DIFFERENTIAL SUBORDINATION RESULTS USING A GENERALIZED SĂLĂGEAN OPERATOR AND RUSCHEWEYH OPERATOR
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Abstract. In the present paper we study the operator using the generalized Sălăgean operator and Ruscheweyh operator, denote by \( DR^n_\lambda \) the Hadamard product of the generalized Sălăgean operator \( D^n_\lambda \) and Ruscheweyh operator \( R^n \), given by

\[
DR^n_\lambda : A \rightarrow A, \quad DR^n_\lambda f(z) = (D^n_\lambda * R^n) f(z)
\]

and \( A_n = \{ f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots, z \in U \} \) is the class of normalized analytic functions with \( A_1 = A \). We obtain several differential subordinations regarding the operator \( DR^n_\lambda \).

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1. Introduction
Denote by \( U \) the unit disc of the complex plane, \( U = \{ z \in \mathbb{C} : |z| < 1 \} \) and \( \mathcal{H}(U) \) the space of holomorphic functions in \( U \).

Let \( A_n = \{ f \in \mathcal{H}(U) : f(z) = z + a_{n+1}z^{n+1} + \ldots, z \in U \} \) with \( A_1 = A \) and \( \mathcal{H}[a,n] = \{ f \in \mathcal{H}(U) : f(z) = a + a_nz^n + a_{n+1}z^{n+1} + \ldots, z \in U \} \) for \( a \in \mathbb{C} \) and \( n \in \mathbb{N} \).

Denote by \( K = \left\{ f \in A : \Re \frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U \right\} \), the class of normalized convex functions in \( U \).

If \( f \) and \( g \) are analytic functions in \( U \), we say that \( f \) is subordinate to \( g \), written \( f \prec g \), if there is a function \( w \) analytic in \( U \), with \( w(0) = 0 \), \( |w(z)| < 1 \), for all \( z \in U \), such that \( f(z) = g(w(z)) \) for all \( z \in U \). If \( g \) is univalent, then \( f \prec g \) if and only if \( f(0) = g(0) \) and \( f(U) \subseteq g(U) \).

Let \( \psi : \mathbb{C}^3 \times U \rightarrow \mathbb{C} \) and \( h \) an univalent function in \( U \). If \( p \) is analytic in \( U \) and satisfies the (second-order) differential subordination

\[
\psi(p(z), zp'(z), z^2p''(z); z) \prec h(z), \quad z \in U,
\]
then \( p \) is called a solution of the differential subordination. The univalent function \( q \) is called a dominant of the solutions of the differential subordination, or more simply a dominant, if \( p \prec q \) for all \( p \) satisfying (1).

A dominant \( \tilde{q} \) that satisfies \( \tilde{q} \prec q \) for all dominants \( q \) of (1) is said to be the best dominant of (1). The best dominant is unique up to a rotation of \( U \).

**Definition 1.** (Al Oboudi [10]) For \( f \in \mathcal{A} \), \( \lambda \geq 0 \) and \( n \in \mathbb{N} \), the operator \( D^n_\lambda \) is defined by \( D^n_\lambda : \mathcal{A} \to \mathcal{A} \),

\[
\begin{align*}
D^0_\lambda f(z) &= f(z) \\
D^1_\lambda f(z) &= (1 - \lambda) f(z) + \lambda z f'(z) = D_\lambda f(z) \\
&\quad \vdots \\
D^{n+1}_\lambda f(z) &= (1 - \lambda) D^n_\lambda f(z) + \lambda z (D^n_\lambda f(z))' = D_\lambda (D^n_\lambda f(z)), \quad z \in U.
\end{align*}
\]

**Remark 1.** If \( f \in \mathcal{A} \) and \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \), then \( D^n_\lambda f(z) = z + \sum_{j=2}^{\infty} [1 + (j - 1) \lambda]^n a_j z^j, \quad z \in U \).

**Remark 2.** For \( \lambda = 1 \) in the above definition we obtain the Sălăgean differential operator [13].

**Definition 2.** (Ruscheweyh [12]) For \( f \in \mathcal{A} \), \( n \in \mathbb{N} \), the operator \( R^n \) is defined by \( R^n : \mathcal{A} \to \mathcal{A} \),

\[
\begin{align*}
R^0 f(z) &= f(z) \\
R^1 f(z) &= z f'(z) \\
&\quad \vdots \\
(n + 1) R^{n+1} f(z) &= z (R^n f(z))' + n R^n f(z), \quad z \in U.
\end{align*}
\]

**Remark 3.** If \( f \in \mathcal{A} \), \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \), then \( R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j, \quad z \in U \).

**Definition 3.** (L. Andrei) Let \( \lambda \geq 0 \) and \( n \in \mathbb{N} \). Denote by \( DR^n_\lambda : \mathcal{A} \to \mathcal{A} \) the operator given by the Hadamard product (the convolution product) of the generalized Sălăgean operator \( D^n_\lambda \) and the Ruscheweyh operator \( R^n \):

\[
DR^n_\lambda f(z) = (D^n_\lambda * R^n) f(z),
\]

for any \( z \in U \) and each nonnegative integer \( n \).

**Remark 4.** If \( f \in \mathcal{A} \) and \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \), then \( DR^n_\lambda f(z) = z + \sum_{j=2}^{\infty} \left[ \frac{(n+j-1)!}{n!(j-1)!} a_j^2 z^j \right], \quad z \in U \).
Remark 5. The operator $DR_n^\lambda$ was studied in [2], [3], [8].

Remark 6. For $\lambda = 1$ we obtain the Hadamard product $SR^n$ of the Sălăgean operator $S_n$ and Ruscheweyh operator $R^n$, which was studied in [4], [5], [6], [7].

Lemma 1. (Hallenbeck and Ruscheweyh [11, Th. 3.1.6, p. 71]) Let $h$ be a convex function with $h(0) = a$, and let $\gamma \in \mathbb{C} \setminus \{0\}$ be a complex number with $\Re \gamma \geq 0$. If $p \in H[a, n]$ and
\[
p(z) + \frac{1}{\gamma}zp'(z) \prec h(z), \quad z \in U,
\]
then
\[
p(z) \prec g(z) \prec h(z), \quad z \in U,
\]
where $g(z) = \frac{\gamma}{n\gamma/n} \int_0^z h(t)t^{\gamma/n-1} dt, z \in U.$

Lemma 2. (Miller and Mocanu [11]) Let $g$ be a convex function in $U$ and let
\[
h(z) = g(z) + \alpha z g'(z), \quad z \in U,
\]
where $\alpha > 0$ and $n$ is a positive integer. If $p(z) = g(0) + p_n z^n + p_{n+1} z^{n+1} + \ldots, z \in U$, is holomorphic in $U$ and
\[
p(z) + \alpha z p'(z) \prec h(z), \quad z \in U,
\]
then
\[
p(z) \prec g(z), \quad z \in U,
\]
and this result is sharp.

2. Main results

Theorem 3. Let $g$ be a convex function, $g(0) = 1$ and let $h$ be the function $h(z) = g(z) + \frac{i}{\gamma} g'(z)$, for $z \in U$.

If $\lambda, \delta \geq 0$, $n \in \mathbb{N}$, $f \in A$ and satisfies the differential subordination

\[
\left( \frac{DR_n^\lambda f(z)}{z} \right)^{\delta-1} (DR_n^\lambda f(z))' \prec h(z), \quad z \in U,
\]

then
\[
\left( \frac{DR_n^\lambda f(z)}{z} \right)^\delta \prec g(z), \quad z \in U,
\]
and this result is sharp.
Proof. For \( f \in A \) and \( f(z) = z + \sum_{j=2}^{\infty} a_j z^j \) we have

\[
DR^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} [1 + (j-1) \lambda]^n a_j^2 z^j, \text{ for } z \in U.
\]

Consider \( p(z) = \left( \frac{DR^n f(z)}{z} \right)^{\delta} = \left( z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} [1 + (j-1) \lambda]^n a_j^2 z^j \right)^{\delta} = \left( 1 + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} [1 + (j-1) \lambda]^n a_j^2 z^j \right)^{\delta} = 1 + p_\delta z^\delta + p_{\delta+1} z^{\delta+1} + \ldots, \text{ for } z \in U.

Differentiating we obtain \( \left( \frac{DR^n f(z)}{z} \right)^{\delta-1} (DR^n f(z))' = p(z) + \frac{1}{\delta} z p'(z), \text{ for } z \in U.
\]

Then \( (2) \) becomes

\[
p(z) + \frac{1}{\delta} z p'(z) \prec h(z) = g(z) + \frac{z}{\delta} g'(z), \text{ for } z \in U.
\]

By using Lemma 2, we have

\[
p(z) \prec g(z), \text{ for } z \in U, \text{ i.e. } \left( \frac{DR^n f(z)}{z} \right)^{\delta} \prec g(z), \text{ for } z \in U.
\]

**Theorem 4.** Let \( h \) be an holomorphic function which satisfies the inequality

\[
\Re \left( 1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2}, \text{ for } z \in U, \text{ and } h(0) = 1.
\]

If \( \lambda, \delta \geq 0, n \in \mathbb{N}, f \in A \) and satisfies the differential subordination

\[
\left( \frac{DR^n f(z)}{z} \right)^{\delta-1} (DR^n f(z))' \prec h(z), \text{ for } z \in U,
\]

then

\[
\left( \frac{DR^n f(z)}{z} \right)^{\delta} \prec q(z), \text{ for } z \in U,
\]

where \( q(z) = \frac{\delta}{z} \int_0^z h(t) t^{\delta-1} dt \). The function \( q \) is convex and it is the best dominant.

**Proof.** Let

\[
p(z) = \left( \frac{DR^n f(z)}{z} \right)^{\delta} = \left( z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} [1 + (j-1) \lambda]^n a_j^2 z^j \right)^{\delta} = \left( 1 + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} [1 + (j-1) \lambda]^n a_j^2 z^j \right)^{\delta} = 1 + \sum_{j=\delta+1}^{\infty} p_j z^{j-1}, \text{ for } z \in U, p \in \mathcal{H}[1, \delta].
\]
Differentiating, we obtain \( (\frac{DR^n_\lambda f(z)}{z})^{\delta-1} (DR^n_\lambda f(z))' = p(z) + \frac{1}{\delta} z p'(z), \ z \in U, \)
and (3) becomes
\[
p(z) + \frac{1}{\delta} z p'(z) \prec h(z), \quad z \in U.
\]

Using Lemma 1, we have
\[
p(z) \prec q(z), \quad z \in U, \text{ i.e. } \left(\frac{DR^n_\lambda f(z)}{z}\right)^\delta \prec q(z) = \frac{\delta}{z^\delta} \int_0^z h(t) t^{\delta-1} dt, \quad z \in U,
\]
and \( q \) is the best dominant.

**Corollary 5.** Let \( h(z) = \frac{1+(2\beta-1)z}{1+z} \) be a convex function in \( U \), where \( 0 \leq \beta < 1 \).

If \( \delta, \lambda \geq 0, \ n \in \mathbb{N}, \ f \in A \) and satisfies the differential subordination
\[
\left(\frac{DR^n_\lambda f(z)}{z}\right)^{\delta-1} (DR^n_\lambda f(z))' \prec h(z), \quad z \in U, \tag{4}
\]
then
\[
\left(\frac{DR^n_\lambda f(z)}{z}\right)^\delta \prec q(z), \quad z \in U,
\]
where \( q \) is given by \( q(z) = (2\beta - 1) + \frac{2(1-\beta)\delta}{z^\delta} \int_0^z \frac{t^{\delta-1}}{1+t} dt, \quad z \in U. \) The function \( q \) is convex and it is the best dominant.

**Proof.** Following the same steps as in the proof of Theorem 4 and considering \( p(z) = \left(\frac{DR^n_\lambda f(z)}{z}\right)^\delta \), the differential subordination (4) becomes
\[
p(z) + \frac{z}{\delta} p'(z) \prec h(z) = \frac{1+(2\beta-1)z}{1+z}, \quad z \in U.
\]

By using Lemma 1 for \( \gamma = \delta \), we have \( p(z) \prec q(z) \), i.e.
\[
\left(\frac{DR^n_\lambda f(z)}{z}\right)^\delta \prec q(z) = \frac{\delta}{z^\delta} \int_0^z h(t) t^{\delta-1} dt = \frac{\delta}{z^\delta} \int_0^z \frac{t^{\delta-1}}{1+t} dt = \frac{\delta}{z^\delta} \int_0^z \left[ (2\beta - 1) t^{\delta-1} + 2(1-\beta) \frac{t^{\delta-1}}{1+t} \right] dt
\]
\[
= (2\beta - 1) + \frac{2(1-\beta)\delta}{z^\delta} \int_0^z \frac{t^{\delta-1}}{1+t} dt, \quad z \in U.
\]
Remark 7. For \( n = 1, \lambda = \frac{1}{2}, \delta = 1 \) we obtain the same example as in [9, Example 7.2.1, p. 273].

Theorem 6. Let \( g \) be a convex function such that \( g(0) = 1 \) and let \( h \) be the function \( h(z) = g(z) + \frac{\lambda}{\delta} g'(z), z \in U \).

If \( \lambda, \delta \geq 0, n \in \mathbb{N}, f \in A \) and the differential subordination
\[
\frac{\delta + 1}{\delta} \frac{D^\alpha f(z)}{(D^\alpha f(z))} + \frac{2}{\delta} \frac{D^\alpha f(z)}{(D^\alpha f(z))} \left[ \frac{(D^\alpha f(z))'}{D^\alpha f(z)} - 2 \frac{(D^{\alpha+1} f(z))'}{D^\alpha f(z)} \right] < h(z),
\]
\( z \in U \), holds, then
\[
z \frac{D^\alpha f(z)}{(D^\alpha f(z))} < g(z), \quad z \in U,
\]
and this result is sharp.

Proof. For \( f \in A, f(z) = z + \sum_{j=2}^{\infty} a_j z^j \) we have
\[
D^\alpha f(z) = z + \sum_{j=2}^{\infty} \frac{\lambda}{\delta} (n+j-1)! [1 + (j-1) \lambda] a_j^j z^j, z \in U.
\]
Consider \( p(z) = z \frac{D^\alpha f(z)}{(D^\alpha f(z))} \) and we obtain
\[
p(z) + \frac{\lambda}{\delta} p'(z) = \frac{\delta + 1}{\delta} \frac{D^\alpha f(z)}{(D^\alpha f(z))} + \frac{2}{\delta} \frac{D^\alpha f(z)}{(D^\alpha f(z))} \left[ \frac{(D^\alpha f(z))'}{D^\alpha f(z)} - 2 \frac{(D^{\alpha+1} f(z))'}{D^\alpha f(z)} \right].
\]
Relation (5) becomes
\[
p(z) + \frac{\lambda}{\delta} p'(z) \prec h(z) = g(z) + \frac{\lambda}{\delta} g'(z), \quad z \in U.
\]
By using Lemma 2, we have
\[
p(z) \prec g(z), \quad z \in U, \text{ i.e. } z \frac{D^\alpha f(z)}{(D^\alpha f(z))} \prec g(z), \quad z \in U.
\]

Theorem 7. Let \( h \) be an holomorphic function which satisfies the inequality
\[
Re \left( 1 + \frac{h''(z)}{h'(z)} \right) > -\frac{1}{2}, z \in U, \text{ and } h(0) = 1.
\]
If \( \lambda, \delta \geq 0, n \in \mathbb{N}, f \in A \) and satisfies the differential subordination
\[
\frac{\delta + 1}{\delta} \frac{D^\alpha f(z)}{(D^\alpha f(z))} + \frac{2}{\delta} \frac{D^\alpha f(z)}{(D^\alpha f(z))} \left[ \frac{(D^\alpha f(z))'}{D^\alpha f(z)} - 2 \frac{(D^{\alpha+1} f(z))'}{D^\alpha f(z)} \right] < h(z),
\]
\( z \in U \), then
\[
z \frac{D^\alpha f(z)}{(D^\alpha f(z))} \prec q(z), \quad z \in U,
\]
where \( q(z) = \frac{\lambda}{\delta} \int_0^z h(t) t^{\delta-1} dt \). The function \( q \) is convex and it is the best dominant.
Proof. Let \( p(z) = z \frac{DR^nf(z)}{(DR^{n+1}f(z))^2} \), \( z \in U, p \in \mathcal{H}[1,1] \).

Differentiating, we obtain

\[
p(z) + \frac{z}{\delta} p'(z) = z^{\delta+1} \frac{DR^n f(z)}{(DR^{n+1}f(z))^2} + \frac{z^2}{\delta} \frac{DR^n f(z)}{(DR^{n+1}f(z))^2} \left[ \frac{(DR^n f(z))'}{DR^n f(z)} - 2 \frac{(DR^{n+1}f(z))'}{DR^{n+1}f(z)} \right],
\]

\( z \in U \), and (6) becomes

\[
p(z) + \frac{z}{\delta} p'(z) \prec h(z), \quad z \in U.
\]

Using Lemma 1, we have

\[
p(z) < q(z), \quad z \in U, \quad \text{i.e.} \quad z \frac{DR^n f(z)}{(DR^{n+1}f(z))^2} < q(z) = \frac{\delta}{z^\delta} \int_0^z h(t) t^{\delta-1} dt, \quad z \in U,
\]

and \( q \) is the best dominant.

**Theorem 8.** Let \( g \) be a convex function such that \( g(0) = 1 \) and let \( h \) be the function \( h(z) = g(z) + \frac{z}{\delta} g'(z), \quad z \in U \).

If \( \lambda, \delta \geq 0, n \in \mathbb{N}, f \in \mathcal{A} \) and the differential subordination

\[
z^{\delta+2} \frac{(DR^n f(z))'}{DR^n f(z)} + \frac{z^3}{\delta} \left[ \frac{(DR^n f(z))''}{DR^n f(z)} - \left( \frac{(DR^n f(z))'}{DR^n f(z)} \right)^2 \right] \prec h(z), \quad z \in U (7)
\]

holds, then

\[
z^{\delta} \frac{(DR^n f(z))'}{DR^n f(z)} \prec g(z), \quad z \in U.
\]

This result is sharp.

Proof. Let \( p(z) = z^2 \frac{(DR^n f(z))'}{DR^n f(z)} \). We deduce that \( p \in \mathcal{H}[0,1] \).

Differentiating, we obtain

\[
p(z) + \frac{z}{\delta} p'(z) = z^2 \frac{2 \delta + 2}{\delta} \frac{(DR^n f(z))'}{DR^n f(z)} + \frac{z^3}{\delta} \left[ \frac{(DR^n f(z))''}{DR^n f(z)} - \left( \frac{(DR^n f(z))'}{DR^n f(z)} \right)^2 \right], \quad z \in U.
\]

Using the notation in (7), the differential subordination becomes

\[
p(z) + \frac{1}{\delta} z^2 p'(z) \prec h(z) = g(z) + \frac{z}{\delta} g'(z).
\]

By using Lemma 2, we have

\[
p(z) \prec g(z), \quad z \in U, \quad \text{i.e.} \quad z^2 \frac{(DR^n f(z))'}{DR^n f(z)} \prec g(z), \quad z \in U,
\]

and this result is sharp.
Theorem 9. Let $h$ be an holomorphic function which satisfies the inequality 
\[ \Re \left( 1 + \frac{zh''(z)}{h(z)} \right) > -\frac{1}{2}, \quad z \in U, \quad \text{and} \quad h(0) = 1. \]
If $\lambda, \delta \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and satisfies the differential subordination 
\[ z^2 \frac{\delta + 2}{\delta} \left( DR^n_\lambda f(z) \right)' + z^3 \left[ \left( DR^n_\lambda f(z) \right)'' - \left( DR^n_\lambda f(z) \right)' \right]^2 < h(z), \quad z \in U, \]  
then 
\[ z^2 \frac{\left( DR^n_\lambda f(z) \right)'}{DR^n_\lambda f(z)} < q(z), \quad z \in U, \]  
where $q(z) = \frac{\delta}{z^2} \int_0^z h(t) t^{\delta - 1} dt$. The function $q$ is convex and it is the best dominant.

Proof. Let $p(z) = z^2 \frac{\left( DR^n_\lambda f(z) \right)'}{DR^n_\lambda f(z)}$, $z \in U$, $p \in \mathcal{H}[0, 1]$.
Differentiating, we obtain 
\[ p(z) + \frac{\delta p'(z)}{\delta} = z^2 \frac{\delta + 2}{\delta} \left( DR^n_\lambda f(z) \right)' + z^3 \left[ \left( DR^n_\lambda f(z) \right)'' - \left( DR^n_\lambda f(z) \right)' \right]^2, \quad z \in U, \text{ and } (8) \]
becomes
\[ p(z) + \frac{1}{\delta} zp'(z) < h(z), \quad z \in U. \]
Using Lemma 1, we have 
\[ p(z) < q(z), \quad z \in U, \quad \text{i.e.} \quad z^2 \frac{\left( DR^n_\lambda f(z) \right)'}{DR^n_\lambda f(z)} < q(z) = \frac{\delta}{z^2} \int_0^z h(t) t^{\delta - 1} dt, \quad z \in U, \]
and $q$ is the best dominant.

Theorem 10. Let $g$ be a convex function such that $g(0) = 1$ and let $h$ be the function 
\[ h(z) = g(z) + zg'(z), \quad z \in U. \]
If $\lambda \geq 0$, $n \in \mathbb{N}$, $f \in \mathcal{A}$ and the differential subordination 
\[ 1 - \frac{DR^n_\lambda f(z) \cdot \left( DR^n_\lambda f(z) \right)'}{\left[ \left( DR^n_\lambda f(z) \right)' \right]^2} < h(z), \quad z \in U \]  
holds, then 
\[ \frac{DR^n_\lambda f(z)}{z \left( DR^n_\lambda f(z) \right)' < g(z), \quad z \in U. \]
This result is sharp.
Proof. Let \( p(z) = \frac{DR_n^\lambda f(z)}{z(DR_n^\lambda f(z))} \). We deduce that \( p \in H[1, 1] \).

Differentiating, we obtain \( 1 - \frac{DR_n^\lambda f(z)\cdot(DR_n^\lambda f(z))^\prime}{[DR_n^\lambda f(z)]^2} = p(z) +zp^\prime(z), \ z \in U \).

Using the notation in (9), the differential subordination becomes
\[
p(z) + zp^\prime(z) \prec h(z) = g(z) + zg^\prime(z).
\]

By using Lemma 2, we have
\[
p(z) \prec g(z), \ z \in U, \ i.e. \ \frac{DR_n^\lambda f(z)}{z(DR_n^\lambda f(z))} \prec g(z), \ z \in U,
\]
and this result is sharp.

**Theorem 11.** Let \( h \) be an holomorphic function which satisfies the inequality
\[
\text{Re} \left( 1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2}, \ z \in U, \text{ and } h(0) = 1.
\]

If \( \lambda \geq 0, \ n \in \mathbb{N}, \ f \in \mathcal{A} \) and satisfies the differential subordination
\[
1 - \frac{DR_n^\lambda f(z)\cdot(DR_n^\lambda f(z))^\prime}{[DR_n^\lambda f(z)]^2} \prec h(z), \ z \in U,
\]
then
\[
\frac{DR_n^\lambda f(z)}{z(DR_n^\lambda f(z))} \prec q(z), \ z \in U,
\]
where \( q(z) = \frac{1}{z} \int_0^z h(t)dt \). The function \( q \) is convex and it is the best dominant.

Proof. Let \( p(z) = \frac{DR_n^\lambda f(z)}{z(DR_n^\lambda f(z))} \), \ z \in U, \ p \in H[0, 1].

Differentiating, we obtain \( 1 - \frac{DR_n^\lambda f(z)\cdot(DR_n^\lambda f(z))^\prime}{[DR_n^\lambda f(z)]^2} = p(z) +zp^\prime(z), \ z \in U, \) and
\[
(10)
\]
becomes
\[
p(z) + zp^\prime(z) \prec h(z), \ z \in U.
\]

Using Lemma 1, we have
\[
p(z) \prec q(z), \ z \in U, \ i.e. \ \frac{DR_n^\lambda f(z)}{z(DR_n^\lambda f(z))} \prec q(z) = \frac{1}{z} \int_0^z h(t)dt, \ z \in U,
\]
and \( q \) is the best dominant.
Corollary 12. Let 
\[ h(z) = \frac{1 + 2\beta - 1}{1 + z} \]
be a convex function in \( U \), where \( 0 \leq \beta < 1 \). If \( \lambda \geq 0, \ n \in \mathbb{N}, \ f \in \mathcal{A} \) and satisfies the differential subordination
\[
1 - \frac{R D^n_{\lambda, \alpha} f(z) \cdot \left( R D^n_{\lambda, \alpha} f(z) \right)''}{\left( R D^n_{\lambda, \alpha} f(z) \right)'} \prec h(z), \ z \in U, \tag{11}
\]
then
\[
\frac{R D^n_{\lambda, \alpha} f(z)}{z \left( R D^n_{\lambda, \alpha} f(z) \right)'} \prec q(z), \ z \in U,
\]
where \( q \) is given by
\[
q(z) = (2\beta - 1) + 2 \left( 1 - \beta \right) \frac{\ln(1+z)}{z}, \ z \in U.
\]
The function \( q \) is convex and it is the best dominant.

Proof. Following the same steps as in the proof of Theorem 11 and considering \( p(z) = \frac{R D^n_{\lambda, \alpha} f(z)}{z \left( R D^n_{\lambda, \alpha} f(z) \right)'} \), the differential subordination (11) becomes
\[
p(z) + z p'(z) \prec h(z) = \frac{1 + (2\beta - 1)z}{1 + z}, \ z \in U.
\]
By using Lemma 1 for \( \gamma = 1 \), we have \( p(z) \prec q(z) \), i.e.
\[
\frac{R D^n_{\lambda, \alpha} f(z)}{z \left( R D^n_{\lambda, \alpha} f(z) \right)'} \prec q(z) = \frac{1}{z} \int_0^z h(t) dt = \frac{1}{z} \int_0^z \frac{1 + (2\beta - 1)t}{1 + t} dt = \frac{1}{z} \int_0^z \left[ (2\beta - 1) + \frac{2(1 - \beta)}{1 + t} \right] dt = (2\beta - 1) + 2 \left( 1 - \beta \right) \frac{\ln(1+z)}{z}, \ z \in U.
\]

Example 1. Let \( h(z) = \frac{1 - z}{1 + z} \) a convex function in \( U \) with \( h(0) = 1 \) and \( Re \left( \frac{2h''(z)}{R'(z)} + 1 \right) > -\frac{1}{2} \).

Let \( f(z) = z + z^2 \), \( z \in U \). For \( n = 1, \ \lambda = \frac{1}{2} \), we obtain \( R^1 f(z) = z f'(z) = z + 2z^2, \ D_{\frac{1}{2}} f(z) = \frac{1}{2} f(z) + \frac{1}{2} z f'(z) = z + \frac{3}{2} z^2, \ D_{\frac{1}{2}}^1 f(z) = z + 3 z^2, \ z \in U. \)

Then \( \left( D_{\frac{1}{2}}^1 f(z) \right)' = 1 + 6z, \ \left( D_{\frac{1}{2}}^1 f(z) \right)'' = 6, \)
\[
\frac{D_{\frac{1}{2}}^1 f(z)}{z \left( D_{\frac{1}{2}}^1 f(z) \right)'} = \frac{z + 3z^2}{z(1 + 6z)} = \frac{1 + 3z}{1 + 6z},
\]
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We have 
\[ q(z) = \frac{1}{2} \int_0^z \frac{1-t}{1+t} \, dt = -1 + \frac{2 \ln(1+z)}{z}. \]

Using Theorem 11 we obtain 
\[ \frac{18z^2 + 6z + 1}{(1+6z)^2} \prec 1 - \frac{z}{1+z}, \quad z \in U, \]
induce 
\[ \frac{1+3z}{1+6z} \prec -1 + \frac{2 \ln(1+z)}{z}, \quad z \in U. \]

**Theorem 13.** Let \( g \) be a convex function such that \( g(0) = 0 \) and let \( h \) be the function 
\[ h(z) = g(z) + zg'(z), \quad z \in U. \]

If \( \lambda \geq 0, \, n \in \mathbb{N}, \, f \in \mathcal{A} \) and the differential subordination 
\[ \left[(DR^n_\lambda f(z))'\right]^2 + DR^n_\lambda f(z) \cdot (DR^n_\lambda f(z))'' \prec h(z), \quad z \in U \] 
(12)
holds, then 
\[ \frac{DR^n_\lambda f(z) \cdot (DR^n_\lambda f(z))'}{z} \prec g(z), \quad z \in U. \]
This result is sharp.

**Proof.** Let \( p(z) = \frac{DR^n_\lambda f(z) \cdot (DR^n_\lambda f(z))'}{z} \). We deduce that \( p \in \mathcal{H}[0,1] \).

Differentiating, we obtain 
\[ \left[(DR^n_\lambda f(z))'\right]^2 + DR^n_\lambda f(z) \cdot (DR^n_\lambda f(z))'' = p(z) + 
zp'(z), \quad z \in U. \]

Using the notation in (12), the differential subordination becomes 
\[ p(z) +zp'(z) \prec h(z) = g(z) + zg'(z). \]

By using Lemma 2, we have 
\[ p(z) \prec g(z), \quad z \in U, \quad \text{i.e.} \quad \frac{DR^n_\lambda f(z) \cdot (DR^n_\lambda f(z))'}{z} \prec g(z), \quad z \in U, \]
and this result is sharp.

**Theorem 14.** Let \( h \) be an holomorphic function which satisfies the inequality 
\[ \text{Re} \left(1 + \frac{zh''(z)}{h'(z)}\right) > -\frac{1}{2}, \quad z \in U, \quad \text{and} \quad h(0) = 0. \]

If \( \lambda \geq 0, \, n \in \mathbb{N}, \, f \in \mathcal{A} \) and satisfies the differential subordination 
\[ \left[(DR^n_\lambda f(z))'\right]^2 + DR^n_\lambda f(z) \cdot (DR^n_\lambda f(z))'' \prec h(z), \quad z \in U, \] 
(13)
then
\[ \frac{DR^n_\lambda f(z) \cdot (DR^n_\lambda f(z))'}{z} \prec q(z), \quad z \in U, \]
where \( q(z) = \frac{1}{z} \int_0^z h(t)dt \). The function \( q \) is convex and it is the best dominant.

**Proof.** Let \( p(z) = \frac{DR^n_\lambda f(z) \cdot (DR^n_\lambda f(z))'}{z}, \quad z \in U, \quad p \in \mathcal{H}[0, 1] \).

Differentiating, we obtain
\[ \left( (DR^n_\lambda f(z))' \right)^2 + DR^n_\lambda f(z) \cdot (DR^n_\lambda f(z))'' = p(z) + zp'(z), \quad z \in U, \]
and (13) becomes
\[ p(z) + zp'(z) \prec h(z), \quad z \in U. \]

Using Lemma 1, we have
\[ p(z) \prec q(z), \quad z \in U, \quad \text{i.e.} \quad \frac{DR^n_\lambda f(z) \cdot (DR^n_\lambda f(z))'}{z} \prec q(z) = \frac{1}{z} \int_0^z h(t)dt, \quad z \in U, \]
and \( q \) is the best dominant.

**Corollary 15.** Let \( h(z) = \frac{1+2(2\beta-1)z}{1+z} \) be a convex function in \( U \), where \( 0 \leq \beta < 1 \).

If \( \lambda \geq 0, \quad n \in \mathbb{N}, \quad f \in \mathcal{A} \) and satisfies the differential subordination
\[ \left( (DR^n_\lambda f(z))' \right)^2 + DR^n_\lambda f(z) \cdot (DR^n_\lambda f(z))'' \prec h(z), \quad z \in U, \quad (14) \]
then
\[ \frac{DR^n_\lambda f(z) \cdot (DR^n_\lambda f(z))'}{z} \prec q(z), \quad z \in U, \]
where \( q \) is given by
\[ q(z) = (2\beta-1) + 2(1-\beta) \ln \left( \frac{1+z}{z} \right), \quad z \in U. \]
The function \( q \) is convex and it is the best dominant.

**Proof.** Following the same steps as in the proof of Theorem 14 and considering
\[ p(z) = \frac{DR^n_\lambda f(z) \cdot (DR^n_\lambda f(z))'}{z}, \quad \text{the differential subordination} \quad (14) \]
becomes
\[ p(z) + zp'(z) \prec h(z) = \frac{1+2(2\beta-1)z}{1+z}, \quad z \in U. \]

By using Lemma 1 for \( \gamma = 1 \), we have \( p(z) \prec q(z) \), i.e.
\[ \frac{DR^n_\lambda f(z) \cdot (DR^n_\lambda f(z))'}{z} \prec q(z) = \frac{1}{z} \int_0^z h(t)dt = \]
\[ = \frac{1}{z} \int_0^z \frac{1+(2\beta-1)t}{1+t} dt = \frac{1}{z} \int_0^z \left[ (2\beta-1) + 2 \frac{(1-\beta)}{1+t} \right] dt \]
\[ = (2\beta-1) + 2(1-\beta) \frac{\ln (1+z)}{z}, \quad z \in U. \]
Example 2. Let \( h(z) = \frac{1-z}{1+z} \) a convex function in \( U \) with \( h(0) = 1 \) and
\[
\text{Re} \left( \frac{z h''(z)}{h'(z)} + 1 \right) > -\frac{1}{2}.
\]
Let \( f(z) = z + z^2, z \in U \). For \( n = 1, \lambda = \frac{1}{2} \), we obtain
\[
DR^1_\lambda f(z) = z + 3z^2, \quad z \in U.
\]
Then
\[
\left( DR^1_\lambda f(z) \right)' = 1 + 6z,
\]
\[
\frac{DR^n_\lambda f(z)(DR^n_\lambda f(z))'}{z} = \frac{(z+3z^2)(1+6z)}{z} = 18z^2 + 9z + 1,
\]
\[
\left[ \left( DR^1_\lambda f(z) \right)' \right]^2 + DR^1_\lambda f(z) \cdot \left( DR^1_\lambda f(z) \right)' = (1 + 6z)^2 + (z + 3z^2) \cdot 6 = 54z^2 + 18z + 1.
\]
We have \( q(z) = \frac{1}{z} \int_0^z \frac{1-3}{1+z} dt = -1 + \frac{2 \ln(1+z)}{z} \).
Using Theorem 14 we obtain
\[
54z^2 + 18z + 1 < \frac{1 - z}{1+z}, \quad z \in U,
\]
induce
\[
18z^2 + 9z + 1 < -1 + \frac{2 \ln(1+z)}{z}, \quad z \in U.
\]

Theorem 16. Let \( g \) be a convex function such that \( g(0) = 0 \) and let \( h \) be the function
\[ h(z) = g(z) + \frac{1}{1+z} g'(z), z \in U. \]
If \( \lambda \geq 0, \delta \in (0, 1), n \in \mathbb{N}, f \in \mathcal{A} \) and the differential subordination
\[
\left( \frac{z}{DR^n_\lambda f(z)} \right)^\delta \frac{DR^{n+1}_\lambda f(z)}{1 - \delta \left( \frac{DR^{n+1}_\lambda f(z)}{DR^n_\lambda f(z)} \right)^\delta} \prec h(z), \quad z \in U
\]
holds, then
\[
\frac{DR^{n+1}_\lambda f(z)}{z} \cdot \left( \frac{z}{DR^n_\lambda f(z)} \right)^\delta \prec g(z), \quad z \in U.
\]
This result is sharp.

Proof. Let \( p(z) = \frac{DR^{n+1}_\lambda f(z)}{z} \cdot \left( \frac{z}{DR^n_\lambda f(z)} \right)^\delta \). We deduce that \( p \in \mathcal{H}[1, 1] \).
Differentiating, we obtain
\[
\left( \frac{z}{DR^n_\lambda f(z)} \right)^\delta \frac{DR^{n+1}_\lambda f(z)}{1 - \delta \left( \frac{DR^{n+1}_\lambda f(z)}{DR^n_\lambda f(z)} \right)^\delta} = p(z) + \frac{1}{1-z} z p'(z), \quad z \in U.
\]
Using the notation in (15), the differential subordination becomes
\[
p(z) + \frac{1}{1-z} z p'(z) \prec h(z) = g(z) + \frac{z}{1-\delta} g'(z).
\]
By using Lemma 2, we have
\[ p(z) \prec g(z), \quad z \in U, \quad \text{i.e.} \quad \frac{DR_{\lambda}^{n+1} f(z)}{z} \cdot \left( \frac{z}{DR_{\lambda}^{n} f(z)} \right)^{\delta} \prec g(z), \quad z \in U, \]
and this result is sharp.

**Theorem 17.** Let \( h \) be an holomorphic function which satisfies the inequality
\[ \Re \left( 1 + \frac{zh''(z)}{h'(z)} \right) > -\frac{1}{2}, \quad z \in U, \quad \text{and} \quad h(0) = 1. \]
If \( \lambda \geq 0, \quad \delta \in (0, 1), \quad n \in \mathbb{N}, \quad f \in A \) and satisfies the differential subordination
\[ \left( \frac{z}{DR_{\lambda}^{n} f(z)} \right)^{\delta} \frac{DR_{\lambda}^{n+1} f(z)}{1 - \delta} \left( \frac{DR_{\lambda}^{n+1} f(z)}{DR_{\lambda}^{n+1} f(z)} - \delta \frac{(DR_{\lambda}^{n+1} f(z))'}{DR_{\lambda}^{n+1} f(z)} \right) \prec h(z), \quad z \in U, \]
then
\[ \frac{DR_{\lambda}^{n+1} f(z)}{z} \cdot \left( \frac{z}{DR_{\lambda}^{n} f(z)} \right)^{\delta} \prec q(z), \quad z \in U, \]
where \( q(z) = \frac{1 - \delta}{2 + \delta} \int_{0}^{z} h(t) t^{-\delta} dt \). The function \( q \) is convex and it is the best dominant.

**Proof.** Let \( p(z) = \frac{DR_{\lambda}^{n+1} f(z)}{z} \cdot \left( \frac{z}{DR_{\lambda}^{n} f(z)} \right)^{\delta}, \quad z \in U, \quad p \in \mathcal{H}[0, 1]. \)
Differentiating, we obtain
\[ \frac{z}{DR_{\lambda}^{n} f(z)} \delta \frac{DR_{\lambda}^{n+1} f(z)}{1 - \delta} \left( \frac{DR_{\lambda}^{n+1} f(z)}{DR_{\lambda}^{n+1} f(z)} - \delta \frac{(DR_{\lambda}^{n+1} f(z))'}{DR_{\lambda}^{n+1} f(z)} \right) = p(z) + \frac{1}{1 - \delta} \frac{z p'(z)}{z}, \quad z \in U, \quad \text{and} \quad (16) \text{ becomes} \]
\[ p(z) + \frac{1}{1 - \delta} \frac{z p'(z)}{z} \prec h(z), \quad z \in U. \]
Using Lemma 1, we have
\[ p(z) \prec q(z), \quad z \in U, \quad \text{i.e.} \quad \frac{DR_{\lambda}^{n+1} f(z)}{z} \cdot \left( \frac{z}{DR_{\lambda}^{n} f(z)} \right)^{\delta} \prec q(z) = \frac{1 - \delta}{2 + \delta} \int_{0}^{z} h(t) t^{-\delta} dt, \]
\( z \in U, \) and \( q \) is the best dominant.

**Remark 8.** For \( \lambda = 1 \) we obtain the same results for the operator \( SR_{\lambda}^{n} \).

**References**


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