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# APPLICATIONS OF THE ROPER-SUFFRIDGE EXTENSION OPERATOR TO ALMOST STARLIKE MAPPINGS OF COMPLEX ORDER $\lambda$

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Abstract. The Roper-Suffridge extension operator provides a way of extending a (locally) univalent function $f \in H(U)$ to a (locally) biholomorphic mapping $F \in H(B^n)$. In this paper we consider certain generalizations of the operator and we show that if $f$ is an almost starlike function of complex order $\lambda$ then $F$ is an almost starlike mapping of complex order $\lambda$. Various particular cases will be also considered.

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## 1. Introduction and preliminaries

Let $\mathbf{C}^n$ denote the space of $n$-complex variables $z = (z_1, \cdots, z_n)$ with respect to the Euclidean inner product $\langle z, w\rangle = \sum_{j=1}^n z_j \overline{w}_j$ and the norm $\|z\| = \langle z, z\rangle^{1/2}$.

For $n \geq 2$, let $\widetilde{z} = (z_2, \cdots, z_n) \in \mathbf{C}^{n-1}$ such that $z = (z_1, \widetilde{z}) \in \mathbf{C}^n$. The open ball $\{z \in \mathbf{C}^n : \|z\| < r\}$ is denoted by $B_r^n$ and the unit ball $B_1^n$ by $B^n$. In the case of one complex variable, let $U_r := B_r^n$ and $U := U_1$. If $G$ is an open set in $\mathbf{C}^n$, let $H(G)$ be the set of holomorphic maps from $G$ into $\mathbf{C}^n$. If $f \in H(B^n)$, we say that $f$ is normalized if $f(0) = 0$ and $Df(0) = I_n$.

Let $S(B^n)$ be the set of normalized biholomorphic mappings on $B^n$. We denote the classes of normalized convex and starlike mappings on $B^n$ by $K(B^n)$ and $S^*(B^n)$ respectively. In one variable, we write $S(U) = S$, $K(U) = K$ and $S^*(U) = S^*$.

**Definition 1.1** *Let $f : B^n \to \mathbf{C}^n$ be a normalized locally biholomorphic mapping and let $A : \mathbf{C}^n \to \mathbf{C}^n$ be a linear operator such that $\mathrm{Re}\langle A(z), z\rangle > 0$ for $z \in \mathbf{C}^n \setminus \{0\}$. We say that $f$ is spirallike with respect to $A$ if*

$$\mathrm{Re}\langle [Df(z)]^{-1} Af(z), z\rangle > 0, \quad z \in B^n \setminus \{0\}.$$

*In the case that $A = e^{i\delta} I_n$, where $|\delta| < \frac{\pi}{2}$, we say that $f$ is spirallike of type $\delta$.*

Note that any spirallike mapping with respect to a linear operator $A$ such that $\mathrm{Re}\langle A(z), z\rangle > 0$ for $z \in \mathbf{C}^n \setminus \{0\}$ is biholomorphic (see [11]). We denote by $\widehat{S}_\delta$, the class of spirallike mappings of type $\delta$, $|\delta| < \frac{\pi}{2}$. Further details about spirallike mappings with respect to linear operators may be found in [11] and [4].

We introduced in [1] a new class of normalized locally biholomorphic mappings as it follows.

**Definition 1.2** *Let $f$ be a normalized locally biholomorphic mapping on $B^n$, and let $\lambda \in \mathbf{C}$, with $\mathrm{Re}\lambda \leq 0$. The function $f$ is said to be an almost starlike mapping of complex order $\lambda$ if*

$$\mathrm{Re}\{(1-\lambda)\langle [Df(z)]^{-1} f(z), z\rangle\} > -\mathrm{Re}\lambda \|z\|^2,\ z \in B^n \setminus \{0\}.$$

It is easy to see that in the case of one variable, the above inequality reduces to the following

$$\mathrm{Re}\left[(1-\lambda)\frac{f(z)}{zf'(z)}\right] \geq -\mathrm{Re}\lambda,\ z \in U.$$

The interest of the study of almost starlikeness of complex order $\lambda$ arises from the fact that every almost starlike mapping $f$ of complex order $\lambda$ is also spirallike with respect to the operator $A = (1-\lambda)I_n$ , and hence $f$ is biholomorphic on $B^n$ (see [1]).

**Remark 1.3** If $\lambda = \frac{\alpha}{\alpha - 1}$ in Definition 1.1, where $\alpha \in [0, 1)$, we obtain the notion of almost starlikeness of order $\alpha$ (see [12]). On the other hand, if $\lambda = i \tan \delta$, $\delta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ we obtain the usual notion of spirallikeness of type $\delta$ and when $\lambda = 0$, we obtain the usual notion of starlikeness.

In 1995, K.A. Roper and T.J. Suffridge [10] introduced an extension operator. This operator is defined for normalized locally univalent functions $f$ on the unit disc $U$ in $\mathbf{C}$ by

$$\Phi_n(f)(z) = \left(f(z_1), \sqrt{f'(z_1)}\widetilde{z}\right)$$

where $z = (z_1, \widetilde{z})$ belongs to the unit ball $B^n$ in $\mathbf{C}^n$, $z_1 \in U$, $\widetilde{z} = (z_2, \cdots, z_n) \in \mathbf{C}^{n-1}$ and the branch of the square root is chosen such that $\sqrt{f'(0)} = 1$.

The Roper-Suffridge extension operator has remarkable properties.

1. If $f$ is a normalized convex function on $U$, then $\Phi_n(f)$ is a normalized convex mapping on $B^n$.

2. If $f$ is a normalized starlike function on $U$, then $\Phi_n(f)$ is a normalized starlike mapping on $B^n$.

3. If $f$ is a normalized Bloch function on $U$, then $\Phi_n(f)$ is a normalized Bloch mapping on $B^n$.

These results were proved by K.A. Roper and T.J. Suffridge [10] and I. Graham and G. Kohr [5].

In [6], I. Graham, G. Kohr and M. Kohr generalized the operator $\Phi_n$, by

$$\Phi_{n,\gamma}(f)(z) = \left(f(z_1), (f'(z_1))^\gamma \widetilde{z}\right),$$

where $\gamma \in \left[0, \frac{1}{2}\right]$, $f$, $z$, $z_1$ and $\widetilde{z}$ are defined as above and the branch of the power function is chosen such that $(f'(z_1))^\gamma \mid_{z_1=0}= 1$. They proved that, if $f$ is a normalized starlike function on $U$, then $\Phi_{n,\gamma}(f)$ is a normalized starlike mapping on $B^n$, if $f$ is a normalized Bloch function on $U$, then $\Phi_{n,\gamma}(f)$ is a normalized Bloch mapping on $B^n$ and the convexity is not preserved on $B^n$ for $\gamma \in \left[0, \frac{1}{2}\right)$.

Other results about the operator $\Phi_{n,\gamma}$ were obtained by using Loewner chains. In [12], the authors showed that if $f$ is an almost starlike function of order $\alpha$ on $U$, then $\Phi_{n,\gamma}(f)$ is an almost starlike mapping of order $\alpha$ on $B^n$. We proved in [1] that if $f$ is an almost starlike function of complex order $\lambda$ on $U$, then $\Phi_{n,\gamma}(f)$ is an almost starlike mapping of complex order $\lambda$ on $B^n$.

I. Graham and G. Kohr introduced a new extension operator in [5], by modifying the coefficient of $\widetilde{z}$. They defined

$$\Phi_{n,\beta}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^\beta \widetilde{z}\right),$$

where $\beta \in [0, 1]$, $f$, $z$, $z_1$ and $\widetilde{z}$ are given as above and the branch of the power function is chosen such that $\left(\frac{f(z_1)}{z_1}\right)^\beta \mid_{z_1=0}= 1$. They proved that the operator $\Phi_{n,\beta}$ is preserving starlikeness and spirallikeness of type $\delta$ and also that it maps a Bloch function $f$ to a Bloch mapping on $B^n$. Convexity is preserved only for $\beta = \frac{1}{2}$. The case $\beta = 1$ was previously considered and discussed by J.A. Pfaltzgraff and T.J. Suffridge [9].

Moreover, this operator preserves almost starlikeness of order $\alpha$ (see [3]) and almost starlikeness of complex order $\lambda$ (see [1]).

In 2002, I. Graham, H. Hamada, G. Kohr and T.J. Suffridge [7] generalized the Roper-Suffridge extension operator follows

$$\Phi_{n,\beta,\gamma}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^\beta (f'(z_1))^\gamma \widetilde{z}\right),$$

where $\beta \geq 0$ and $\gamma \geq 0$, $f$, $z$, $z_1$ and $\widetilde{z}$ are given as above and the branches of the power functions are chosen such that $\left(\frac{f(z_1)}{z_1}\right)^\beta \mid_{z_1=0}= 1$ and $(f'(z_1))^\gamma \mid_{z_1=0}= 1$. They proved the next result (see [7]).

**Theorem 1.4** *Assume* $\beta \in [0,1]$ *and* $\gamma \in \left[0, \frac{1}{2}\right]$ *such that* $\beta + \gamma \leq 1$. *Let* $f$ *be a locally univalent function on* $U$. *If* $f \in S^*$ *then* $\Phi_{n,\beta,\gamma}(f) \in S^*(B^n)$.

The proof uses the characterisation of starlikeness in terms of Loewner chains. However, only for $\beta = 0$ and $\gamma = 1/2$, that is only when the extension operator is the original Roper-Suffridge operator, convexity is preserved. It was shown in [3] that the operator preserves almost starlikeness of order $\alpha$ and in [1] we proved, by using the theory of Loewner chains, that the operator maps an almost starlike function of complex order $\lambda$ to an almost starlike mapping of complex order $\lambda$ on $B^n$.

We next give the following two lemmas. Lemma 1.5 gives the well-known Herglotz representation and Lemma 1.6 can be obtained easily.

**Lemma 1.5**[2] *Let* $f$ *be a holomorphic function on the unit disc* $U$. *Then* $\mathrm{Re} f(z) \geq 0$, $\forall\, z \in U$, *if and only if there exists an increasing function,* $\mu$, *on* $[0, 2\pi]$, *which satisfies* $\mu(2\pi) - \mu(0) = \mathrm{Re} f(0)$, *such that*

$$f(z) = \int_0^{2\pi} \frac{1 + ze^{-i\theta}}{1 - ze^{-i\theta}} d\mu(\theta) + i\mathrm{Im} f(0), \ z \in U.$$

**Lemma 1.6** *Suppose* $w \in \mathbf{C}$, *then we have*

1. $\mathrm{Re}(1 - w^2)(1 - \overline{w})^2 = (1 - |w|^2)|1 - w|^2$;
2. $\mathrm{Re}(1 + 2w - w^2)(1 - \overline{w})^2 = (1 - |w|^2)^2 - 2|w|^2|1 - w|^2$.

The aim of the present paper is to show that the operator $\Phi_{n,\beta,\gamma}$ preserves the notion of almost starlikeness of complex order $\lambda$ from dimension one into the $n$-dimensional case. This provides a way to obtain concrete examples of almost starlike mappings of complex order on the unit ball $B^n$. Various particular cases will be also considered.

## 2.Main results

We begin this section with the main result of this paper.

**Theorem 2.1** *Let* $\lambda \in \mathbf{C}$ *with* $\mathrm{Re}\lambda \leq 0$ *and let* $f$ *be an almost starlike function of complex order* $\lambda$ *on the unit disc* $U$. *Then* $F$ *is an almost starlike mapping of complex order* $\lambda$ *on* $B^n$, *where*

$$F(z) = \Phi_{n,\beta,\gamma}(f)(z) = \left( f(z_1), \left( \frac{f(z_1)}{z_1} \right)^{\beta} (f'(z_1))^{\gamma} \tilde{z} \right)$$

*$z = (z_1, \widetilde{z}) \in B^n$, $z_1 \in U$, $\widetilde{z} = (z_2, \cdots, z_n) \in \mathbf{C}^{n-1}$, $\beta \in [0,1]$, $\gamma \in \left[0, \frac{1}{2}\right]$ such that $\beta + \gamma \leq 1$, $f(z_1) \neq 0$ for $z_1 \in U \setminus \{0\}$ and the branches of the power functions $\left(\frac{f(z_1)}{z_1}\right)^\beta$, $(f'(z_1))^\gamma$ are chosen to satisfy the conditions $\left(\frac{f(z_1)}{z_1}\right)^\beta |_{z_1=0}= 1$ and $(f'(z_1))^\gamma |_{z_1=0}= 1$ respectively.*

*Proof.* By the definition of almost starlike mapping of complex order $\lambda$, we need to prove that the following inequality holds,

$$\mathrm{Re}\{(1-\lambda)\langle [DF(z)]^{-1}F(z), z\rangle\} \geq -\mathrm{Re}\lambda \|z\|^2 \tag{1}$$

or equivalently

$$\mathrm{Re}(1-\lambda)\overline{z}[DF(z)]^{-1}F(z) \geq -\mathrm{Re}\lambda \|z\|^2 \tag{2}$$

for all $z \in B^n$.

For $z = (z_1, \widetilde{z})$ we have two cases.

**Case I:** If $\widetilde{z} = 0$, then we can get the conclusion easily.

**Case II:** Suppose $\widetilde{z} \neq 0$.

Obviously, the mapping $F$ is holomorphic at every point $z = (z_1, \widetilde{z}) \in \overline{B}^n$ with $\widetilde{z} \neq 0$. Let $z = \zeta u$, $u \in \mathbf{C}^n$, $\|u\| = 1$, and $\zeta \in \overline{U} \setminus \{0\}$, then we have

$$\mathrm{Re}(1-\lambda)\overline{z}[DF(z)]^{-1}F(z) \geq -\mathrm{Re}\lambda \|z\|^2 \Leftrightarrow$$

$$\frac{\mathrm{Re}(1-\lambda)\overline{z}[DF(z)]^{-1}F(z) + \mathrm{Re}\lambda \|z\|^2}{\|z\|^2} \geq 0 \Leftrightarrow$$

$$\frac{\mathrm{Re}(1-\lambda)\overline{\zeta}\overline{u}[DF(\zeta u)]^{-1}F(\zeta u) + \mathrm{Re}\lambda |\zeta|^2}{|\zeta|^2} \geq 0 \Leftrightarrow$$

$$\mathrm{Re}\frac{(1-\lambda)\overline{u}[DF(\zeta u)]^{-1}F(\zeta u)}{\zeta} + \mathrm{Re}\lambda \geq 0.$$

Since the expression

$$\mathrm{Re}\frac{(1-\lambda)\overline{u}[DF(\zeta u)]^{-1}F(\zeta u)}{\zeta} + \mathrm{Re}\lambda$$

is the real part of a holomorphic function with respect to $\zeta$, it is a harmonic function. By the minimum principle for harmonic functions, we know that it attains its minimum on $|\zeta| = 1$, so we only need to prove (1) for all $z = (z_1, \widetilde{z}) \in \partial B^n$, $\widetilde{z} \neq 0$.

From

$$F(z) = \begin{pmatrix} f(z_1) \\ \left(\frac{f(z_1)}{z_1}\right)^\beta (f'(z_1))^\gamma \widetilde{z} \end{pmatrix},$$

we get

$$DF(z) = \begin{pmatrix} f'(z_1) & 0 \\ \left(\frac{f(z_1)}{z_1}\right)^{\beta} (f'(z_1)^{\gamma} \left[\beta\left(\frac{f'(z_1)}{f(z_1)} - \frac{1}{z_1}\right) + \gamma\frac{f''(z_1)}{f'(z_1)}\right]\widetilde{z} & \left(\frac{f(z_1)}{z_1}\right)^{\beta} (f'(z_1))^{\gamma} I_{n-1} \end{pmatrix}$$

and

$$[DF(z)]^{-1} = \begin{pmatrix} \frac{1}{f'(z_1)} & 0 \\ \left[\beta\left(\frac{1}{z_1 f'(z_1)} - \frac{1}{f(z_1)}\right) - \gamma\frac{f''(z_1)}{(f'(z_1))^2}\right]\widetilde{z} & \left(\frac{f(z_1)}{z_1}\right)^{-\beta} (f'(z_1))^{-\gamma} I_{n-1} \end{pmatrix}.$$

Therefore

$$(1-\lambda)[DF(z)]^{-1}F(z) + \mathrm{Re}\lambda\, z =$$

$$= \begin{pmatrix} (1-\lambda)\frac{f(z_1)}{f'(z_1)} + \mathrm{Re}\lambda\, z_1 \\ (1-\lambda)\left[\beta\left(\frac{f(z_1)}{z_1 f'(z_1)} - 1\right) - \gamma\frac{f''(z_1)f(z_1)}{(f'(z_1))^2}\right]\widetilde{z} + (1-\lambda+\mathrm{Re}\lambda)\widetilde{z} \end{pmatrix}$$

and hence

$$\mathrm{Re}\{(1-\lambda)\overline{z}[DF(z)]^{-1}F(z) + \mathrm{Re}\lambda\|z\|^2\} = \mathrm{Re}\{(1-\lambda)|z_1|^2\frac{f(z_1)}{z_1 f'(z_1)} + \mathrm{Re}\lambda|z_1|^2 +$$

$$(1-\lambda)\|\widetilde{z}\|^2\left[\beta\left(\frac{f(z_1)}{z_1 f'(z_1)} - 1\right) - \gamma\frac{f''(z_1)f(z_1)}{(f'(z_1))^2}\right] + (1-\lambda+\mathrm{Re}\lambda)\|\widetilde{z}\|^2\}$$

$$= \mathrm{Re}\{\left[(1-\lambda)|z_1|^2 + (1-\lambda)\|\widetilde{z}\|^2\beta\right]\frac{f(z_1)}{z_1 f'(z_1)} + \mathrm{Re}\lambda|z_1|^2 + \tag{3}$$

$$[1-\lambda+\mathrm{Re}\lambda-\beta(1-\lambda)]\,\|\widetilde{z}\|^2 - \gamma\|\widetilde{z}\|^2(1-\lambda)\frac{f''(z_1)f(z_1)}{(f'(z_1))^2}\}$$

$$= \mathrm{Re}\{\left[|z_1|^2 + (1-|z_1|)^2\beta\right](1-\lambda)\frac{f(z_1)}{z_1 f'(z_1)} + \mathrm{Re}\lambda|z_1|^2 +$$

$$[1-\lambda+\mathrm{Re}\lambda-\beta(1-\lambda)]\,(1-|z_1|^2) - \gamma(1-|z_1|^2)(1-\lambda)\frac{f''(z_1)f(z_1)}{(f'(z_1))^2}\}.$$

Let

$$p(z_1) = (1-\lambda)\frac{f(z_1)}{z_1 f'(z_1)} + \mathrm{Re}\lambda,$$

then

$$\begin{aligned} p'(z_1) &= (1-\lambda)\frac{(f'(z_1))^2 z_1 - f(z_1)f'(z_1) - f(z_1)f''(z_1)z_1}{(z_1 f'(z_1))^2} \\ &= (1-\lambda)\left[\frac{1}{z_1} - \frac{f(z_1)}{z_1^2 f'(z_1)} - \frac{f(z_1)f''(z_1)}{z_1(f'(z_1))^2}\right], \end{aligned}$$

thus we have

$$(1-\lambda)\frac{f(z_1)f''(z_1)}{(f'(z_1))^2} = 1-\lambda+\mathrm{Re}\lambda - p(z_1) - z_1 p'(z_1). \tag{4}$$

In addition, we know that $p \in H(U)$ and $\mathrm{Re}\, p(z_1) \geq 0$, $\forall\, z_1 \in U$, then, by Lemma 1.5, there exists an increasing function $\mu$, on $[0, 2\pi]$, which satisfies $\mu(2\pi) - \mu(0) = \mathrm{Re}\ p(0) = p(0) = 1 - \lambda + \mathrm{Re}\lambda$, such that

$$p(z_1) = \int_0^{2\pi} \frac{1 + z_1 e^{-i\theta}}{1 - z_1 e^{-i\theta}} d\mu(\theta),\ \ z_1 \in U \tag{5}$$

and

$$p'(z_1) = \int_0^{2\pi} \frac{2e^{-i\theta}}{(1 - z_1 e^{-i\theta})^2} d\mu(\theta). \tag{6}$$

Substituting (4), (5), and (6) into (3), we have

$$\mathrm{Re}\{(1-\lambda)\overline{z}[DF(z)]^{-1}F(z) + \mathrm{Re}\lambda\|z\|^2\}$$

$$= \mathrm{Re}\{[|z_1|^2 + \beta(1-|z_1|^2)](p(z_1) - \mathrm{Re}\lambda) + \mathrm{Re}\lambda|z_1|^2 +$$

$$[1-\lambda+\mathrm{Re}\lambda - \beta(1-\lambda)]\,(1-|z_1|^2) - \gamma(1-|z_1|^2)\left[1-\lambda+\mathrm{Re}\lambda - p(z_1) - z_1 p'(z_1)\right]\}$$

$$= \mathrm{Re}\{p(z_1)[|z_1|^2 + \beta(1-|z_1|^2) + \gamma(1-|z_1|^2)] + z_1 p'(z_1)\gamma(1-|z_1|^2) +$$

$$(1-|z_1|^2)(1-\lambda+\mathrm{Re}\lambda)(1-\beta-\gamma)\}$$

$$\geq \mathrm{Re}\{[|z_1|^2 + (\beta+\gamma)(1-|z_1|^2)]p(z_1) + \gamma(1-|z_1|^2)z_1 p'(z_1)\}$$

$$= \mathrm{Re}\int_0^{2\pi}\{[|z_1|^2 + (\beta+\gamma)(1-|z_1|^2)]\frac{1+z_1 e^{-i\theta}}{1-z_1 e^{-i\theta}} + 2\gamma(1-|z_1|^2)\frac{z_1 e^{-i\theta}}{(1-z_1 e^{-i\theta})^2}\}d\mu(\theta)$$

$$= \mathrm{Re}\int_0^{2\pi} \frac{1}{(1-z_1 e^{-i\theta})^2}\{[|z_1|^2 + (\beta+\gamma)(1-|z_1|^2)](1 - z_1^2 e^{-2i\theta}) +$$

$$2\gamma(1-|z_1|^2)z_1 e^{-i\theta}\}d\mu(\theta).$$

If we take $w = z_1 e^{-i\theta}$ in the above equality, then we have $w \in U$. Considering that $\mu$ is an increasing function, we need only to prove that

$$\mathrm{Re}\frac{1}{(1-w)^2}\{[|w|^2 + (\beta+\gamma)(1-|w|^2)](1-w^2) + 2\gamma(1-|w|^2)w\} \geq 0,\ w \in U.$$

By Lemma 1.6, we have

$$\mathrm{Re}\frac{1}{(1-w)^2}\{[|w|^2 + (\beta+\gamma)(1-|w|^2)](1-w^2) + 2\gamma(1-|w|^2)w\} =$$

$$= \mathrm{Re}\frac{1}{(1-w)^2}\{|w|^2(1-w^2) + \beta(1-|w|^2)(1-w^2) + \gamma(1-|w|^2)(1-2w-w^2)\}$$

$$= \frac{1}{|1-w|^4}\mathrm{Re}\{|w|^2(1-w^2)(1-\overline{w})^2 + \beta(1-|w|^2)(1-w^2)(1-\overline{w})^2 +$$

$$\gamma(1-|w|^2)(1+2w-w^2)(1-\overline{w})^2\}$$

$$= \frac{1}{|1-w|^4}\{|w|^2(1-|w|^2)|1-w|^2 + \beta(1-|w|^2)^2|1-w|^2 +$$

$$\gamma(1-|w|^2)[(1-|w|^2)^2 - 2|w|^2|1-w|^2]\}$$

$$= \frac{1-|w|^2}{|1-w|^4}\{(1-2\gamma)|w|^2|1-w|^2 + \beta(1-|w|^2)|1-w|^2 + \gamma(1-|w|^2)^2\} \geq 0.$$

We can now conclude that $F$ is an almost starlike mapping of complex order $\lambda$ on $B^n$.

**Remark 2.2** We mention that if $\lambda = \alpha/(1-\alpha)$, where $\alpha \in [0,1)$, in Theorem 2.1, we deduce that the operator $\Phi_{n,\beta,\gamma}$ preserves the usual notion of almost starlikeness of order $\alpha$ (see [3]).

We next present certain particular cases of Theorem 2.1.

If we take $\lambda = i\tan\delta$, with $|\delta| < \frac{\pi}{2}$ in Theorem 2.1, we obtain:

**Corollary 2.3***Let $f$ be a spirallike function of type $\delta$ on the unit disc $U$, then*

$$F(z) = \Phi_{n,\beta,\gamma}(f)(z) = \left(f(z_1), \left(\frac{f(z_1)}{z_1}\right)^{\beta}(f'(z_1))^{\gamma}\widetilde{z}\right)$$

*is a spirallike mapping of type $\delta$ on $B^n$, where $z = (z_1, \widetilde{z}) \in B^n$, $z_1 \in U$, $\widetilde{z} = (z_2, \cdots, z_n) \in \mathbf{C}^{n-1}$ , $\beta \in [0,1]$, $\gamma \in \left[0, \frac{1}{2}\right]$ such that $\beta + \gamma \leq 1$, $f(z_1) \neq 0$ when $z_1 \in U \setminus \{0\}$ and the branches of the power functions $\left(\frac{f(z_1)}{z_1}\right)^{\beta}$, $(f'(z_1))^{\gamma}$ are chosen to satisfy $\left(\frac{f(z_1)}{z_1}\right)^{\beta}|_{z_1=0}= 1$ and $(f'(z_1))^{\gamma}|_{z_1=0}= 1$ respectively.*

Taking $\lambda = 0$ in Theorem 2.1, we obtain the following result due to Graham, Hamada, Kohr, and Suffridge.

**Corollary 2.4** *[7]* Let $f$ be a starlike function on the unit disc $U$, then

$$F(z) = \Phi_{n,\beta,\gamma}(f)(z) = \left( f(z_1), \left( \frac{f(z_1)}{z_1} \right)^{\beta} (f'(z_1))^{\gamma} \widetilde{z} \right)$$

is a starlike mapping on $B^n$, where $z = (z_1, \widetilde{z}) \in B^n$, $z_1 \in U$, $\widetilde{z} = (z_2, \cdots, z_n) \in \mathbf{C}^{n-1}$, $\beta \in [0,1]$, $\gamma \in \left[0, \frac{1}{2}\right]$ such that $\beta + \gamma \leq 1$ , $f(z_1) \neq 0$ when $z_1 \in U \setminus \{0\}$ and the branches of the power functions $\left(\frac{f(z_1)}{z_1}\right)^{\beta}$, $(f'(z_1))^{\gamma}$ are chosen to satisfy $\left(\frac{f(z_1)}{z_1}\right)^{\beta} |_{z_1=0}= 1$ and $(f'(z_1))^{\gamma} |_{z_1=0}= 1$ respectively.

For $\beta = 0$ in Theorem 2.1, we get the following result:

**Corollary 2.5** *Let* $\lambda \in \mathbf{C}$ *with* $\mathrm{Re}\lambda \leq 0$ *and let* $f$ *be an almost starlike function of complex order* $\lambda$ *on the unit disc* $U$*, then*

$$F(z) = \Phi_{n,\gamma}(f)(z) = (f(z_1), (f'(z_1))^{\gamma} \widetilde{z})$$

*is an almost starlike mapping of complex order* $\lambda$ *on* $B^n$*, where* $\gamma \in \left[0, \frac{1}{2}\right]$*, and the branch of the power function* $(f'(z_1))^{\gamma}$ *is chosen to satisfy* $(f'(z_1))^{\gamma} |_{z_1=0}= 1$.

If we take $\gamma = 0$ in Theorem 2.1, we obtain the next particular case.

**Corollary 2.6** *Let* $\lambda \in \mathbf{C}$, $\mathrm{Re}\lambda \leq 0$ *and let* $f$ *be an almost starlike function of complex order* $\lambda$*, on the unit disc* $U$*, then*

$$F(z) = \Phi_{n,\beta}(f)(z) = \left( f(z_1), \left( \frac{f(z_1)}{z_1} \right)^{\beta} \widetilde{z} \right)$$

*is an almost starlike mapping of complex order* $\lambda$ *on* $B^n$*, where* $\beta \in [0,1]$ *and the branch of the power function is chosen such that* $\left(\frac{f(z_1)}{z_1}\right)^{\beta} |_{z_1=0}= 1$.

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