g(R(e_i, X)Y, e_i)$$

and

$$r = \sum_{i=1}^{n} \epsilon_i S(e_i, e_i),$$

where we put $\epsilon_i = g(e_i, e_i)$, that is, $\epsilon_1 = -1, \epsilon_2 = \cdots = \epsilon_n = 1$.

A Lorentzian almost paracontact manifold $M^n$ equipped with the structure $(\phi, \xi, \eta, g)$ is called Lorentzian paracontact manifold if

$$\Omega(X, Y) = \frac{1}{2} \left\{ (\nabla_X \eta)Y + (\nabla_Y \eta)X \right\}.$$

A Lorentzian almost paracontact manifold $M^n$ equipped with the structure $(\phi, \xi, \eta, g)$ is called an LP-Sasakian manifold [7] if

$$(\nabla_X \phi)Y = g(\phi X, \phi Y)\xi + \eta(Y)\phi^2 X.$$
In an LP-Sasakian manifold the 1-form $\eta$ is closed. Also in [7], it is proved that if an $n$-dimensional Lorentzian manifold $(M^n, g)$ admits a timelike unit vector field $\xi$ such that the 1-form $\eta$ associated to $\xi$ is closed and satisfies

$$\langle \nabla_X \nabla_Y \eta \rangle Z = g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z),$$
then $M^n$ admits an LP-Sasakian structure. Further, on such an LP-Sasakian manifold $M^n (\phi, \xi, \eta, g)$, the following relations hold [7]:

$$\eta(R(X, Y)Z) = \langle g(Y, Z)\eta(X) - g(X, Z)\eta(Y) \rangle,$$  
(6)

$$S(X, \xi) = (n - 1)\eta(X),$$  
(7)

$$R(X, Y)\xi = [\eta(Y)X - \eta(X)Y],$$  
(8)

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X,$$  
(9)

$$(\nabla_X \phi)(Y) = [g(X, Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X],$$  
(10)
for all vector fields $X, Y, Z$, where $R, S$ denote respectively the curvature tensor and the Ricci tensor of the manifold. Also since the vector field $\eta$ is closed in an LP-Sasakian manifold, we have ([8],[7])

$$(\nabla_X \eta)Y = \Omega(X, Y),$$  
(11)

$$\Omega(X, \xi) = 0,$$  
(12)

$$\nabla_X \xi = \phi X,$$  
(13)
for any vector field $X$ and $Y$.

3. Significance of the associated scalars in an LP-Sasakian $\eta$-Einstein manifold

We can express (1) as follows:

$$S(X, \xi) = (a - b)g(X, \xi).$$  
(14)
From (14), we conclude that \((a - b)\) is an eigen value of the Ricci operator \(Q\) defined by \(S(X,Y) = g(QX,Y)\) and \(\xi\) is an eigen vector corresponding to this eigen value.

Let \(V\) be any other vector orthogonal to \(\xi\) so that

\[
\eta(V) = 0.
\]  
(15)

From (1), we obtain

\[
S(X,V) = ag(X,V) + b\eta(X)\eta(V),
\]  
(16)

Hence in virtue of (15), we get

\[
S(X,V) = ag(X,V).
\]  
(17)

From (17), we see that \(a\) is an eigen value of the Ricci operator \(Q\) and \(V\) is an eigen vector corresponding to this eigen value. If the manifold under consideration is \(n\)-dimensional and \(V\) is any vector orthogonal to \(\xi\), it follows from a known result in linear algebra [12] that the eigen value \(a\) is of multiplicity \((n - 1)\). Hence the multiplicity of the eigen value \((a - b)\) must be 1. Therefore we can state the following:

**Theorem 3.1.** In an LP-Sasakian \(\eta\)-Einstein manifold of dimension \(n\), the Ricci operator \(Q\) has only two distinct eigen values \((a - b)\) and \(a\) of which the former is simple and the later is of multiplicity \((n - 1)\).

### 4. \(\eta\)-EINSTEIN MANIFOLDS

This section deals with \(\eta\)-Einstein LP-Sasakian manifolds.

From (1) we have

\[
S(\phi X,Y) = ag(\phi X,Y),
\]  
(18)

\[
S(\xi,\xi) = -a + b.
\]  
(19)

**Theorem 4.1.** The Ricci curvature of an \(\eta\)-Einstein LP-Sasakian manifold in the direction of \(\xi\) is equal to \(-(n - 1)\).

*Proof.* Substituting \(\xi\) for \(X\) in (7) we have the theorem.

**Theorem 4.2.** The functions \(a\) and \(b\) of the defining equation (1) are constants, provided \(\text{tr} \phi = 0\).
Proof. Equation (19) and (7) imply

\[-a + b = 1 - n.\]  

(20)

So we need only to show that \(a\) is constant. Taking a frame field we get from (1),

\[
\sum_{i=1}^{n} \epsilon_i S(e_i, e_i) = a \sum_{i=1}^{n} \epsilon_i g(e_i, e_i) + b \sum_{i=1}^{n} \epsilon_i \eta(e_i) \eta(e_i),
\]

which gives

\[r = na - b,
\]

where \(r\) is the scalar curvature of the manifold. Now differentiating the above equation we have

\[dr(X) = nda(X) - db(X) = (n + 1)da(X).\]  

(21)

Again from (1) we have

\[QX = aX + b\eta(X)\xi.\]  

(22)

Differentiating (22) along \(Y\), we get

\[(\nabla_Y Q)X = (Ya)X + (Yb)\eta(X)\xi + bg(\phi X, Y)\xi + b\eta(X)\phi Y.\]  

(23)

Contracting the above equation with respect to \(Y\), we get

\[(\text{div} Q)X = Xa + (\xi b)\eta(X) + bn(\xi) tr\phi.\]  

(24)

Using the identity [11] \((\text{div} Q)X = \frac{dr(X)}{2}\), (21) and \(tr\phi = 0\), we get

\[(n - 1)da(X) = 2db(\xi)\eta(X).\]  

(25)

Putting \(X = \xi\) in it, we get

\[(n - 1)da(\xi) = -2db(\xi) = 2da(\xi),\]

which gives \(da(\xi) = 0\) and hence \(db(\xi) = 0\). Consequently (25) yields \(da(X) = 0\).

We now obtain a necessary and sufficient condition for an LP-Sasakian manifold to be an \(\eta\)-Einstein manifold. In an LP-Sasakian manifold, the following relation holds [1]
\[ R(X,Y)\phi Z = \phi R(X,Y)Z + g(Y,Z)\phi X \]
\[ -g(X,Z)\phi Y + g(X,\phi Z)Y - g(Y,\phi Z)X \]
\[ +2[g(X,\phi Z)\eta(Y) - g(Y,\phi Z)\eta(X)]\xi \]
\[ +2[\eta(Y)\phi X - \eta(X)\phi Y]\eta(Z). \quad (26) \]

Taking a frame field and contracting (26) with respect to \( X \), we get

\[ S(Y,\phi Z) = (C_1^1\overline{R})(Y,Z) \]
\[ +[g(Y,Z) + 2\eta(Y)\eta(Z)]tr\phi - (n+1)g(Y,\phi Z), \quad (27) \]

where \( C_1^1 \) denotes contraction at the first slot and \( \overline{R} = \phi R \).

Since \( (C_1^1\overline{R})(Y,Z) = (C_1^1\overline{R})(Z,Y) \), from the above it is obvious that

\[ S(Y,\phi Z) = S(Z,\phi Y). \quad (28) \]

**Theorem 4.3.** In order that an LP-Sasakian manifold to be an \( \eta \)-Einstein manifold it is necessary and sufficient that the symmetric tensor \( (C_1^1\overline{R}) \) and \( \Omega \) should be linearly dependent, provided \( tr\phi = 0 \).

**Proof.** At first we assume that \( (C_1^1\overline{R}) \) and \( \Omega \) are linearly dependent. Then from (27) we have

\[ S(Y,\phi Z) = \lambda g(Y,\phi Z), \]

where \( \lambda \) is a scalar. Now using Theorem 4.2 we can easily seen that the manifold is a \( \eta \)-Einstein manifold.

Conversely, let the manifold is an \( \eta \)-Einstein manifold. Then we have

\[ S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z). \]

Replacing \( Y \) by \( \phi Y \) in the above equation we obtain

\[ S(Y,\phi Z) = ag(Y,\phi Z). \quad (29) \]

Using (29) in (27) we see that \( (C_1^1\overline{R}) \) and \( \Omega \) are linearly dependent.
5. Examples

Example 5.1: [14] A conformally flat LP-Sasakian manifold is an \( \eta \)-Einstein manifold.

Example 5.2: [4] A \( \phi \)-conformally flat LP-Sasakian manifold is an \( \eta \)-Einstein manifold.

Example 5.3: Let \((M^{n-1}, \tilde{g})\) be a hypersurface of \((M^n, g)\). If \(A\) is the (1,1) tensor corresponding to the normal valued second fundamental tensor \(H\), then we have ([3],p.41),

\[
\tilde{g}(A_\xi(X), Y) = g(H(X, Y), \xi)
\]

where \(\xi\) is the unit normal vector field and \(X, Y\) are tangent vector fields.

Let \(H_\xi\) be the symmetric (0,2)tensor associated with \(A_\xi\) in the hypersurface defined by

\[
\tilde{g}(A_\xi(X), Y) = (H_\xi(X, Y)).
\]

A hypersurface of a Riemannian manifold \((M^n, g)\) is called quasi-umbilical ([3], p.147) if its second fundamental tensor has the form

\[
H_\xi(X, Y) = \alpha g(X, Y) + \beta \omega(X)\omega(Y)
\]

where \(\omega\) is a 1-form, the vector field corresponding to the 1-form \(\omega\) is a unit vector field, and \(\alpha, \beta\) are scalars. If \(\alpha = 0\) (respectively \(\beta = 0\) or \(\alpha = \beta = 0\)) holds, then it is called cylindrical (respectively umbilical or geodesic).

Now from (30), (31) and (32) we obtain

\[
g(H(X, Y), \xi) = \alpha g(X, Y)g(\xi, \xi) + \beta \omega(X)\omega(Y)g(\xi, \xi)
\]

which implies that

\[
H(X, Y) = \alpha g(X, Y)\xi + \beta \omega(X)\omega(Y)\xi
\]

since \(\xi\) is the only unit normal vector field.

We have the following equation of Gauss ([3], p.45) for any vector fields \(X, Y, Z, W\) tangent to the hypersurface

\[
g(R(X, Y)Z, W) = \tilde{g}(\tilde{R}(X, Y)Z, W) - g(H(X, W), H(Y, Z)) + g(H(Y, W), H(X, Z)),
\]

where \(\tilde{R}\) is the curvature tensor of the hypersurface.
Let us assume that the hypersurface is quasi-umbilical. Then from (33) and (34) it follows that
\[
g(R(X, Y)Z, W) = \tilde{g}(\tilde{R}(X, Y)Z, W) + \alpha^2 [g(Y, W)g(X, Z)
- g(X, W)g(Y, Z)] + \alpha \beta g(Y, W)\omega(X)\omega(Z)
+ g(X, Z)\omega(Y)\omega(W) - g(X, W)\omega(Y)\omega(Z)
- g(Y, Z)\omega(X)\omega(W)].
\] (35)

We know that every LP-Sasakian space form is of constant curvature 1 [14]. Hence we have
\[
R(X, Y)Z = g(Y, Z)X - g(X, Z)Y
\]
which implies that
\[
g(R(X, Y)Z, W) = g(Y, Z)g(X, W) - g(X, Z)g(Y, W).
\] (36)

Using (36) in (35) we have
\[
\tilde{g}(\tilde{R}(X, Y)Z, W) = (\alpha^2 - 1)[g(X, W)g(Y, Z) - g(X, Z)g(Y, W)]
- \alpha \beta [g(Y, W)\omega(X)\omega(Z) + g(X, Z)\omega(Y)\omega(W)
- g(X, W)\omega(Y)\omega(Z) - g(Y, Z)\omega(X)\omega(W)].
\] (37)

Let \( \{e_i\} \), \( i = 1, 2, \ldots, n \) be an orthonormal frame at any point of the manifold. Then putting \( X = W = \{e_i\} \) in (37) and taking summation over \( i \), we get
\[
\sum_{i=1}^{n} e_i \tilde{g}(\tilde{R}(e_i, Y)Z, e_i) = (\alpha^2 - 1) \sum_{i=1}^{n} \epsilon_i [g(e_i, e_i)g(Y, Z) - g(e_i, Z)g(Y, e_i)]
- \alpha \beta \sum_{i=1}^{n} \epsilon_i [g(Y, e_i)\omega(e_i)\omega(Z) + g(e_i, Z)\omega(Y)\omega(e_i)
- g(e_i, e_i)\omega(Y)\omega(Z) - g(Y, Z)\omega(e_i)\omega(e_i)],
\]
which implies that
\[
\tilde{S}(Y, Z) = [(\alpha^2 - 1)(n - 1) - \alpha \beta]g(Y, Z) + \alpha \beta (n - 2)\omega(Y)\omega(Z).
\] (38)

Thus a quasi-umbilical hypersurface of an LP-Sasakian space form is \( \eta \)-Einstein.
References


Uday Chand De,
Department of Pure Mathematics,
Calcutta University,
35 Ballygunge Circular Road
Kol 700019, West Bengal, India. email uc_de@yahoo.com

Krishnendu De,
Konnagar High School (H.S.),
68 G.T. Road (West), Konnagar, Hooghly,
Pin. 712235, West Bengal, India. email krishnendu_de@yahoo.com