Acta Universitatis Apulensis
ISSN: 1582-5329

No. 28/2011
pp. 341-350

DIFFERENTIAL SUBORDINATION AND SUPERORDINATION FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING AN EXTENDED INTEGRAL OPERATOR

R. M. El-Ashwah and M. K. Aouf

Abstract. In this paper we derived some subordination, superordination and sandwich results for certain normalized analytic functions in the open unit disc, which are acted upon by a class of extended integral operator.

2000 *Mathematics Subject Classification*: 30C45.

1. Introduction

Let $H(U)$ be the class of analytic functions in the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ and let $H[a, k]$ consisting of functions of the form:

$$f(z) = a + a_k z^k + a_{k+1} z^{k+1} ... \qquad (a \in \mathbb{C}). \tag{1.1}$$

Also, let A_1 be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.2}$$

If $f, g \in H(U)$, we say that f is subordinate to g, written symbolically as $f(z) \prec g(z)$, if there exists a Schwarz function w, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ $(z \in U)$ such that $f(z) = g(w(z))$. In particular, if the function g is univalent in U, then we have the following equivalence (cf., e.g., [9]; see also [10, p.4]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \textit{ and } f(U) \subset g(U).$$

Supposing that p, h are two analytic functions in U, let

$$\varphi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}.$$

If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent functions in U and if p satisfies the second-order superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \tag{1.3}$$

then p is called to be a solution of the differential superordination (1.3). (If f is subordinate to F, then F is superordinate to f). An analytic function q is called a subordinant of (1.3), if $q(z) \prec p(z)$ for all the functions p satisfying (1.3). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all the subordinants q of (1.3), is called the best subordinant (cf., e.g., [9], see also [10]).

Recently, Miller and Mocanu [9] obtained sufficient conditions on the functions h, q and φ for which the following implication holds:

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z). \tag{1.4}$$

For $\nu > -1$ and $f(z) \in A_1$, we recall the generalized Bernardi-Libera-Livingston integral operator $L_\nu f(z)$ (see [1], [7] and [8]) as:

$$L_\nu f(z) = \frac{\nu+1}{z^\nu} \int_0^z t^{\nu-1} f(t)dt. \tag{1.5}$$

In [2] Catas extended the multiplier transformation and defined the operator $I^m(\lambda, \ell)f(z)$ on A_1 by the following series:

$$I^m(\lambda, \ell)f(z) = z + \sum_{k=2}^{\infty} \left[\frac{1+\ell+\lambda(k-1)}{1+\ell}\right]^m a_k z^k$$

$$(\lambda \geq 0; \ell \geq 0; m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}; \mathbb{N} = \{1, 2, ...\}; z \in U). \tag{1.6}$$

We note that $I^0(1,0)f(z) = f(z)$ and $I^1(1,0)f(z) = zf'(z)$.

Now, we define the integral operator $J^m(\lambda, \ell)f(z)$ $(\lambda > 0; \ell \geq 0; m \in \mathbb{N}_0)$ as follows:

$$\begin{aligned}
J^0(\lambda, \ell)f(z) &= f(z), \\
J^1(\lambda, \ell)f(z) &= \left(\frac{1+\ell}{\lambda}\right) z^{1-\left(\frac{1+\ell}{\lambda}\right)} \int_0^z t^{\left(\frac{1+\ell}{\lambda}\right)-2} f(t)dt \quad (f \in A_1; z \in U), \\
J^2(\lambda, \ell)f(z) &= \left(\frac{1+\ell}{\lambda}\right) z^{1-\left(\frac{1+\ell}{\lambda}\right)} \int_0^z t^{\left(\frac{1+\ell}{\lambda}\right)-2} J^1(\lambda, \ell)f(t)dt \quad (f \in A_1; z \in U),
\end{aligned}$$

and, in general,

$$\begin{aligned} J^m(\lambda,\ell)f(z) &= \left(\frac{1+\ell}{\lambda}\right) z^{1-\left(\frac{1+\ell}{\lambda}\right)} \int_0^z t^{\left(\frac{1+\ell}{\lambda}\right)-2} J^{m-1}(\lambda,\ell)f(t)dt \\ &= J^1(\lambda,\ell)\left(\frac{z}{1-z}\right) * J^1(\lambda,\ell)(\frac{z}{1-z}) * ......J^1(\lambda,\ell)(\frac{z}{1-z}) * f(z) \\ & \lfloor - - - - - - - - - - - m - times - - - - - - - - - - - \rfloor \end{aligned}$$

$$(f \in A_1; m \in \mathbb{N}; z \in U). \tag{1.7}$$

We note that if $f(z) \in A_1$, then from (1.1) and (1.7), we have

$$J^m(\lambda,\ell)f(z) = z + \sum_{k=2}^{\infty} \left[\frac{1+\ell}{1+\ell+\lambda(k-1)}\right]^m a_k z^k$$

$$(\lambda > 0; \ell \geq 0; m \in \mathbb{N}_0; z \in U). \tag{1.8}$$

From (1.8), it is easy verify that

$$\lambda z(J^{m+1}(\lambda,\ell)f(z))^{'} = (1+\ell)J^m(\lambda,\ell)f(z) - (1+\ell-\lambda)J^{m+1}(\lambda,\ell)f(z)(\lambda > 0). \tag{1.9}$$

The operator $J^m(\lambda,\ell)f(z)$ was introduced by El-Ashwah and Aouf [4, with $p = 1$].

We note that:
(i) $J^m(1,1)f(z) = I^m f(z)$ (see Flett [5] and Uralegaddi and Somanatha [15]);
(ii) $J^m(1,0)f(z) = I^m f(z)(m \in N_0)$ (see Salagean [13]);
(iii) $J^\alpha(1,1)f(z) = I^\alpha f(z)(\alpha > 0)$ (see Jung et al. [6]);
(iv) $J^m(\lambda,0) = J_\lambda^{-m} f(z)$ (see Patel [12]).

2. Preliminaries

In order to prove our subordination and superordination results, we make use of the following known definition and results.

Definition 1 [11]. *Denote by Q the set of all functions $f(z)$ that are analytic and injective on $\overline{U}\backslash E(f)$, where*

$$E(f) = \left\{\zeta : \zeta \in \partial U \text{ and } \lim_{z\to\zeta} f(z) = \infty\right\} \tag{2.1}$$

and are such that $f^{'}(\zeta) \neq 0$ for $\zeta \in \partial U \backslash E(f)$.

Lemma 1 [14]. *Let q be a convex univalent function in U and let $\psi \in \mathbb{C}, \delta \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$ with*

$$\operatorname{Re}\left\{1 + \frac{zq^{''}(z)}{q^{'}(z)}\right\} > \max\left\{0, -\operatorname{Re}\left(\frac{\psi}{\delta}\right)\right\}.$$

If $p(z)$ is analytic in U and

$$\psi p(z) + \delta z p^{'}(z) \prec \psi q(z) + \delta z q^{'}(z), \tag{2.2}$$

then

$$p(z) \prec q(z)$$

and q is the best dominant.

Lemma 2 [11]. *Let q be convex univalent in U and $\delta \in \mathbb{C}$. Further assume that* $\operatorname{Re}(\overline{\delta}) > 0$. *If $p(z) \in H[q(0), 1] \cap Q$ and $p(z) + \delta z p^{'}(z)$ is univalent in U, then*

$$q(z) + \delta z q^{'}(z) \prec p(z) + \delta z p^{'}(z), \tag{2.3}$$

implies

$$q(z) \prec p(z)$$

and q is the best subordinant.

3.Main Results

Unless otherwise mentioned we shall assume throughout the paper that $\lambda > 0, \ell \geq 0, m \in \mathbb{N}_0$ and $z \in U$.

Theorem 1. *Let q be convex univalent in U, with $q(0) = 1$, $\gamma \in \mathbb{C}^*$. Further, assume that*

$$\operatorname{Re}\left\{1 + \frac{zq^{''}(z)}{q^{'}(z)}\right\} > \max\left\{0, -\operatorname{Re}\left(\frac{\ell+1}{\lambda\gamma}\right)\right\}. \tag{3.1}$$

If $f \in A_1$, $J^m(\lambda, \ell)f(z) \neq 0$ for $0 < |z| < 1$, and

$$\frac{J^{m+1}(\lambda,\ell)f(z)}{J^m(\lambda,\ell)f(z)} + \gamma\left\{1 - \frac{J^{m-1}(\lambda,\ell)f(z)J^{m+1}(\lambda,\ell)f(z)}{\left(J^m(\lambda,\ell)f(z)\right)^2}\right\}$$

$$\prec q(z) + \left(\frac{\lambda\gamma}{\ell+1}\right) z q^{'}(z), \tag{3.2}$$

then

$$\frac{J^{m+1}(\lambda,\ell)f(z)}{J^m(\lambda,\ell)f(z)} \prec q(z)$$

and q is the best dominant of subordination (3.2).
Proof. Define a function p by

$$p(z)=\frac{J^{m+1}(\lambda,\ell)f(z)}{J^{m}(\lambda,\ell)f(z)} \quad (z\in U). \tag{3.3}$$

Then the function p is analytic in U and $p(0)=1$. Therefore, differentiating (3.3) logarithmically with respect to z and using the identity (1.9) in the resulting equation, we have

$$\frac{J^{m+1}(\lambda,\ell)f(z)}{J^{m}(\lambda,\ell)f(z)}+\gamma\left\{1-\frac{J^{m-1}(\lambda,\ell)f(z)J^{m+1}(\lambda,\ell)f(z)}{\left(J^{m}(\lambda,\ell)f(z)\right)^{2}}\right\}$$
$$=p(z)+\left(\frac{\lambda\gamma}{\ell+1}\right)zp^{'}(z),$$

that is,

$$p(z)+\left(\frac{\lambda\gamma}{\ell+1}\right)zp^{'}(z)\prec q(z)+\left(\frac{\lambda\gamma}{\ell+1}\right)zq^{'}(z)$$

and therefore, the theorem follows by applying Lemma 1.

Putting $q(z)=\dfrac{1+Az}{1+Bz}(A,B\in\mathbb{C},A\neq B$ and $|B|\leq 1)$ in Theorem 1, we obtain the following corollary.

Corollary 1. *If $f(z)\in A_1$, $\operatorname{Re}\left\{\dfrac{1-Bz}{1+Bz}\right\}>\max\left\{0,-\operatorname{Re}\left(\frac{\ell+1}{\lambda\gamma}\right)\right\}$ and $\gamma\in\mathbb{C}^*$ satisfy*

$$\frac{J^{m+1}(\lambda,\ell)f(z)}{J^{m}(\lambda,\ell)f(z)}+\gamma\left\{1-\frac{J^{m-1}(\lambda,\ell)f(z)J^{m+1}(\lambda,\ell)f(z)}{\left(J^{m}(\lambda,\ell)f(z)\right)^{2}}\right\}$$
$$\prec\frac{1+Az}{1+Bz}+\left(\frac{\lambda\gamma}{\ell+1}\right)\frac{(A-B)z}{(1+Bz)^{2}},$$

then

$$\frac{J^{m+1}(\lambda,\ell)f(z)}{J^{m}(\lambda,\ell)f(z)}\prec\frac{1+Az}{1+Bz} \tag{3.4}$$

and $q(z)=\frac{1+Az}{1+Bz}$ is the best dominant.

Putting $A=1$ and $B=-1$ in Corollary 1, we have

Corollary 2. *Let $f(z)\in A_1$, and $\gamma\in\mathbb{C}^*$ satisfy*

$$\frac{J^{m+1}(\lambda,\ell)f(z)}{J^{m}(\lambda,\ell)f(z)}+\gamma\left\{1-\frac{J^{m-1}(\lambda,\ell)f(z)J^{m+1}(\lambda,\ell)f(z)}{\left(J^{m}(\lambda,\ell)f(z)\right)^{2}}\right\}$$

$$\prec \frac{1+z}{1-z} + \left(\frac{2\lambda\gamma}{\ell+1}\right)\frac{z}{(1-z)^2},$$

then

$$\text{Re}\left\{\frac{J^{m+1}(\lambda,\ell)f(z)}{J^m(\lambda,\ell)f(z)}\right\} > 0.$$

Now, by appealing to Lemma 2, it can be easily prove the following theorem.

Theorem 2. *Let q be convex univalent in U, with $q(0) = 1$. Let $\gamma \in \mathbb{C}$ with* $\text{Re}(\gamma) > 0$. If $f \in A_1, \dfrac{J^{m+1}(\lambda,\ell)f(z)}{J^m(\lambda,\ell)f(z)} \in H[q(0),1] \cap Q$,

$$\frac{J^{m+1}(\lambda,\ell)f(z)}{J^m(\lambda,\ell)f(z)} + \gamma\left\{1 - \frac{J^{m-1}(\lambda,\ell)f(z)J^{m+1}(\lambda,\ell)f(z)}{(J^m(\lambda,\ell)f(z))^2}\right\}$$

is univalent in U, and

$$q(z) + \left(\frac{\lambda\gamma}{\ell+1}\right)zq^{'}(z) \prec \frac{J^{m+1}(\lambda,\ell)f(z)}{J^m(\lambda,\ell)f(z)} + \gamma\Big\{1- \frac{J^{m-1}(\lambda,\ell)f(z)J^{m+1}(\lambda,\ell)f(z)}{(J^m(\lambda,\ell)f(z))^2}\Big\}, \tag{3.5}$$

then

$$q(z) \prec \frac{J^{m+1}(\lambda,\ell)f(z)}{J^m(\lambda,\ell)f(z)}, \tag{3.6}$$

and q is the best subordinant.

Combining Theorem 1 and Theorem 2, we obtain the following sandwich thereom.

Theorem 3. *Let q_1 be convex univalent in U, with $q_1(0) = 1$. Let $\gamma \in \mathbb{C}^*$ with* $\text{Re}(\gamma) > 0$, *q_2 be univalent in U, $q_2(0) = 1$ and satisfies (3.1). If* $f \in A_1, \dfrac{J^{m+1}(\lambda,\ell)f(z)}{J^m(\lambda,\ell)f(z)} \in H[q(0),1] \cap Q$,

$$\frac{J^{m+1}(\lambda,\ell)f(z)}{J^m(\lambda,\ell)f(z)} + \gamma\left\{1 - \frac{J^{m-1}(\lambda,\ell)f(z)J^{m+1}(\lambda,\ell)f(z)}{(J^m(\lambda,\ell)f(z))^2}\right\}$$

is univalent in U and

$$\begin{aligned} q_1(z) + \left(\frac{\lambda\gamma}{\ell+1}\right)zq_1^{'}(z) &\prec \frac{J^{m+1}(\lambda,\ell)f(z)}{J^m(\lambda,\ell)f(z)} + \gamma\left\{1 - \frac{J^{m-1}(\lambda,\ell)f(z)J^{m+1}(\lambda,\ell)f(z)}{(J^m(\lambda,\ell)f(z))^2}\right\} \\ &\prec q_2(z) + \left(\frac{\lambda\gamma}{\ell+1}\right)zq_2^{'}(z), \end{aligned}$$

then

$$q_1(z) \prec \frac{J^{m+1}(\lambda,\ell)f(z)}{J^m(\lambda,\ell)f(z)} \prec q_2(z) \tag{3.7}$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

Theorem 4. *Let q be convex univalent in U, with $q(0) = 1, \gamma \in \mathbb{C}^*$.* Further, assume that (3.1) holds. If $f \in A_1$ satisfies

$$(1+\gamma)\frac{zJ^m(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2} + \gamma\frac{zJ^{m-1}(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2} -$$

$$2\gamma\frac{z(J^m(\lambda,\ell)f(z))^2}{(J^{m+1}(\lambda,\ell)f(z))^3} \prec q(z) + \left(\frac{\lambda\gamma}{\ell+1}\right)zq'(z), \tag{3.8}$$

then

$$\frac{zJ^m(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2} \prec q(z) \tag{3.9}$$

and $q(z)$ is the best dominant.

Proof. Define the function p by

$$p(z) = \frac{zJ^m(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2} \quad (z \in U). \tag{3.10}$$

Differentiating (3.10) logarithmically with respect to z, we obtain

$$\frac{zp'(z)}{p(z)} = \left(\frac{\ell+1}{\lambda}\right) + \left(\frac{\ell+1}{\lambda}\right)\frac{J^{m-1}(\lambda,\ell)f(z)}{J^m(\lambda,\ell)f(z)} - 2\left(\frac{\ell+1}{\lambda}\right)\frac{J^m(\lambda,\ell)f(z)}{J^{m+1}(\lambda,\ell)f(z)}.$$

Then, simple computations show that

$$p(z) + \left(\frac{\lambda\gamma}{\ell+1}\right)zp'(z) = (1+\gamma)\frac{zJ^m(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2} +$$

$$+\gamma\frac{zJ^{m-1}(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2} - 2\gamma\frac{z(J^m(\lambda,\ell)f(z))^2}{(J^{m+1}(\lambda,\ell)f(z))^3}.$$

Applying Lemma 1, the theorem follows.

Taking $q(z) = \dfrac{1+Az}{1+Bz}$ $(A, B \in \mathbb{C}, A \neq B$ and $|B| \leq 1)$ in Theorem 4, we obtain the following corollary.

Corollary 3. *If $f(z) \in A_1$ and $\gamma \in \mathbb{C}^*$ satisfy*

$$(1+\gamma)\frac{zJ^m(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2} + \gamma\frac{zJ^{m-1}(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2} -$$

$$2\gamma\frac{z(J^m(\lambda,\ell)f(z))^2}{(J^{m+1}(\lambda,\ell)f(z))^3} \prec \frac{1+Az}{1+Bz} + \left(\frac{\lambda\gamma}{\ell+1}\right)\frac{(A-B)}{(1+Bz)^2},$$

then

$$\frac{J^m(\lambda,\ell)f(z)}{J^{m+1}(\lambda,\ell)f(z)} \prec \frac{1+Az}{1+Bz} \tag{3.12}$$

and $q(z) = \dfrac{1+Az}{1+Bz}$ *is the best dominant.*

Theorem 5. *Let* q *be convex univalent in* U, with $q(0) = 1$. Let $\gamma \in \mathbb{C}$. with $\text{Re}(\gamma) > 0$. If $f(z) \in A_1$, $\dfrac{zJ^m(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2} \in H[q(0),1] \cap Q$,

$$(1+\gamma)\frac{zJ^m(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2} + \gamma\frac{zJ^{m-1}(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2} - 2\gamma\frac{z(J^m(\lambda,\ell)f(z))^3}{(J^{m+1}(\lambda,\ell)f(z))^3}$$

is univalent in U, *and*

$$q(z) + \left(\frac{\lambda\gamma}{\ell+1}\right)zq^{'}(z) \prec (1+\gamma)\frac{zJ^m(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2} +$$

$$\gamma\frac{zJ^{m-1}(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2} - 2\gamma\frac{z(J^m(\lambda,\ell)f(z))^2}{(J^{m+1}(\lambda,\ell)f(z))^3} \tag{3.13}$$

then

$$q(z) \prec \frac{zJ^m(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2}, \tag{3.14}$$

and q *is the best subordinant.*

Proof The proof is similar to the proof of Theorem 3 and using Lemma 2.

Combining Theorem 4 and Theorem 5, we get the following sandwich theorem.

Theorem 6. *Let* q_1 *be convex univalent in* U, *with* $q_1(0) = 1$. *Let* $\gamma \in \mathbb{C}^*$ *with* $\text{Re}(\gamma) > 0, q_2$ *be univalent in* U, $q_2(0) = 1$ *and satisfies (3.1). If* $f \in A_1$, $\dfrac{zJ^m(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2} \in H[q(0),1] \cap Q$,

$$(1+\gamma)\frac{zJ^m(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2} + \gamma\frac{zJ^{m-1}(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2} - 2\gamma\frac{z(J^m(\lambda,\ell)f(z))^2}{(J^{m+1}(\lambda,\ell)f(z))^3}$$

is univalent in U *and*

$$q_1(z) + \left(\frac{\lambda\gamma}{\ell+1}\right)zq_1^{'}(z) \prec (1+\gamma)\frac{zJ^m(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2} +$$

$$\gamma\frac{zJ^{m-1}(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2}-2\gamma\frac{z(J^{m}(\lambda,\ell)f(z))^2}{(J^{m+1}(\lambda,\ell)f(z))^3}\prec$$

$$q_2(z)+\left(\frac{\lambda\gamma}{\ell+1}\right)zq_2^{'}(z),$$

then

$$q_1(z)\prec\frac{zJ^{m}(\lambda,\ell)f(z)}{(J^{m+1}(\lambda,\ell)f(z))^2}\prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinant and the best dominant.

Remark. Putting $\ell=0$ and $\lambda=1$ in the above results, we obtain the results obtained by Cotirlā [3].

References

[1] D. Bernardi, *Convex and starlike univalent functions*, Trans. Amer. Math. Soc, 135(1969), 429-446.
[2] A. Catas, *A note on a certain subclass of analytic functions defined by multiplier transformations*, in Proc. Book of the International Symposium on Geometric Function Theory and Applications, Istanbul, Turkey, (August 2007), 241-250.
[3] L. A. Cotirlā, *Differential subordinations and superordinations for analytic functions defined by an integral operator*, Carpathian J. Math., 25(2009), no. 1, 49 - 54.
[4] R. M. El-Ashwah and M. K. Aouf, *Some properties of new integral operator*, Acta Univ. Apulensis, (2010), no. 24, 51-61.
[5] T. M. Flett, *The dual of an inequality of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl. 38 (1972), 746-765.
[6] T. B. Jung, Y. C. Kim and H. M. Srivastava,*The Hardy space of analytic functions associated with certain one-parameter families of integral operator*, J. Math. Anal. Appl. 176(1993), 138–147.
[7] R. J. Libera , *Some classes of regular functions*, Proc. Amer. Math. Soc. 16(1965), 755-758.
[8] A. E. Livingston, *On the radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc. 17(1966), 352-357.
[9] S. S. Miller and P. T. Mocanu, *Differential subordinations and univalent functions*, Michigan Math. J. 28(1981), no.2, 157-171.
[10] S. S. Miller and P. T. Mocanu, *Differential subordinations: Theory and Applications*, Series on Mongraphs and Textbooks in Pure and Applied Mathematics, Vol. 225, Marcel Dekker Inc., New York. and Basel, 2000.

[11] S. S. Miller and P. T. Mocanu, *Subordinants of differential superordinations*, Complex Variables, 48(2003), no.10, 815-826.
[12] J. Patel, *Inclusion relations and convolution properties of certain subclasses of analytic functions defined by a generalized Salagean operator*, Bull. Belg. Math. Soc. Simon Stevin 15(2008), 33-47.
[13] G. S. Salagean, *Subclasses of univalent functions*, Lecture Notes in Math. (Springer-Verlag) 1013(1983), 362-372.
[14] T. N. Shanmugam, V. Radichandran and S. Sivasubramanian, *Differential sandwich theorems for some subclasses of analytic functions*, Austral. J. Math. Anal. Appl. 3(2006), no.1, Art. 8, 1-11.
[15] B. A. Uralegaddi and C. Somanatha, Certain classes of univalent functions, *In Current Topics in Analytic Function Theory*, (Edited by H. M. Srivastava and S. Owa), World Scientific Publishing Company, Singapore, 1992, 371-374.

R. M. El-Ashwah
Department of Mathematics
Faculty of Science (Damietta Branch)
Mansoura University
New Damietta 34517, Egypt
E-mail :*r_elashwah@yahoo.com*

M. K. Aouf
Department of Mathematics
Faculty of Science
Mansoura University
Mansoura 35516, Egypt.
E-mail : *mkaouf127@yahoo.com*