SOME RESULTS OF $p$–VALENT FUNCTIONS DEFINED BY INTEGRAL OPERATORS

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ABSTRACT. In this paper, we derive some properties for $G_{p,n,l,\delta}(z)$ and $F_{p,n,l,\delta}(z)$ considering the classes $M\beta (p, \beta_{i}, \mu_{i})$, $K\beta (p, \beta_{i}, \mu_{i})$ and $N\beta (\gamma)$. Two new subclasses $KDF_{p,n,l}(\beta, \mu, \delta_{1}, \delta_{2}, ..., \delta_{n})$ and $KDG_{p,n,l}(\beta, \mu, \delta_{1}, \delta_{2}, ..., \delta_{n})$ are defined. Necessary and sufficient conditions for a family of functions $f_{i}$ and $g_{i}$, respectively, to be in the $KDF_{p,n,l}(\beta, \mu, \delta_{1}, \delta_{2}, ..., \delta_{n})$ and $KDG_{p,n,l}(\beta, \mu, \delta_{1}, \delta_{2}, ..., \delta_{n})$ are defined. As special cases, the properties of $\int_{0}^{\alpha} \prod_{i=1}^{n} (f'(t))^{\delta_{i}} dt$ and $\int_{0}^{\alpha} \prod_{i=1}^{n} (f(t))^{\delta_{i}} dt$ are given.

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1. Introduction and preliminaries

Let $A_{p}$ denote the class of the form

$$f(z) = z^{p} + \sum_{m=p+1}^{\infty} a_{m}z^{m}, \quad (p \in \mathbb{N} = \{1, 2, ..., \}),$$

which are analytic in the open disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Also denote $T_{p}$ the subclass of $A_{p}$ consisting of functions whose nonzero coefficients, from the second one, are negative and has the form

$$f(z) = z^{p} - \sum_{m=p+1}^{\infty} a_{m}z^{m}, \quad a_{m} \geq 0, \quad (p \in \mathbb{N} = \{1, 2, ..., \}).$$

Also $A_{1} = A$, $T_{1} = T$.

A function $f \in A_{p}$ is said to be $p$–valently starlike of order $\alpha (0 \leq \alpha < p)$ if and only if

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha, \quad (z \in U).$$

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We denote by \( S_p^*(\alpha) \), the class of all such functions. On the other hand, a function \( f \in \mathcal{A}_p \) is said to be \( p \)-valently convex of order \( \alpha \) \((0 \leq \alpha < p)\) if and only if
\[
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad (z \in \mathcal{U}).
\]

Let \( C_p(\alpha) \) denote the class of all those functions which are \( p \)-valently convex of order \( \alpha \) in \( \mathcal{U} \).

Note that \( S_p^*(0) = S_p^* \) and \( C_p(0) = C_p \) are, respectively, the classes of \( p \)-valently starlike and \( p \)-valently convex functions in \( \mathcal{U} \). Also, we note that \( S_p^*(0) = S^* \) and \( C_1 = C \) are, respectively, the usual classes of starlike and convex functions in \( \mathcal{U} \).

Let \( \mathcal{N}_p(\gamma) \) be the subclass of \( \mathcal{A}_p \) consisting of the functions \( f \) which satisfy the inequality
\[
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \gamma, \quad (z \in \mathcal{U}), \quad \gamma > p.
\]

Also \( \mathcal{N}_1(\gamma) = \mathcal{N}(\gamma) \). For \( p = 1 \), this class was studied by Owa (see [12]) and Mohammed (see [9]).

For a function \( f \in \mathcal{A}_p \), we define the following operator
\[
D^0 f(z) = f(z) \\
D^1 f(z) = \frac{1}{p} zf'(z) \\
\vdots \\
D^k f(z) = D \left( D^{k-1} f(z) \right),
\]
where \( k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). The differential operator \( D^k \) was introduced by Shenan et al. (see [18]). When \( p = 1 \), we get Sălăgean differential operator (see [15]).

We note that if \( f \in \mathcal{A}_p \), then
\[
D^k f(z) = z^p + \sum_{m=p+1}^{\infty} \frac{m}{p}^{k} a_m z^m, \quad (p \in \mathbb{N} = \{1, 2, \ldots\}) (z \in \mathcal{U}).
\]

We also note that if \( f \in \mathcal{T}_p \), then
\[
D^k f(z) = z^p - \sum_{m=p+1}^{\infty} \frac{m}{p}^{k} a_m z^m, \quad (p \in \mathbb{N} = \{1, 2, \ldots\}) (z \in \mathcal{U}).
\]
Let $MT(p, \beta, \mu)$ be the subclass of $A_p$ consisting of the functions $f$ which satisfy the analytic characterization

$$\left| \frac{z(D^i f(z))'}{D^i f(z)} - p \right| < \beta \left| \frac{z(D^i f(z))'}{D^i f(z)} + p \right|, \quad (5)$$

for some $0 < \beta \leq p$, $0 \leq \mu < p$, $l_i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $z \in \mathcal{U}$. For $p = 1$, $l_1 = l_2 = ... = l_n = 0$ for all $i = \{1, 2, ..., n\}$ this class was studied (see [2]).

**Definition 1.** A function $f \in A_p$ is said to be in the class $KD(p, \beta, \mu)$ if satisfies the following inequality:

$$\Re \left\{ 1 + \frac{z(D^i f(z))^\mu}{(D^i f(z))^{\mu'}} \right\} \geq \mu \left| 1 + \frac{z(D^i f(z))^\mu}{(D^i f(z))^{\mu'}} - p \right| + \beta, \quad (6)$$

for some $0 \leq \beta < p$, $\mu \geq 0$, $l_i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and $z \in \mathcal{U}$. For $p = 1$, $l_1 = l_2 = ... = l_n = 0$ for all $i = \{1, 2, ..., n\}$ this class was studied (see [17], [9]).

**Definition 2.** Let $l = (l_1, l_2, ..., l_n) \in \mathbb{N}_0^n$, $\delta = (\delta_1, \delta_2, ..., \delta_n) \in \mathbb{R}_+^n$, for all $i = \{1, 2, ..., n\}$, $n \in \mathbb{N}$. We define the following general integral operators

$$\mathcal{F}_{p,n}^{l,\delta} (f_1, f_2, ..., f_n) : A_p^n \rightarrow A_p,$$

$$\mathcal{F}_{p,n}^{l,\delta} (f_1, f_2, ..., f_n) = \mathcal{F}_{p,n,l,\delta}(z),$$

$$\mathcal{F}_{p,n,l,\delta}(z) = \int_0^z pt^{p-1} \prod_{i=1}^n \left( \frac{D^l f_i(t)}{t^{lp}} \right)^{\delta_i} dt, \quad (7)$$

and

$$\mathcal{G}_{p,n}^{l,\delta} (g_1, g_2, ..., g_n) : A_p^n \rightarrow A_p,$$

$$\mathcal{G}_{p,n}^{l,\delta} (g_1, g_2, ..., g_n) = \mathcal{G}_{p,n,l,\delta}(z),$$

$$\mathcal{G}_{p,n,l,\delta}(z) = \int_0^z pt^{p-1} \prod_{i=1}^n \left( \frac{(D^l g_i(t))'}{pt^{p-1}} \right)^{\delta_i} dt, \quad (8)$$

where $f_i, g_i \in A_p$ for all $i = \{1, 2, ..., n\}$ and $D$ is defined by (4).

**Remark 1.** (7) integral operator was studied and introduced by Saltük et al. (see [16]). We note that if $l_1 = l_2 = ... = l_n = 0$ for all $i = \{1, 2, ..., n\}$, then the integral operator $\mathcal{F}_{p,n,l,\delta}(z)$ reduces to the operator $F_p(z)$ which was studied by Frasin (see [6]). Upon setting $p = 1$ in the operator (7), we can obtain the integral operator $D^k F(z)$ which was studied by Breaz (see [5]) and Breaz (see [4]). For $p = 1$ and $l_1 = l_2 = ... = l_n = 0$ in (7), the integral operator $\mathcal{F}_{p,n,l,\delta}(z)$ reduces to the operator $F_n(z)$ which was studied by Breaz, Breaz (see [2]) and Mohammed (see [17], [9]).
[10]). Observe that \( p = n = 1, l_1 = 0 \) and \( \delta_1 = \delta \), we obtain the integral operator \( I_\delta(f)(z) \) which was studied by Pescar and Owa (see [13]), D. Breaz (see [5]) and Mohammed (see [11]) for \( \delta_1 = \delta \in [0, 1] \) special case of the operator \( I_\delta(f)(z) \) was studied by Miller, Mocanu and Reade (see [8]). For \( p = n = 1, l_1 = 0 \) and \( \delta_1 = 1 \) in (7), we have Alexander integral operator \( I(f)(z) \) in (see [1]).

**Remark 2.** (8) integral operator was studied and introduced by Saltık et al. (see [16]). For \( l_1 = l_2 = ... = l_n = 0 \) in (8) the integral operator \( G_{p,n,l,\delta}(z) \) reduces to the operator \( G_p(z) \) which was studied by Frasin (see [6]). For \( p = 1 \) and \( l_1 = l_2 = ... = l_n = 0 \) in (8), the integral operator \( G_{p,n,l,\delta}(z) \) reduces to the operator \( G_{\delta_1,\delta_2,\ldots,\delta_n}(z) \) which was studied by Breaz, Breaz and Owa (see [3]) and Mohammed (see [10]). Observe \( p = n = 1, l_1 = 0 \) and \( \delta_1 = \delta \), we obtain the integral operator \( G(z) \) which was introduced and studied by Pfaltzgraff (see [14]), Mohammed (see [11]), D. Breaz (see [5]) and Kim and Merkes (see [7]).

Now, by using the equations (7) and (8) and the Definition 1 we introduce the following two new subclasses of \( KD(p, \beta, \mu) \).

**Definition 3.** A family of functions \( f_i, i = \{1, 2, ..., n\} \) is said to be in the class \( KD F_{p,n,l}(\beta, \mu, \delta_1, \delta_2, ..., \delta_n) \) if satisfies the inequality:

\[
\Re \left\{ 1 + \frac{z(D^h F_{p,n,l,\delta}(z))''}{(D^h F_{p,n,l,\delta}(z))'''} \right\} \geq \mu \left| 1 + \frac{z(D^h F_{p,n,l,\delta}(z))''}{(D^h F_{p,n,l,\delta}(z))'''} - p \right| + \beta, \tag{9}
\]

for some \( 0 \leq \beta < p, \mu \geq 0, l_i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( z \in U \) where \( F_{p,n,l,\delta} \) is defined in (7).

**Definition 4.** A family of functions \( g_i, i = \{1, 2, ..., n\} \) is said to be in the class \( KD G_{p,n,l}(\beta, \mu, \delta_1, \delta_2, ..., \delta_n) \) if satisfies the inequality:

\[
\Re \left\{ 1 + \frac{z(D^h G_{p,n,l,\delta}(z))''}{(D^h G_{p,n,l,\delta}(z))'''} \right\} \geq \mu \left| 1 + \frac{z(D^h G_{p,n,l,\delta}(z))''}{(D^h G_{p,n,l,\delta}(z))'''} - p \right| + \beta, \tag{10}
\]

for some \( 0 \leq \beta < p, \mu \geq 0, l_i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) and \( z \in U \) where \( G_{p,n,l,\delta} \) is defined in (8).

2. **Sufficient conditions of the operator \( F_{p,n,l,\delta}(z) \)**

First, in this section we prove a sufficient condition for the integral operator \( F_{p,n,l,\delta}(z) \) to be in the class \( N_p(\eta) \).

**Theorem 1.** Let \( l = (l_1, l_2, ..., l_n) \in \mathbb{N}_0^n, \delta = (\delta_1, \delta_2, ..., \delta_n) \in \mathbb{R}_+^n, 0 \leq \mu_i < p, 0 < \beta_i \leq p \) and \( f_i \in A_p \) for all \( i = \{1, 2, ..., n\} \). If \( \left| \frac{(D^h f_i)(z)}{D^h f_i(z)} \right| < M_i \) and \( f_i \in MT(p, \beta_i, \mu_i) \), then the integral operator
$F_{p,n,l,\delta}(z) = \int_0^z pt^{p-1} \prod_{i=1}^n \left( \frac{D_i f_i(t)}{t^p} \right) \delta_i \, dt,$

is in $\mathcal{N}_p(\eta)$, where

$$\eta = \sum_{i=1}^n \delta_i \beta_i (p + \mu_i M_i) + p. \quad (11)$$

**Proof.** From the definition (7), we observe that $F_{p,n,l,\delta}(z) \in \mathcal{A}_p$. On the other hand, it is easy to see that

$$F'_{p,n,l,\delta}(z) = pz^{p-1} \prod_{i=1}^n \left( D_i f_i(z) \right) \delta_i. \quad (12)$$

Now, we differentiate (12) logarithmically and multiply by $z$, we obtain

$$1 + \frac{z F''_{p,n,l,\delta}(z)}{F'_{p,n,l,\delta}(z)} = p + \sum_{i=1}^n \delta_i \left( \frac{z (D_i f_i)'(z)}{D_i f_i(z)} - p \right).$$

We calculate the real part from both terms of the above expression and obtain

$$\Re \left\{ 1 + \frac{z F''_{p,n,l,\delta}(z)}{F'_{p,n,l,\delta}(z)} \right\} = \sum_{i=1}^n \delta_i \Re \left\{ \frac{z (D_i f_i)'(z)}{D_i f_i(z)} - p \right\} + p.$$

Since $\Re w \leq |w|$, then

$$\Re \left\{ 1 + \frac{z F''_{p,n,l,\delta}(z)}{F'_{p,n,l,\delta}(z)} \right\} \leq \sum_{i=1}^n \delta_i \left| \frac{z (D_i f_i)'(z)}{D_i f_i(z)} - p \right| + p.$$

Since $f_i \in MT(p, \beta_i, \mu_i)$ for all $i = \{1, 2, ..., n\}$, we have

$$\Re \left\{ 1 + \frac{z F''_{p,n,l,\delta}(z)}{F'_{p,n,l,\delta}(z)} \right\} \leq \sum_{i=1}^n \delta_i \beta_i \mu_i \left| \frac{z (D_i f_i)'(z)}{D_i f_i(z)} \right| + p \sum_{i=1}^n \delta_i \beta_i + p. \quad (13)$$

Since (13) and $\left| \frac{D_i f_i}'(z)}{D_i f_i(z)} \right| < M_i$, we obtain
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\[ \Re \left\{ 1 + \frac{z F''_{p,n,l,\delta}(z)}{F'_{p,n,l,\delta}(z)} \right\} < \sum_{i=1}^{n} \delta_i \beta_i M_i + p \sum_{i=1}^{n} \delta_i \beta_i + p = \sum_{i=1}^{n} \delta_i \beta_i (p + \mu_i M_i) + p. \]

Hence \( F_{p,n,l,\delta}(z) \in N_p(\eta), \eta = \sum_{i=1}^{n} \delta_i \beta_i (p + \mu_i M_i) + p. \)

**Remark 3.** For \( p = 1, \delta_i = 0 \) for all \( i = \{1, 2, ..., n\} \) in Theorem 1, we obtain Theorem 1 (see [9]).

Putting \( p = 1 \) in Theorem 1, we have

**Corollary 1.** Let \( l = (l_1, l_2, ..., l_n) \in \mathbb{N}_0^n, \delta = (\delta_1, \delta_2, ..., \delta_n) \in \mathbb{R}_+^n, 0 \leq \mu_i < 1, 0 < \beta_i \leq 1 \) and \( f_i \in A \) for all \( i = \{1, 2, ..., n\} \). If \( \left| \frac{(D_{l_i} f_i)'(z)}{D_{l_i} f_i(z)} \right| < M \) and \( f_i \in MT(1, \beta_i, \mu_i) \) then the integral operator

\[
F_{1,n,l,\delta}(z) = \int_0^z \prod_{i=1}^{n} \left( \frac{D_{l_i} f_i(t)}{t} \right)^{\delta_i} dt,
\]

is in \( N(\eta) \), where

\[
\eta = \sum_{i=1}^{n} \delta_i \beta_i (1 + \mu_i M) + 1.
\]

Putting \( p = n = 1, l_1 = 0, \delta_1 = \delta, \mu_1 = \mu, \beta_1 = \beta, M_1 = M \) and \( f_1 = f \) in Theorem 1, we have

**Corollary 2.** Let \( \delta \in \mathbb{R}_+, 0 \leq \mu < 1, 0 < \beta \leq 1 \) and \( f \in A \). If \( \left| \frac{f'(z)}{f(z)} \right| < M \) and \( f \in MT(1, \beta, \mu) \) then the integral operator \( \int_0^z \left( \frac{f(t)}{t} \right)^{\delta} \) is in \( N(\eta) \), where \( \eta = \delta \beta (1 + \mu M) + 1. \)

3. Sufficient conditions of the operator \( G_{p,n,l,\delta}(z) \)

Next, in this section we give a condition for the integral \( G_{p,n,l,\delta}(z) \) to be \( p \)-valently convex.

**Theorem 2.** Let \( l = (l_1, l_2, ..., l_n) \in \mathbb{N}_0^n, \delta = (\delta_1, \delta_2, ..., \delta_n) \in \mathbb{R}_+^n, \mu_i \geq 0, g_i \in KD(p, \beta_i, \mu_i) \) and let \( \beta_i \geq 0 \) be real number with the property \( 0 \leq \beta_i < p \) for all \( i = \{1, 2, ..., n\} \). Moreover suppose that \( 0 < \sum_{i=1}^{n} \delta_i (p - \beta_i) \leq p \), then the integral operator

\[
G_{p,n,l,\delta}(z) = \int_0^z p t^{p-1} \prod_{i=1}^{n} \left( \frac{D_{l_i} g_i(t)}{p t^{p-1}} \right)^{\delta_i} dt,
\]

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is convex order of \( \sigma = p - \sum_{i=1}^{n} \delta_i (p - \beta_i) \).

Proof. From the definition (8), we observe that \( G_{p,n,l,d}(z) \in A_p \). On the other hand, it is easy to see that

\[
G'_{p,n,l,d}(z) = pz^{p-1} \prod_{i=1}^{n} \left( \frac{(D^i g_i(z))'}{p^{2p-1}} \right) \delta_i.
\]  

(14)

Now, we differentiate (14) logarithmically and make the similar operators to the proof of the Theorem 2, we obtain

\[
1 + \frac{zG''_{p,n,l,d}(z)}{G'_{p,n,l,d}(z)} = p + \sum_{i=1}^{n} \delta_i \left( z \frac{(D^i g_i(z))''}{(D^i g_i(z))'} - p + 1 \right).
\]

We calculate the real part from both terms of the above expression and obtain

\[
\Re \left\{ 1 + \frac{zG''_{p,n,l,d}(z)}{G'_{p,n,l,d}(z)} \right\} = \sum_{i=1}^{n} \delta_i \Re \left\{ 1 + \frac{z (D^i g_i(z))''}{(D^i g_i(z))'} - p \right\} + p \sum_{i=1}^{n} \delta_i + p.
\]

Since \( g_i \in KD (p, \beta_i, \mu_i) \) for all \( i = \{1, 2, ..., n\} \), we have

\[
\Re \left\{ 1 + \frac{zG''_{p,n,l,d}(z)}{G'_{p,n,l,d}(z)} \right\} > \sum_{i=1}^{n} \delta_i \left( \mu_i \left| 1 + \frac{z (D^i g_i(z))''}{(D^i g_i(z))'} - p \right| + \beta_i \right) - p \sum_{i=1}^{n} \delta_i + p.
\]

Since \( \delta_i \mu_i \left| 1 + \frac{z (D^i g_i(z))''}{(D^i g_i(z))'} - p \right| > 0 \), we obtain

\[
\Re \left\{ 1 + \frac{zG''_{p,n,l,d}(z)}{G'_{p,n,l,d}(z)} \right\} \geq p - \sum_{i=1}^{n} \delta_i (p - \beta_i),
\]

which implies that \( G_{p,n,l,d}(z) \) is \( p \)-valently convex of order \( \sigma = p - \sum_{i=1}^{n} \delta_i (p - \beta_i) \).

**Remark 4.** Setting \( p = 1, l_i = 0 \) and \( g_i = f_i \) for all \( i = \{1, 2, ..., n\} \) in Theorem 2, we have obtain Theorem 2 in (see [9]).

Putting \( p = 1 \) in Theorem 2, we have

**Corollary 3.** Let \( l = (l_1, l_2, ..., l_n) \in \mathbb{N}_0^n, \delta = (\delta_1, \delta_2, ..., \delta_n) \in \mathbb{R}_+^n, \mu_i \geq 0, g_i \in KD (1, \beta_i, \mu_i) \) and let \( \beta_i \geq 0 \) be real number with the property \( 0 \leq \beta_i < 1 \) for all \( i = \{1, 2, ..., n\} \). Moreover suppose that \( 0 < \sum_{i=1}^{n} \delta_i (1 - \beta_i) \leq 1 \), then the integral operator \( G_{l,n,l,d}(z) = \int_0^x \prod_{i=1}^{n} \left( (D^i g_i(t))' \right) \delta_i \, dt \) is convex order of \( \sigma = 1 - \sum_{i=1}^{n} \delta_i (1 - \beta_i) \).
Putting $p = n = 1, l_1 = 0, \delta_1 = \delta, \mu_1 = \mu, \beta_1 = \beta$ and $g_1 = g$ in Theorem 2, we have

**Corollary 4.** Let $\delta \in \mathbb{R}^+, \mu \geq 0, g \in KD (1, \beta, \mu)$ and let $\beta \geq 0$ be real number with the property $0 \leq \beta < 1$. Moreover suppose that $0 < \delta (1 - \beta) \leq 1$, then the integral operator $G_{1,1,0,\delta}(z) = \int_0^z (g'(t))^{\delta} \, dt$ is convex order of $\sigma = 1 - \delta (1 - \beta)$.

4. **A necessary and sufficient condition for a family of analytic functions** $f_i \in KD_{p,n,l}(\beta, \mu, \delta_1, \delta_2, \ldots, \delta_n)$

In this section, we give a necessary and sufficient condition for a family of functions $f_i \in KD_{p,n,l}(\beta, \mu, \delta_1, \delta_2, \ldots, \delta_n)$. Before embarking on the proof of our result, let us calculate the expression $\frac{z^{F''_{p,n,l,\delta}(z)}}{F'_{p,n,l,\delta}(z)}$, required for proving our result.

Recall that, from (7), we have

$$F'_{p,n,l,\delta}(z) = pz^{p-1} \prod_{i=1}^{n} \left( \frac{Df_i(z)}{z^{p}} \right)^{\delta_i}. \quad (15)$$

Now, we differentiate (15) logarithmically and multiply by $z$, we obtain

$$1 + \frac{z^{F''_{p,n,l,\delta}(z)}}{F'_{p,n,l,\delta}(z)} - p = \sum_{i=1}^{n} \left( z \frac{Df_i(z)}{Df_i(z)} - p \delta_i \right).$$

Let $D^i f_i(z) = z^p - \sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)^i a_{m,i} z^m,$

$$(D^i f_i)'(z) = p z^{p-1} - \sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)^i m a_{m,i} z^{m-1}$$

and we get

$$1 + \frac{z^{F''_{p,n,l,\delta}(z)}}{F'_{p,n,l,\delta}(z)} - p = \sum_{i=1}^{n} \left[ \frac{p z^p - \sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)^i m a_{m,i} z^m}{z^p - \sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)^i a_{m,i} z^m} - p \right], \quad (16)$$

Theorem 3. Let the function $f_i \in T_p$ for $i \in \{1, 2, \ldots, n\}$. Then the functions
$f_i \in KDF_{p,n,l}(\beta, \mu, \delta_1, \delta_2, \ldots, \delta_n)$ for $i \in \{1, 2, \ldots, n\}$ if and only if

$$\sum_{i=1}^{n} \left[ \sum_{m=p+1}^{\infty} \delta_i \left( \frac{m}{p} \right)^{l_i} (m-p) (\mu+1) a_{m,i} \right] \leq p - \beta.$$  \hfill (17)

Proof. First consider

$$\mu \left| 1 + \frac{zF''_{p,n,l,\delta}(z)}{F_{p,n,l,\delta}(z)} - p \right| - \Re \left\{ 1 + \frac{zF''_{p,n,l,\delta}(z)}{F_{p,n,l,\delta}(z)} \right\} \leq (\mu + 1) \left| 1 + \frac{zF''_{p,n,l,\delta}(z)}{F_{p,n,l,\delta}(z)} - p \right| .$$

From (16), we obtain

$$(\mu + 1) \left| 1 + \frac{zF''_{p,n,l,\delta}(z)}{F_{p,n,l,\delta}(z)} - p \right| ,$$

$$= (\mu + 1) \sum_{i=1}^{n} \frac{\left[ \sum_{m=p+1}^{\infty} \delta_i \left( \frac{m}{p} \right)^{l_i} (m-p) |a_{m,i}| z^{m-p} \right] |z|^m}{1 - \sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)^{l_i} |a_{m,i}| |z|^{m-p}} ,$$

$$\leq (\mu + 1) \sum_{i=1}^{n} \frac{\left[ \sum_{m=p+1}^{\infty} \delta_i \left( \frac{m}{p} \right)^{l_i} (m-p) a_{m,i} \right] |z|^m}{1 - \sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)^{l_i} a_{m,i} |z|^{m-p}} .$$

If (17) holds then the above expression is bounded by $p - \beta$ and consequently

$$\mu \left| 1 + \frac{zF''_{p,n,l,\delta}(z)}{F_{p,n,l,\delta}(z)} - p \right| - \Re \left\{ 1 + \frac{zF''_{p,n,l,\delta}(z)}{F_{p,n,l,\delta}(z)} \right\} < -\beta ,$$

which equivalent to

$$\Re \left\{ 1 + \frac{zF''_{p,n,l,\delta}(z)}{F_{p,n,l,\delta}(z)} \right\} \geq \mu \left| 1 + \frac{zF''_{p,n,l,\delta}(z)}{F_{p,n,l,\delta}(z)} - p \right| + \beta .$$
Hence \( f_i \in KD_{p,n,l}(\beta, \mu, \delta_1, \delta_2, \ldots, \delta_n) \) for \( i \in \{1, 2, \ldots, n\} \).

Conversely, let \( f_i \in KD_{p,n,l}(\beta, \mu, \delta_1, \delta_2, \ldots, \delta_n) \) for \( i \in \{1, 2, \ldots, n\} \) and prove that (17) holds. If \( f_i \in KD_{p,n,l}(\beta, \mu, \delta_1, \delta_2, \ldots, \delta_n) \) for \( i \in \{1, 2, \ldots, n\} \) and \( z \) is real, we get from (7) and (16)
which give the required result.

**Remark 5.** Setting $p = 1$, $l_i = 0$ for $i \in \{1, 2, ..., n\}$ in Theorem 3, we have obtain Theorem 3 in (see [9]).

Putting $p = 1$ in Theorem 3, we have

**Corollary 5.** Let the function $f_i \in T$ for $i \in \{1, 2, ..., n\}$. Then the functions $f_i \in KDF_{1,n,l} (\beta, \mu, \delta_1, \delta_2, ..., \delta_n)$ for $i \in \{1, 2, ..., n\}$ if and only if
\[
\sum_{i=1}^{n} \sum_{m=2}^{\infty} \frac{\delta_i (m - 1) \mu (m + 1) a_{m,i}}{1 - \sum_{m=2}^{\infty} (m)^{l_i} a_{m,i}} \leq 1 - \beta.
\]

Putting $p = n = 1$, $l_1 = 0$, $\delta_1 = \delta$ and $f_1 = f$ in Theorem 3, we have

**Corollary 6.** Let the function $f \in T$. Then the functions $f \in KDF_{1,1,0} (\beta, \mu, \delta)$ if and only if
\[
\sum_{m=2}^{\infty} \frac{\delta (m - 1) \mu (m + 1) a_{m,1}}{1 - \sum_{m=2}^{\infty} a_{m,1}} \leq 1 - \beta.
\]

5. A necessary and sufficient condition for a family of analytic functions

$g_i \in KDG_{p,n,l}(\beta, \mu, \delta, \delta_1, ..., \delta_n)$

In this section, we give a necessary and sufficient condition for a family of functions $g_i \in KDG_{p,n,l}(\beta, \mu, \delta, \delta_1, ..., \delta_n)$. Let us calculate the expression $\frac{zG''_{p,n,l,\delta}(z)}{G'_{p,n,l,\delta}(z)}$, required for proving our result.

Recall that, from (8), we have
\[
G'_{p,n,l,\delta}(z) = p2^{p-1} \prod_{i=1}^{n} \left( \frac{(D^{l_i}g_i(z))'}{p2^{p-1}} \right)^{\delta_i}.
\]  
(18)

Now, we differentiate (18) logarithmically and multiply by $z$, we obtain
\[
1 + \frac{zG''_{p,n,l,\delta}(z)}{G'_{p,n,l,\delta}(z)} - p = \sum_{i=1}^{n} \delta_i \left( \frac{z(D^{l_i}g_i(z))''}{(D^{l_i}g_i(z))'} - p + 1 \right).
\]

Let $D^{l_i}g_i(z) = z^p - \sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)^{l_i} a_{m,i} z^m, (D^{l_i}g_i)'(z) = p2^{p-1} - \sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)^{l_i} ma_{m,i} z^{m-1}$ and
Theorem 4. Let the function \( g_i \in T_p \) for \( i \in \{1, 2, ..., n\} \). Then the functions \( g_i \in KD_{p,n,l} (\beta, \mu, \delta_1, \delta_2, ..., \delta_n) \) for \( i \in \{1, 2, ..., n\} \) if and only if

\[
\sum_{i=1}^{n} \delta_i \left[ \sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)^{l_i} m (m - p) a_{m,i} z^{m-p} \right] \leq p - \beta. \tag{20}
\]

Proof. First consider

\[
\mu \left| 1 + \frac{zG''_{p,n,l,\delta}(z)}{G'_{p,n,l,\delta}(z)} - p \right| - \Re \left\{ 1 + \frac{zG''_{p,n,l,\delta}(z)}{G'_{p,n,l,\delta}(z)} \right\} \leq (\mu + 1) \left| 1 + \frac{zG''_{p,n,l,\delta}(z)}{G'_{p,n,l,\delta}(z)} - p \right|. \]

From (19), we obtain
\[
(\mu + 1) \left| 1 + \frac{zG''_{p,n,l,\delta}(z)}{G'_{p,n,l,\delta}(z)} - p \right| = (\mu + 1) \sum_{i=1}^{n} \delta_i \left[ \frac{\sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)_{l_i} m (m - p) a_{m,i} z^{m-p}}{p - \sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)_{l_i} m a_{m,i} z^{m-p}} \right],
\]

\[
\leq (\mu + 1) \sum_{i=1}^{n} \delta_i \left[ \frac{\sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)_{l_i} m (m - p) |a_{m,i}| |z|^{m-p}}{p - \sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)_{l_i} m |a_{m,i}| |z|^{m-p}} \right],
\]

If (20) holds then the above expression is bounded by \( p - \beta \) and consequently

\[
\mu \left| 1 + \frac{zG''_{p,n,l,\delta}(z)}{G'_{p,n,l,\delta}(z)} - p \right| - \Re \left\{ 1 + \frac{zG''_{p,n,l,\delta}(z)}{G'_{p,n,l,\delta}(z)} \right\} < -\beta,
\]

which equivalent to

\[
\Re \left\{ 1 + \frac{zG''_{p,n,l,\delta}(z)}{G'_{p,n,l,\delta}(z)} \right\} \geq \mu \left| 1 + \frac{zG''_{p,n,l,\delta}(z)}{G'_{p,n,l,\delta}(z)} - p \right| + \beta.
\]

Hence \( g_i \in KDG_{p,n,l} (\beta, \mu, \delta_1, \delta_2, ..., \delta_n) \) for \( i \in \{1, 2, ..., n\} \).

Conversely, let \( g_i \in KDG_{p,n,l} (\beta, \mu, \delta_1, \delta_2, ..., \delta_n) \) for \( i \in \{1, 2, ..., n\} \) and prove that (20) holds. If \( g_i \in KDG_{p,n,l} (\beta, \mu, \delta_1, \delta_2, ..., \delta_n) \) for \( i \in \{1, 2, ..., n\} \) and \( z \) is real, we get from (8) and (19)
\[ p - \sum_{i=1}^{n} \delta_i \left[ \frac{\sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)^l_i m (m - p) a_{m,i} z^{m-p}}{p - \sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)^l_i m a_{m,i} z^{m-p}} \right], \]

\[ \geq \mu \sum_{i=1}^{n} \delta_i \left[ \frac{\sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)^l_i m (m - p) a_{m,i} z^{m-p}}{p - \sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)^l_i m a_{m,i} z^{m-p}} \right] + \beta, \]

\[ \geq \mu \sum_{i=1}^{n} \delta_i \left[ \frac{\sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)^l_i m (m - p) a_{m,i} z^{m-p}}{p - \sum_{m=p+1}^{\infty} \left( \frac{m}{p} \right)^l_i m a_{m,i} z^{m-p}} \right] + \beta. \]

That is equivalent to

\[ \sum_{i=1}^{n} \delta_i \mu \left( \frac{m}{p} \right)^l_i m (m - p) a_{m,i} z^{m-p} \]

\[ \geq \sum_{i=1}^{n} \delta_i \left( \frac{m}{p} \right)^l_i m (m - p) a_{m,i} z^{m-p} \]

\[ \leq p - \beta. \]

The above inequality reduce to

\[ \sum_{i=1}^{n} \delta_i \left( \frac{m}{p} \right)^l_i (\mu + 1) m (m - p) a_{m,i} z^{m-p} \]

\[ \leq p - \beta. \]

Let \( z \to 1^- \) along the real axis, then we get

\[ \sum_{i=1}^{n} \delta_i \left( \frac{m}{p} \right)^l_i m (m - p) a_{m,i} z^{m-p} \]

\[ \leq p - \beta, \]

which give the required result.

**Remark 6.** Setting \( p = 1, \ l_i = 0 \) for \( i \in \{1, 2, ..., n\} \) in Theorem 4, we have obtain Theorem 4 in [see [9]].

Putting \( p = 1 \) in Theorem 4, we have
Corollary 7. Let the function $g_i \in T$ for $i \in \{1, 2, ..., n\}$. Then the functions $g_i \in KD\mathcal{G}_{1,n,l}(\beta, \mu, \delta_1, \delta_2, ..., \delta_n)$ for $i \in \{1, 2, ..., n\}$ if and only if

$$
\sum_{i=1}^{n} \left[ \sum_{m=2}^{\infty} \frac{\delta_i (\mu + 1) m (m-1) a_{m,i}}{1 - \sum_{m=2}^{\infty} (m)^{l_i} m a_{m,i}} \right] \leq 1 - \beta.
$$

Putting $p = n = 1$, $l_1 = 0$, $\delta_1 = \delta$ and $g_1 = g$ in Theorem 4, we have

Corollary 8. Let the function $g \in T$. Then the functions $g \in KD\mathcal{G}_{1,1,0}(\beta, \mu, \delta)$ if and only if

$$
\sum_{m=2}^{\infty} \frac{\delta (\mu+1)m(m-1)a_{m,1}}{1 - \sum_{m=2}^{\infty} ma_{m,1}} \leq 1 - \beta.
$$

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