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A NOTION OF OPEN GENERALIZED MORPHISM THAT CARRIES AMENABILITY

Mădălina Roxana Buneci

Abstract. The purpose of this paper is to reformulate in the setting of topological groupoids the concept of morphism introduced by Zakrzewski [8]. We shall also introduce a notion of openness for these generalized morphisms and we shall prove that open generalized morphisms of locally compact Hausdorff second countable groupoids carry topological amenability.

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1.Introduction

There are many equivalent definitions of the (algebraic) groupoid. The shortest definition is : a groupoid is a small category with inverses. Since we shall use the notion of amenability for groupoids introduced by Jean Renault in [6] and extensively studied in [1], we shall prefer the same definition as in [6] (and [1]):

A *groupoid* is a set Γ endowed with a *product map (multiplication)*

$$(x, y) \to xy \ \left[: \Gamma^{(2)} \to \Gamma\right]$$

where $\Gamma^{(2)}$ is a subset of $\Gamma \times \Gamma$ called the *set of composable pairs*, and an *inverse map*

$$x \to x^{-1} \ [: \Gamma \to \Gamma]$$

such that the following conditions hold:

1. If $(x, y) \in \Gamma^{(2)}$ and $(y, z) \in \Gamma^{(2)}$, then $(xy, z) \in \Gamma^{(2)}$, $(x, yz) \in \Gamma^{(2)}$ and $(xy)\, z = x\,(yz)$.
2. $(x^{-1})^{-1} = x$ for all $x \in \Gamma$.
3. For all $x \in \Gamma$, $(x, x^{-1}) \in \Gamma^{(2)}$, and if $(z, x) \in \Gamma^{(2)}$, then $(zx)\, x^{-1} = z$.
4. For all $x \in \Gamma$, $(x^{-1}, x) \in \Gamma^{(2)}$, and if $(x, y) \in \Gamma^{(2)}$, then $x^{-1}(xy) = y$.

The maps r and d on Γ, defined by the formulae $r(x) = xx^{-1}$ and $d(x) = x^{-1}x$, are called the *range* and the (*domain*) *source* maps. It follows easily from the definition that they have a common image called the unit space of G, which is denoted $\Gamma^{(0)}$. Its elements are units in the sense that $xd(x) = r(x)x = x$. It is useful to note that a pair (x, y) lies in $\Gamma^{(2)}$ precisely when $d(x) = r(y)$, and that the cancellation laws hold (e.g. $xy = xz$ iff $y = z$). The fibers of the range and the source maps are denoted $\Gamma^u = r^{-1}(\{u\})$ and $\Gamma_v = d^{-1}(\{v\})$, respectively.

Let Γ and G be groupoids. A function $\varphi : \Gamma \to G$ is a (groupoid) *homomorphism* if $(\varphi(\gamma_1), \varphi(\gamma_2)) \in G^{(2)}$ and $\varphi(\gamma_1)\varphi(\gamma_2) = \varphi(\gamma_1\gamma_2)$ whenever $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$. Let us note that since $\varphi\left(\gamma_1^{-1}\right)\varphi(\gamma_1)\varphi(\gamma_2) = \varphi\left(\gamma_1^{-1}\gamma_1\gamma_2\right) = \varphi(\gamma_2)$, $\varphi\left(\gamma_1^{-1}\right) = \varphi(\gamma_1)^{-1}$. Hence $\varphi\left(\gamma_1\gamma_1^{-1}\right) = \varphi(\gamma_1)\varphi(\gamma_1)^{-1} \in G^{(0)}$. In the sequel we shall denote by $\varphi^{(0)} : \Gamma^{(0)} \to G^{(0)}$ the restriction of φ to $\Gamma^{(0)}$.

A *topological groupoid* consists of a groupoid Γ and a topology compatible with the groupoid structure. This means that:

1. $x \to x^{-1}$ $[: \Gamma \to \Gamma]$ is continuous.
2. (x, y) $\left[: \Gamma^{(2)} \to \Gamma\right]$ is continuous where $\Gamma^{(2)}$ has the induced topology from $\Gamma \times \Gamma$.

Let Γ and G be topological groupoids. By a *topological homomorphism* from Γ to G we shall mean a homomorphism $\varphi : \Gamma \to G$ which is continuous.

We shall be exclusively concerned with topological groupoids which are locally compact Hausdorff (and we shall call them locally compact Hausdorff groupoids).

Zakrzewski introduced in [8] a notion of morphism between groupoids which reduces to a group homomorphism if groupoids are groups and to an ordinary map in the reverse direction if groupoids are sets. A morphism in the sense of Zakrzewski is a relation satisfying some additional properties.

The purpose of this paper is to reformulate the notion of groupoid morphism given by Zakrzewski in terms of groupoid actions and to extend it to the setting of topological groupoids. We shall also define a notion of open morphism and we shall prove that open generalized morphisms of locally compact Hausdorff second countable groupoids carry topological amenability.

2. Actions of groupoids

Definition 1. *Let Γ be a groupoid and X be a set. We say Γ acts (to the left) on X if there is a map $\rho : X \to \Gamma^{(0)}$ (called a momentum map) and a map $(\gamma, x) \to \gamma \cdot x$ from*

$$\Gamma *_{\rho} X = \{(\gamma, x) : d(\gamma) = \rho(x)\}$$

to X, called (left) action, such that:

1. $\rho(\gamma \cdot x) = r(\gamma)$ *for all* $(\gamma, x) \in \Gamma *_{\rho} X$.
2. $\rho(x) \cdot x = x$ *for all* $x \in X$.
3. *If* $(\gamma_2, \gamma_1) \in \Gamma^{(2)}$ *and* $(\gamma_1, x) \in \Gamma *_{\rho} X$, *then* $(\gamma_2\gamma_1) \cdot x = \gamma_2 \cdot (\gamma_1 \cdot x)$.

If Γ is a topological groupoid and X is a topological space, then we say that a left action is continuous if the mappings ρ and $(\gamma, x) \to \gamma \cdot x$ are continuous, where $\Gamma *_{\rho} X$ is endowed with the relative product topology coming from $\Gamma \times X$.

The difference with the definition of action in [5] is that we do not assume that the momentum map is surjective and open.

The action is called *free* if $(\gamma, x) \in \Gamma *_{\rho} X$ and $\gamma \cdot x = x$ implies $\gamma \in \Gamma^{(0)}$.

The continuous action is called *proper* if the map $(\gamma, x) \to (\gamma \cdot x, x)$ from $\Gamma *_{\rho} X$ to $X \times X$ is proper (i.e. the inverse image of each compact subset of $X \times X$ is a compact subset of $\Gamma *_{\rho} X$).

In the same manner, we define a *right action* of Γ on X, using a continuous map $\sigma : X \to \Gamma^{(0)}$ and a map $(x, \gamma) \to x \cdot \gamma$ from

$$X *_{\sigma} \Gamma = \{(x, \gamma) : \sigma(x) = r(\gamma)\}$$

to X.

The simplest example of proper and free action is the case when the groupoid Γ acts upon itself by either right or left translation (multiplication).

Definition 2. *Let Γ_1, Γ_2 be two groupoids and X be set. Let us assume that Γ_1 acts to the left on X with momentum map $\rho : X \to \Gamma_1^{(0)}$, and that Γ_2 acts to the right on X with momentum map $\sigma : X \to \Gamma_2^{(0)}$. We say that the action commute if*

1. $\rho(x \cdot \gamma_2) = \rho(x)$ *for all* $(x, \gamma_2) \in X *_{\sigma} \Gamma_2$ *and* $\sigma(\gamma_1 \cdot x) = \sigma(x)$ *for all* $(\gamma_1, x) \in \Gamma_1 *_{\rho} X$.

2. $\gamma_1 \cdot (x \cdot \gamma_2) = (\gamma_1 \cdot x) \cdot \gamma_2$ *for all* $(\gamma_1, x) \in \Gamma_1 *_\rho X$*,* $(x, \gamma_2) \in X *_\sigma \Gamma_2$.

3. Open generalized groupoid morphisms

Definition 3. *Let* Γ *and* G *be two groupoids. By an algebraic generalized morphism from* Γ *to* G *we mean a left action of* Γ *on* G *which commutes with the multiplication on* G*.*

The generalized morphism is said continuous (or topological generalized morphism) if the action of Γ *on* G *is continuous (assuming that* Γ *and* G *are topological spaces).*

Piotr Stachura pointed out to us that if we have a morphism in the sense of the preceding definition and if $\rho : G \to \Gamma$ is the momentum map of the left action, then $\rho = \rho \circ r$. Indeed, for any $x \in G$, we have $\rho(x) = \rho(xx^{-1}) = \rho(r(x))$ because of the fact that left action of Γ on G commutes with the multiplication on G.

Therefore an algebraic morphism $h : \Gamma \longrightarrow\!\!\triangleright\, G$ is given by two maps

1. $\rho_h : G^{(0)} \to \Gamma^{(0)}$
2. $(\gamma, x) \to \gamma \cdot_h x$ from $\Gamma \star_h G$ to G, where

$$\Gamma \star_h G = \{(\gamma, x) \in \Gamma \times G : d(\gamma) = \rho_h(r(x))\},$$

satisfying the following conditions:

(1) $\rho_h(r(\gamma \cdot_h x)) = r(\gamma)$ for all $(\gamma, x) \in \Gamma \star_h G$.

(2) $\rho_h(r(x)) \cdot_h x = x$ for all $x \in G$.

(3) $(\gamma_1\gamma_2) \cdot_h x = \gamma_1 \cdot_h (\gamma_2 \cdot_h x)$ for all $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$ and all $(\gamma_2, x) \in \Gamma \star_h G$.

(4) $d(\gamma \cdot_h x) = d(x)$ for all $(\gamma, x) \in \Gamma \star_h G$.

(5) $(\gamma \cdot_h x_1) x_2 = \gamma \cdot_h (x_1 x_2)$ for all $(\gamma, x_1) \in \Gamma \star_h G$ and $(x_1, x_2) \in G^{(2)}$.

In the case continuous generalized morphism the map ρ_h is a continuous map. The map ρ_h is not necessarily open or surjective. However, the image of ρ_h is always a saturated subset of $\Gamma^{(0)}$. Indeed, let $v \sim u = \rho_h(t)$ and let $\gamma \in \Gamma$ be such that $r(\gamma) = v$ and $d(\gamma) = u$. Then v belongs to the image of ρ_h because $v = r(\gamma) = \rho_h(r(\gamma \cdot_h t))$.

Remark 4. *Let h be an algebraic generalized morphism from Γ to G (in the sense of Definition 3). Then h is determined by ρ_h and the restriction of the action to*

$$\left\{(\gamma, t) \in \Gamma \times G^{(0)} : d(\gamma) = \rho_h(t)\right\}.$$

Indeed, using the condition 5, one obtains

$$\gamma \cdot_h x = (\gamma \cdot_h r(x))\, x$$

Let us also note that

$$(\gamma_1\gamma_2) \cdot_h x = ((\gamma_1\gamma_2) \cdot_h r(x))\, x = \gamma_1 \cdot_h (\gamma_2 \cdot_h r(x))\, x$$
$$= (\gamma_1 \cdot_h r(\gamma_2 \cdot_h r(x)))\, (\gamma_2 \cdot_h r(x))\, x.$$

Consequently, for any $\gamma \in \Gamma$ and any $t \in G^{(0)}$ with $\rho_h(t) = d(\gamma)$, we have

$$\left(\gamma^{-1} \cdot_h r(\gamma \cdot_h t)\right)(\gamma \cdot_h t) = \left(\gamma^{-1}\gamma\right) \cdot_h t = d(\gamma) \cdot_h t = \rho_h(t) \cdot_h t = t.$$

Thus for any $\gamma \in \Gamma$ and any $t \in G^{(0)}$ with $\rho_h(t) = d(\gamma)$,

$$(\gamma \cdot_h t)^{-1} = \gamma^{-1} \cdot_h r(\gamma \cdot_h t).$$

Therefore, algebraically, the notion of generalized morphism in the sense of Definition 3 is the same with that introduced in [8, p. 351]. In order to prove the equivalence of these definitions, we can use [7, Proposition 2.7/p. 5], taking $f = \rho_h$ and $g(\gamma, t) = \gamma \cdot_h t$.

Remark 5. *Let $h : \Gamma \longrightarrow\!\!\!\!\triangleright\, G$ be a continuous generalized morphism of locally compact Hausdorff groupoids (in the sense of Definition 3). Then G is left Γ-space under the action $(\gamma, x) \to \gamma \cdot_h x$, and a right G-space under the multiplication on G. G is a correspondence in the sense of [3] if and only if the left action of Γ on G is proper and ρ_h is open and injective. G is a regular bibundle in the sense of [2, Definition 6/p.103] if and only if the action of Γ is free and transitive along the fibres of d (this means that for all $u \in G^{(0)}$ and x satisfying $d(x) = u$, there is $\gamma \in \Gamma$ such that $\gamma \cdot_h u = x$). Therefore, the* *notion of generalized morphism introduced in Definition 3 is not cover by the notions used in [3] and [2].*

Remark 6. *Let $\varphi : \Gamma \to G$ be a groupoid homomorphism (in the usual sense). Let us assume that $\varphi^{(0)} : \Gamma^{(0)} \to G^{(0)}$ is a surjective map. Then φ can be viewed as a generalized morphism in the sense of Definition 3. Indeed, let $\rho_h : G^{(0)} \to \Gamma^{(0)}$ be a right inverse for $\varphi^{(0)}$, and let us define*

$$\gamma \cdot_h x = \varphi(\gamma)\, x$$

for any $(\gamma, x) \in \{(\gamma, x) \in \Gamma \times G : d(\gamma) = \rho_h(r(x))\}$. Thus we obtain a generalized morphism in the sense of Definition 3. Similarly, any topological homomorphism $\varphi : \Gamma \to G$ for which $\varphi^{(0)} : \Gamma^{(0)} \to G^{(0)}$ admits a continuous right inverse can be viewed as a continuous generalized morphism from Γ to G.

Lemma 7. *Let Γ and G be two groupoids and let h be an algebraic generalized morphism from Γ to G (in the sense of Definition 3). Then the function defined by*

$$r((\gamma_1\gamma_2) \cdot_h x) = r(\gamma_1 \cdot_h (\gamma_2 \cdot_h r(x)))$$

for any $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$ and $(\gamma_1, x) \in \left\{(\gamma, t) \in \Gamma \times G^{(0)} : d(\gamma) = \rho_h(t)\right\}$ gives an action of Γ on $G^{(0)}$.

Proof. Let $(\gamma_1, \gamma_2) \in \Gamma^{(2)}$ and $(\gamma_1, x) \in \left\{(\gamma, t) \in \Gamma \times G^{(0)} : d(\gamma) = \rho_h(t)\right\}$. Using the computation in the Remark 4, we obtain

$$\begin{aligned} r((\gamma_1\gamma_2) \cdot_h x) &= r(((\gamma_1\gamma_2) \cdot_h r(x))\, x) \\ &= r(((\gamma_1\gamma_2) \cdot_h r(x))) = r(\gamma_1 \cdot_h (\gamma_2 \cdot_h r(x))) \end{aligned}$$

Definition 8. *Let Γ and G be topological groupoids and let h be a continuous generalized morphism from Γ to G (in the sense of Definition 3). Then h is said to be open if the following conditions are satisfied:*

1. *The map $(\gamma, t) \to \gamma \cdot_h t$ from $\Gamma *_h G^{(0)} = \left\{(\gamma, t) \in \Gamma \times G^{(0)} : d(\gamma) = \rho_h(t)\right\}$ to G is open, where $\Gamma *_h G^{(0)}$ is endowed with the relative product topology coming from $\Gamma \times G^{(0)}$.*

2. *The map $\rho_h : G^{(0)} \to \Gamma^{(0)}$ has a left continuous inverse (that is a continuous map p_h:$\Gamma^{(0)} \to G^{(0)}$ such that $p_h(\rho_h(t)) = t$ for all $t \in G^{(0)}$).*

Remark 9. *The continuous generalized morphism constructed from a groupoid homomorphism $\varphi : \Gamma \to G$ as in Remark 6 is open (in the sense of Definition 8) if and only if φ is open (as a map from Γ to G).*

Let Γ and G be topological groupoids and let h be a continuous open generalized morphism from Γ to G (in the sense of Definition 8). Let us denote by

$$h(\Gamma) = \{\gamma \cdot_h t : (\gamma, t) \in \Gamma *_h G\}.$$

It is easy to see that $h(\Gamma)$ is an open wide subgroupoid of G.

Definition 10. *Let Γ and G be topological groupoids and let h be a continuous open generalized morphism from Γ to G (in the sense of Definition 8). The subgroupoid $h(\Gamma)$ of G defined above is said the image of Γ by h.*

Lemma 11. *Let Γ and G be locally compact Hausdorff groupoids and let h be a continuous open generalized morphism from Γ to G (in the sense of Definition 8). If Γ and $G^{(0)}$ are second countable, then there is a map $\sigma_h : h(\Gamma) \to \Gamma$ such that*

1. *$\sigma_h(x) \cdot_h d(x) = x$ for all $x \in h(\Gamma)$.*
2. *$\sigma_h(K)$ is relatively compact in Γ for any compact subset K of G.*

*Proof.*Since h be a continuous open generalized morphism from Γ to G, and since Γ is a second countable locally compact space, it follows that $h(\Gamma)$ is a second countable locally compact space.

The map $(\gamma, t) \to \gamma \cdot_h t$ from $\Gamma *_h G^{(0)}$ to $h(\Gamma)$ is a continuous open surjection between locally compact Hausdorff second countable spaces. According to Mackey's Lemma [4, Lemma 1.1/p.102] there is a regular cross section $\sigma : h(\Gamma) \to \Gamma *_h G^{(0)}$. In particular, this means that $\sigma(K)$ is relatively compact in Γ for any compact subset K of $h(\Gamma)$. Let $\sigma_h(x) = p_1(\sigma(x))$ for all $x \in h(\Gamma)$, where p_1 is the first projection. Then $.\sigma_h(x) \cdot_h d(x) = x$ for all $x \in h(\Gamma)$, and $\sigma_h(K)$ is relatively compact in Γ for any compact subset K of G.

4. Open generalized morphisms and topological amenability

The notion of amenability for groupoids was introduced in [6] and was extensively studied in [1].

Definition 12. *A locally compact Hausdorff groupoid* Γ *is said topologically amenable if there exists an approximate invariant continuous mean for the range map* $r : \Gamma \to \Gamma^{(0)}$. *That is a net* $(m_i)_i$ *of continuous systems of probability measures for* r *such that* $\left\| \gamma^{-1} m_i^{r(\gamma)} - m_i^{d(\gamma)} \right\|_1$ *goes to zero uniformly on compact subsets of* Γ. *(see [1, Definition 2.2.1/p.42] and [1, Definition 2.2.8/p. 45])*

A continuous system of probability measures for r [1, Definition 1.1.1/p.19] is a family $m = \left\{ m^u : u \in \Gamma^{(0)} \right\}$ of probability measures on Γ such that

1. the support of m^u is contained in Γ^u;
2. for every continuous function with compact support $f : \Gamma \to \mathbf{C}$, the function
$$u \overset{m(f)}{\to} \int f(\gamma)\, dm^u(\gamma)$$
is continuous.

In fact it is not hard to prove that for every continuous bounded function $f : \Gamma \to \mathbf{C}$, the function

$$u \overset{m(f)}{\to} \int f(\gamma)\, dm^u(\gamma)$$

is continuous. Indeed, let $u_0 \in \Gamma^{(0)}$, let K be a compact neighborhood of u_0 and let $\varepsilon > 0$. According to [1, Lemma 2.2.3/p. 42] there exists a compact subset L of Γ such that $m^u(\Gamma - L) < \frac{\varepsilon}{3M}$ for each $u \in K$, where $M \geq |f(\gamma)|$ for all $\gamma \in \Gamma$. Let $a : \Gamma \to [0, 1]$ be a continuous function with compact support such that $a(\gamma) = 1$ for all $\gamma \in L$. Since $m = \left\{ m^u : u \in \Gamma^{(0)} \right\}$ is a continuous system of probability measures, it follows that there is a neighborhood $V \subset K$ of u_0 such that

$$\left| \int a(\gamma) f(\gamma)\, dm^u(\gamma) - \int a(\gamma) f(\gamma)\, dm^{u_0}(\gamma) \right| < \frac{\varepsilon}{3}$$

for all $u \in V$. Thus for all $u \in V \subset K$ we have

$$\left| \int f(\gamma)\, dm^u(\gamma) - \int f(\gamma)\, dm^{u_0}(\gamma) \right|$$

$$\begin{aligned}
&\leq \left|\int a(\gamma) f(\gamma)\, dm^{u}(\gamma) - \int a(\gamma) f(\gamma)\, dm^{u_0}(\gamma)\right| + \\
&\quad + \left|\int (1-a(\gamma)) f(\gamma)\, dm^{u}(\gamma) - \int (1-a(\gamma)) f(\gamma)\, dm^{u_0}(\gamma)\right| \\
&\leq \frac{\varepsilon}{3} + \left|\int_{\Gamma - L} f(\gamma)\, dm^{u}(\gamma)\right| + \left|\int_{\Gamma - L} f(\gamma)\, dm^{u_0}(\gamma)\right| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}
\end{aligned}$$

THEOREM 13. *Let Γ and G be locally compact Hausdorff second countable groupoids. Let $h : \Gamma \longrightarrow\!\!\!\triangleright\, G$ be a continuous open generalized morphism. If Γ is topologically amenable, then $h(\Gamma)$ is topologically amenable.*

Proof. Let $(m_i)_i$ be an approximate invariant continuous mean for $r : \Gamma \rightarrow \Gamma^{(0)}$. Let f be a continuous function with compact support on G and let us define

$$n_i^t(f) = m_i^{\rho_h(t)}(f_t),$$

where $f_t(\gamma) = f\left(\left(\gamma^{-1} \cdot_h t\right)^{-1}\right) = f\left(\gamma \cdot_h r\left(\gamma^{-1} \cdot_h t\right)\right) = f\left(\gamma \cdot_h r\left(\gamma^{-1} \cdot_h p_h(r(\gamma))\right)\right)$ for all $\gamma \in \Gamma^{\rho_h(t)}$ (p_h is the left inverse for ρ_h). The function F defined by $F(\gamma) = f_t(\gamma)$ for all $\gamma \in \Gamma^{\rho_h(t)}$ is continuous and bounded on Γ. Therefore, for each i, $n_i = \left\{n_i^t,\, t \in G^{(0)}\right\}$ is a continuous system of probability measures for $r : G \rightarrow G^{(0)}$.

Let us show that $(n_i)_i$ is an approximate invariant continuous mean for $r : G \rightarrow G^{(0)}$. Let $\varepsilon > 0$ and let K be a compact subset of G. Let σ_h be a map as in Lemma 11 and let $i(\varepsilon)$ such that for each $i \geq i_\varepsilon$

$$\left\|\gamma m_i^{d(\gamma)} - m_i^{r(\gamma)}\right\|_1 < \varepsilon$$

for all $\gamma \in \sigma_h(K)$.

Let f be a continuous function with compact support on G satisfying $|f(x)| \leq \varepsilon$ and let $x \in K$. For each i we have

$$\begin{aligned}
& n_i^{d(x)}(y \rightarrow f(xy)) \\
&= m_i^{\rho_h(d(x))}\left(\gamma \rightarrow f\left(x\left(\gamma \cdot_h r\left(\gamma^{-1} \cdot_h d(x)\right)\right)\right)\right) \\
&= m_i^{\rho_h(d(x))}\left(\gamma \rightarrow f\left((\sigma_h(x) \cdot_h d(x))\left(\gamma \cdot_h r\left(\gamma^{-1} \cdot_h d(x)\right)\right)\right)\right) \\
&= m_i^{\rho_h(d(x))}\left(\gamma \rightarrow f\left(\sigma_h(x) \cdot_h \left(\gamma \cdot_h r\left(\gamma^{-1} \cdot_h d(x)\right)\right)\right)\right) \\
&= m_i^{\rho_h(d(x))}\left(\gamma \rightarrow f\left((\sigma_h(x)\gamma) \cdot_h r\left(\gamma^{-1} \cdot_h d(x)\right)\right)\right)
\end{aligned}$$

On the other hand

$$\begin{aligned} & n_i^{r(x)}(f) \\ &= m_i^{\rho_h(r(x))}\left(\gamma \to f\left(\gamma \cdot_h r\left(\gamma^{-1} \cdot_h r(x)\right)\right)\right) \\ &= m_i^{r(\sigma_h(x))}\left(\gamma \to f\left(\gamma \cdot_h r\left(\left(\sigma_h(x)^{-1}\gamma\right)^{-1} \cdot_h d(x)\right)\right)\right) \end{aligned}$$

because $r(\sigma_h(x)) = \rho_h(r(\sigma_h(x) \cdot_h d(x))) = \rho_h(r(x))$ and

$$\begin{aligned} r\left(\left(\sigma_h(x)^{-1}\gamma\right)^{-1} \cdot_h d(x)\right) &= r\left(\left(\gamma^{-1}\sigma_h(x)\right) \cdot_h d(x)\right) \\ &= r\left(\gamma^{-1} \cdot_h r(\sigma_h(x) \cdot_h d(x))\right) \\ &= r\left(\gamma^{-1} \cdot_h r(x)\right). \end{aligned}$$

Thus

$$\begin{aligned} \left|xn_i^{d(x)}(f) - n_i^{r(x)}(f)\right| &\leq \left\|\sigma_h(x)\, m_i^{\rho_h(d(x))} - m_i^{r(\sigma_h(x))}\right\|_1 \\ &= \left\|\sigma_h(x)\, m_i^{d(\sigma_h(x))} - m_i^{r(\sigma_h(x))}\right\|_1 \\ &< \varepsilon \end{aligned}$$

for each $i \geq i(\varepsilon)$ and each $x \in K$. Therefore$\left\|xn_i^{d(x)} - n_i^{r(x)}\right\|_1$ converges to zero uniformly on the compact set K.

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References

[1] C. Anantharaman-Delaroche and J. Renault, *Amenable groupoids*, Monographie de L'Enseignement Mathematique No **36**, Geneve, 2000.

[2] N.P. Landsman, *Operator algebras and Poisson manifolds associated to groupoids*, Comm. Math. Phys. **222** (2001), 97-116.

[3] M. Macho-Stadler and M. O'uchi, *Correspondences of groupoid C^**-algebras, J. Operator Theory **42** (1999), 103-119..

[4] G. Mackey, *Induced representations of locally compact groups. I*, Ann. of Math., **55**(1952), 101-139.

[5] P. Muhly, J. Renault and D. Williams, *Equivalence and isomorphism for groupoid C*-algebras*, J. Operator Theory **17**(1987), 3-22.

[6] J. Renault, *A groupoid approach to C^*- algebras*, Lecture Notes in Math. Springer-Verlag, **793**, 1980.
[7] P. Stachura, *Differential groupoids and C^*-algebras*, arXiv:math-QA/9905097.
[8] S. Zakrzewski, *Quantum and Classical pseudogroups I*, Comm. Math. Phys. **134** (1990), 347-370.

Mădălina Roxana Buneci
Department of Automatics
University Constantin Brâncuşi of Târgu-Jiu
Bld. Republicii 1, 210152, Târgu-Jiu, Romania
e-mail: *ada@utgjiu.ro*