ON COMPLETE SPACELIKE HYPERSURFACES IN ANTI-DE SITTER SPACE $H_1^{n+1}(-1)$

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ABSTRACT. In this paper, we investigate complete spacelike hypersurfaces with constant mean curvature in anti-de Sitter space $H_1^{n+1}(-1)$. Some rigidity theorems are obtained for these hypersurfaces.

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1. Introduction

Let $M_1^{n+1}(c)$ denote an $(n+1)$-dimensional Lorentzian manifold of constant curvature $c$, which is called a Lorentzian space form. Then an $(n+1)$-dimensional Lorentzian space form $M_1^{n+1}(c)$ is said to be a de Sitter space $S_1^{n+1}(c)$, a Lorentzian Minkowski space $L_1^{n+1}$ or an anti-de Sitter space $H_1^{n+1}(c)$ respectively, according to its sectional curvature $c > 0$, $c = 0$ or $c < 0$. A hypersurface $M$ in a Lorentzian space form $M_1^{n+1}(c)$ is said to be spacelike if the induced metric on $M$ from that of $M_1^{n+1}(c)$ is positive definite.

In recent years, the study of spacelike hypersurfaces in semi-Riemannian ambients has got increasing interesting motivated by their importance in problems related to Physics, more specifically in the theory of general relativity.

E. Calabi [1] first studied the Bernstein problem for maximal spacelike entire graphs in $R_1^{n+1}$, $n \leq 4$, and proved that it must be hyperplane. Later S.Y. Cheng and S.T. Yau [2] showed that this conclusion remains true for arbitrary $n$. In [4] T. Ishihara proved that complete maximal spacelike hypersurfaces of $M_1^{n+1}(c)$, $c \geq 0$, are totally geodesic. Further, in the same paper, T. Ishihara also proved the following result:

**Theorem 1.1.**[4]. Let $M^n$ be an $n$-dimensional complete maximal spacelike hypersurface in anti-de Sitter space $H_1^{n+1}(-1)$, then the norm square of the second fundamental form of $M$ satisfies $S \leq n$ and $S = n$ if and only if $M^n = H^n(-\frac{n}{m}) \times H^{n-m}(-\frac{n}{n-m})$, $(1 \leq m \leq n - 1)$.

In [3], L.F. Cao and G.X. Wei gave a new characterization of hyperbolic cylinder $M^n = H^n(-\frac{n}{m}) \times H^{n-m}(-\frac{n}{n-m})$ in anti-de Sitter space $H_1^{n+1}(-1)$.  

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Theorem 1.2.[3]. Let $M^n$ be an $n$-dimensional ($n \geq 3$) complete maximal spacelike hypersurface with two distinct principal curvatures $\lambda$ and $\mu$ in anti-de Sitter space $H_1^{n+1}(-1)$. If $\inf |\lambda - \mu| > 0$, then $M^n = H^m(-\frac{a}{m}) \times H^{n-m}(-\frac{n}{n-m})$, $(1 \leq m \leq n-1)$. 

In [5], C.X.Nie studied complete spacelike hypersurfaces with constant mean curvature in anti-de Sitter space $H_1^{n+1}(-1)$ and gave the following result:

**Theorem 1.3.** Let $M^n$ be an $n$-dimensional ($n \geq 3$) complete spacelike hypersurface with constant mean curvature and two distinct principal curvatures $\lambda$ and $\mu$ in anti-de Sitter space $H_1^{n+1}(-1)$. If $\inf |\lambda - \mu| > 0$, then $M^n = H^m(-\frac{1}{\sqrt{a^2}}) \times H^{n-m}(-\frac{1}{\sqrt{a^2}})$, $(1 \leq m \leq n-1)$.

In this note, we also investigate complete spacelike hypersurfaces with constant mean curvature in $H_1^{n+1}(-1)$. More precisely, we prove the following results:

**Theorem 1.4.** Let $M^n$ ($n \geq 3$) be a complete spacelike hypersurface with constant mean curvature $H$ in $H_1^{n+1}(-1)$. Assume that $M^n$ has $n-1$ principal curvatures with the same sign everywhere. If the Ricci curvature $Ric_M$ of $M^n$ and $S$ satisfy the following:

$$
Ric_M \geq -\frac{n(n-2)}{n-1} [1 + \frac{n^2 H^2}{2(n-1)} - \frac{\sqrt{n^2 H^4 + 4(n-1)H^2}}{2(n-1)}] = -C_-(H)
$$

$$
S \leq n + \frac{n^3 H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)} \sqrt{n^2 H^4 + 4(n-1)H^2} = S_+(H),
$$

then $S$ is constant, $S = S_+(H)$ and $M^n = H^1(-\frac{1}{\sqrt{a^2}}) \times H^{n-1}(-\frac{1}{\sqrt{a^2}})$ with $a^2 \leq \frac{1}{n}$.

**Corollary 1.5.** Let $M^n$ ($n \geq 3$) be a complete maximal spacelike hypersurface in $H_1^{n+1}(-1)$. Assume that $M^n$ has $n-1$ principal curvatures with the same sign everywhere. If $Ric_M \geq -\frac{n(n-2)}{n-1}$, then $S = n$ and $M^n = H^1(-n) \times H^{n-1}(-\frac{n}{n-1})$.

**Theorem 1.6.** Let $M^n$ ($n \geq 3$) be a complete spacelike hypersurface with constant mean curvature $H$ in $H_1^{n+1}(-1)$. Assume that $M^n$ has $n-1$ principal curvatures with the same sign everywhere. If $-C_-(H) \leq Ric_M \leq 0$, then $S$ is constant, $S = S_+(H)$ and $M^n = H^1(-\frac{1}{\sqrt{a^2}}) \times H^{n-1}(-\frac{1}{\sqrt{a^2}})$ with $a^2 \leq \frac{1}{n}$.

2. Preliminaries

Let $M^n$ be an $n$-dimensional spacelike hypersurface of $H_1^{n+1}(-1)$. We choose a local field of semi-Riemannian orthonormal frames $\{e_1, \ldots, e_n, e_{n+1}\}$ in $H_1^{n+1}(-1)$ such that, restricted to $M^n$, $e_1, \ldots, e_n$ are tangent to $M^n$. Let $\omega_1, \ldots, \omega_{n+1}$ be
its dual frame field such that the semi-Riemannian metric of $H^{n+1}_1(c)$ is given by
\[ ds^2 = \sum_{A=1}^{n+1} \epsilon_A (\omega_A)^2, \]
where $\epsilon_i = 1$, $i = 1, \cdots, n$ and $\epsilon_{n+1} = -1$. Then the structure equations of $S^{n+1}_1(1)$ are given by
\begin{align*}
  d\omega_A &= \sum_B \epsilon_B \omega_{AB} \wedge \omega_B, \quad \omega_A + \omega_{BA} = 0, \\
  d\omega_{AB} &= \sum_C \epsilon_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2} \sum_{CD} K_{ABCD} \omega_C \wedge \omega_D, \\
  K_{ABCD} &= -\epsilon_A \epsilon_B (\delta_{AC} \delta_{BD} - \delta_{AD} \delta_{BC}).
\end{align*}
We restrict these forms to $M^n$, then $\omega_{n+1} = 0$ and the Riemannian metric of $M^n$ is written as $ds^2 = \sum_i \omega_i^2$. Since
\[ 0 = d\omega_{n+1} = \sum_i \omega_{n+1,i} \wedge \omega_i, \]
by Cartan’s lemma we may write
\[ \omega_{n+1,i} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}. \]
From these formulas, we obtain the structure equations of $M^n$:
\begin{align*}
  d\omega_i &= \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \\
  d\omega_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \\
  R_{ijkl} &= -(\delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk}) - (h_{ik} h_{jl} - h_{il} h_{jk}),
\end{align*}
where $R_{ijkl}$ are the components of curvature tensor of $M^n$. We call
\[ B = \sum_{ij} h_{ij} \omega_i \otimes \omega_j \otimes e_{n+1} \]
the second fundamental form of $M^n$. From the above equation, we have
\[ R = -n(n-1) - n^2 H^2 + S, \]
where $R$ is the scalar curvature and $S$ is the norm square of the second fundamental form and $H$ is the mean curvature, then we have
\[ S = \sum_{ij} h^2, \quad H = \frac{1}{n} \sum_i h_{ii}. \]
Now, we compute some local formulas. For any fixed point $x$ in $M$, we can choose a local frame field $\{e_1, \cdots, e_n\}$, such that
\[
h_{ij}(x) = \lambda_i(x) \delta_{ij}, \quad i, j = 1, \cdots, n.
\]
where $\lambda_i$ are principal curvatures.

**Example 1.** Let $M = H^1(-\frac{1}{a^2}) \times H^{n-1}(-\frac{1}{1-a^2})$ $(a > 0)$ be a spacelike hypersurface of $H^{n+1}_1(-1)$. Then $M$ has two distinct constant principal curvatures
\[
\lambda_1 = \frac{\sqrt{1-a^2}}{a}, \quad \lambda_2 = \cdots = \lambda_n = -\frac{a}{\sqrt{1-a^2}}.
\]
and constant mean curvature $H = \frac{1}{n} \sum \lambda_i = \frac{1-na^2}{n\sqrt{1-a^2}}$.

If $a^2 < \frac{1}{n}$, then we have
\[
S = n + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)} \sqrt{n^2H^4 + 4(n-1)H^2} = S_+(H)
\]
and the infremum of Ricci curvature of $M^n$ is given by
\[
-C_+(H) = -\frac{n(n-2)}{n-1} \left[1 + \frac{n^2H^2}{2(n-1)} - \frac{\sqrt{n^2H^4 + 4(n-1)H^2}}{2(n-1)}\right].
\]
If $a^2 = \frac{1}{n}$, then we have $H = 0$, $S = n$.
and the infremum of Ricci curvature of $M^n$ is given by $-\frac{n(n-2)}{n-1}$.
If $1 > a^2 > \frac{1}{n}$, then we have
\[
S = n + \frac{n^3H^2}{2(n-1)} - \frac{n(n-2)}{2(n-1)} \sqrt{n^2H^4 + 4(n-1)H^2} = S_-(H)
\]
and the infremum of Ricci curvature of $M^n$ is given by
\[
-C_-(H) = -\frac{n(n-2)}{n-1} \left[1 + \frac{n^2H^2}{2(n-1)} + \frac{\sqrt{n^2H^4 + 4(n-1)H^2}}{2(n-1)}\right].
\]

3. **Proof of Theorems**

By renumbering the principal directions $e_1, \cdots, e_n$, if necessary, we may assume that the principal curvature satisfy
\[
\lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1
\]
Then we have

\[ S = \sum_{i=1}^{n} \lambda_i^2, \quad nH = \sum_i \lambda_i \]  
\[ R_{ijkl} = -\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk} - \lambda_i\lambda_j\delta_{ik}\delta_{jl} + \lambda_i\lambda_j\delta_{il}\delta_{jk} \]  
\[ Ric_{ii} = \sum_{k=1}^{n} R_{ikik} = -(n-1) - nH\lambda_i + \lambda_i^2 \]

Set

\[ P(t) = t^2 - nHt - (n-1), \]

It has two real roots \( \Lambda_{\pm} = \frac{nH + \sqrt{n^2H^2 + 4(n-1)}}{2} \). From (11) and (12), we have

\[ Ric_{ii} = P(\lambda_i). \]

In the next part, we give the proof of Theorem 1.4.

**Proof of Theorem 1.4:**

Assume \( H \geq 0 \). From (1) and (8), we have

\[ R = -n(n-1) - n^2H^2 + S \]
\[ \leq -n(n-1) - n^2H^2 + n + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)} \sqrt{n^2H^4 + 4(n-1)H^2} \]
\[ = -(n-1)C_-(H). \]

By using the conditions \( R = \sum_i Ric_{ii} \) and \( Ric_{ii} \geq -C_-(H) \), we have \( Ric_{ii} \leq 0 \) for \( i \in \{1, \cdots, n\} \). From (13), we have

\[ P(\lambda_i) \leq 0, \]

for \( i = 1, \cdots, n \). So we have

\[ \Lambda_- \leq \lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_1 \leq \Lambda_+. \]

Denote \( \mu = \frac{nH - \sqrt{n^2H^2 + 4(n-1)}}{2(n-1)} \), we have \( P(\mu) = P(nH - \mu) = -C_-(H) \). Since \( M^n \) has \( (n-1) \) principal curvatures with the same sign everywhere and \( Ric_{ii} \geq -C_-(H) \), then we have the following possible case.

**Case A:**

\[ \Lambda_- \leq \lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 \leq \mu < 0 < nH - \mu \leq \lambda_1 \leq \Lambda_+. \]
Case B:

\[ \Lambda_- \leq \lambda_n \leq \mu < 0 < nH - \mu \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 \leq \lambda_1 \leq \Lambda_+. \]

If the principal curvatures satisfy Case A, then we have

\[ \lambda_n \leq \lambda_{n-1} \leq \cdots \leq \lambda_2 \leq \mu < 0, \]

On the other hand, we have

\[ \sum_{i=2}^{n} \lambda_i = nH - \lambda_1 \geq nH - \Lambda_+ = \frac{nH - \sqrt{n^2H^2 + 4(n-1)}}{2} = (n-1)\mu. \]

So we have

\[ \lambda_n = \cdots = \lambda_2 = \mu, \quad \lambda_1 = \frac{nH + \sqrt{n^2H^2 + 4(n-1)}}{2}, \]

\[ S = n + \frac{n^3H^2}{2(n-1)} + \frac{n(n-2)}{2(n-1)} \sqrt{n^2H^4 + 4(n-1)H^2} \]

and

\[ \inf |\lambda_1 - \lambda_2| = \frac{(2n - 3)nH + n\sqrt{n^2H^2 + 4(n-1)}}{2(n-1)} > 0. \]

then from Theorem 1.3 and Example 1, we know that \( M^n = H^1(-\frac{1}{a^2}) \times H^{n-1}(-\frac{1}{1-a^2}) \) with \( a^2 \leq \frac{1}{n}. \)

If the principal curvatures satisfy Case B, then we have

\[ \sum_{i=1}^{n-1} \lambda_i = nH - \lambda_n \]

\[ \leq nH - \frac{nH - \sqrt{n^2H^2 + 4(n-1)}}{2} = \frac{nH + \sqrt{n^2H^2 + 4(n-1)}}{2}. \tag{14} \]

On other hand, we have

\[ \sum_{i=1}^{n-1} \lambda_i \geq (n-1)(nH - \mu) = \frac{(2n - 3)nH + \sqrt{n^2H^2 + 4(n-1)}}{2} \tag{15} \]
From (14) and (15), we have
\[
\frac{(2n-3)nH + \sqrt{n^2H^2 + 4(n-1)}}{2} \leq \frac{nH + \sqrt{n^2H^2 + 4(n-1)}}{2},
\]
so
\[H \leq 0.\]

Since \(H \geq 0\), then \(H = 0\). So the case B turns into the following:
\[-\sqrt{n-1} \leq \lambda_n \leq -\frac{1}{\sqrt{n-1}} < 0 < \frac{1}{\sqrt{n-1}} \leq \lambda_{n-1} \leq \cdots \leq \lambda_1 \leq \sqrt{n-1}.\] (16)
then we have
\[(n-1)\frac{1}{\sqrt{n-1}} = \sqrt{n-1} \leq \sum_{i=1}^{n-1} \lambda_i = -\lambda_n \leq \sqrt{n-1}.\] (17)
From (16) and (17), we have
\[\lambda_1 = \cdots = \lambda_{n-1} = \frac{1}{\sqrt{n-1}}\]
and
\[\lambda_n = -\sqrt{n-1}.\]
So
\[S = \sum_{i=1}^{n} \lambda_i^2 = n\]
From Theorem 1.1 and \(S = n\), we know that \(M^n = H^1(-n) \times H^{n-1}(-\frac{n}{n-1})\). Thus we complete the proof of Theorem 1.4.

**Proof of Corollary 1.5:** Since \(M^n\) is a complete maximal spacelike hypersurface of \(H_{1}^{n+1}(-1)\), then we know that \(S \leq n\) from Theorem 1.1. So we know that \(M^n\) satisfies the following:
\[\text{Ric}_M \geq -\frac{n(n-2)}{n-1}\]
and
\[S \leq n.\]
From Theorem 1.4, we know that $S$ is constant, $S = n$ and $M^n = H^1(-n) \times H^{n-1}(-\frac{n}{n-1})$. This completes the proof of Corollary 1.5.

**Proof of Theorem 1.6:** Since $-C_-(H) \leq \text{Ric}_M \leq 0$, then we have

$$-C_-(H) \leq P(\lambda_i) = \lambda_i^2 - nH\lambda_i - (n - 1) \leq 0$$

So we know that the principal curvatures satisfy the Case A or Case B. From the proof of Theorem 1.4, we know that Theorem 1.6 is true.

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