Acta Universitatis Apulensis
ISSN: 1582-5329

No. 28/2011
pp. 115-124

MULTIPLE SOLUTIONS FOR A FOURTH ORDER ELLIPTIC EQUATION WITH HARDY TYPE POTENTIAL

Nguyen Thanh Chung

Abstract. Consider the fourth order elliptic equation with Hardy type potential

$$\begin{cases} \Delta^2 u &= \frac{\mu}{|x|^4} a(x)u + \lambda b(x) f(u) \quad \text{in } \Omega, \\ u &= 0, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 5)$, is a bounded domain with smooth boundary $\partial\Omega$, $0 \in \Omega$, ν is the outward unit normal to $\partial\Omega$, the weighted function $a : \Omega \to \mathbb{R}$ may change sign, λ, μ are two parameters. Under suitable conditions on the nonlinearities, a multiplicity result is given using a variant of the three critical point theorem by G. Bonanno [3].

2000 *Mathematics Subject Classification*: 35J65, 35J20.

1. Introduction and Preliminaries

In this article, we are concerned with a class of fourth order elliptic equations with Hardy type potential

$$\begin{cases} \Delta^2 u &= \frac{\mu}{|x|^4} a(x)u + g(\lambda, x, u) \quad \text{in } \Omega, \\ u &= 0, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 5)$ is a bounded domain with smooth boundary $\partial\Omega$, $0 \in \Omega$, ν is the outward unit normal to $\partial\Omega$, λ, μ are two parameters, $0 \leq \mu < \mu^\star$, where $\mu^\star = \left(\frac{N(N-4)}{4}\right)^2$ is the best constant in the Hardy inequality i.e.

$$\int_\Omega \frac{|\varphi|^2}{|x|^4} dx \leq \frac{1}{\mu^\star} \int_\Omega |\Delta \varphi|^2 dx \tag{1.2}$$

for all $\varphi \in C_0^\infty(\Omega)$, see [12].

We point out the fact that if $\mu = 0$, problem (1.1) has been intensively studied in the last decades. In the papers [4, 5, 7, 9, 11], the authors studied the problems of p-biharmonic type, in which p is a constant. The topic involving $p(x)$-biharmonic type operators has been studied in recent years, see [1, 2].

In the case $\mu > 0$, problem (1.1) has been studied in some papers, we refer to [10, 12, 13]. In [12], Y. Yao et al. studied problem (1.1) in the special case $a(x) \equiv 1$, and $g(\lambda, x, u) = \lambda f(x)u$. They showed that if $f \in \hat{f}$, with

$$\hat{f} = \left\{ f : \Omega \to \mathbb{R}^+ : \lim_{|x|\to 0} |x|^4 f(x) = 0, f \in L^\infty_{loc}(\Omega\backslash\{0\}) \right\},$$

then for any $0 \leq \mu < \mu^\star$, the problem admits a non-trivial solution in $W_0^{2,2}(\Omega)$. In [13], the authors studied the existence of a non-trivial solution of the problem in the critical case:

$$\begin{cases} \Delta^2 u = \mu \frac{|u|^{q-2}u}{|x|^s} u + |u|^{2_\star - 2}u & \text{in } \Omega, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where $2 \leq q \leq 2_\star(s) = \frac{2(N-s)}{N-4} \leq 2_\star = \frac{2N}{N-4}$, $N \geq 5$, $0 < s < 4$. Very recently, Y. Wang et al. [10] studied the problem

$$\begin{cases} \Delta^2 u = \mu \frac{|u|^{2_\star(s)-2}u}{|x|^s} u + \lambda b(x)|u|^{r-2}u & \text{in } \mathbb{R}, \\ u \in W_0^{2,2}(\mathbb{R}^N), & N \geq 5, \end{cases} \tag{1.4}$$

where $1 < r < 2_\star = \frac{2N}{N-4}$, $N \geq 5$, and $0 \leq b(x) \in L^q(\mathbb{R}^N)$ with $q = \frac{2_\star}{2_\star - r}$, $meas(\{b(x) > 0\}) > 0$. Using variational techniques, the authors showed the existence of infinitely many solutions of (1.4) under suitable conditions on the parameters μ and λ.

In this paper, we consider the fourth order elliptic problem (1.1) in the case when $g(\lambda, x, u) = \lambda b(x)f(u)$, i.e.,

$$\begin{cases} \Delta^2 u = \frac{\mu}{|x|^4} a(x)u + \lambda b(x)f(u) & \text{in } \Omega, \\ u = 0, \quad \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

in which the function $f : \mathbb{R} \to \mathbb{R}$ is superlinear at zero and sublinear at infinity, the weighted function $a : \Omega \to \mathbb{R}$ may change sign, i.e., there exists a positive constant $A_0 > 0$ such that

$$-A_0 \leq a(x) \leq A_0 \text{ for all } x \in \overline{\Omega}, \tag{1.6}$$

the function $b \in L^\infty(\Omega)$, $b(x) \geq 0$ for all $x \in \overline{\Omega}$, there exists $R_0 > 0$ such that

$$R_0 < \operatorname{dist}(0, \partial\Omega) \text{ and } b_{R_0} = \inf_{|x|\leq R_0} b(x) > 0. \tag{1.7}$$

In order to state the main result of this paper, we assume $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying the following conditions:

$(f1)$ f is sublinear at infinity, i.e.,

$$\lim_{|t|\to\infty}\frac{f(t)}{t}=0;$$

$(f2)$ f is superlinear at zero, i.e.,

$$\lim_{t\to 0}\frac{f(t)}{t}=0;$$

$(f3)$ There exists $t_0\in\mathbb{R}$, such that $F(t_0)>0$, where $F(t)=\int_0^t f(s)ds$.

It should be noticed that the term $|u|^{r-2}u$ is not superlinear at zero if $1<r<2$ and it is not sublinear at infinity if $2<r<2_\star(s)=\frac{2(N-s)}{N-4}$, so the situation introduced here is different from [10]. Moreover, by the presence of the functions a and b, especially a may change sign in Ω, the obtained result in this work is better than that of [6], eventually with the Laplace operator $-\Delta$.

Let $W_0^{2,2}(\Omega)$ be the usual Sobolev space with respect to the norm $\|u\|_{2,2}=\Big(\int_\Omega|\Delta u|^2dx\Big)^{\frac{1}{2}}$. We denote by S_q the best constant in the embedding $W_0^{2,2}(\Omega)\hookrightarrow L^q(\Omega)$.

Definition 1.1. A function $u\in W_0^{2,2}(\Omega)$ is said to be a weak solution of problem (1.5) if and only if

$$\int_\Omega \Delta u\Delta v dx-\mu\int_\Omega\frac{a(x)}{|x|^4}uvdx-\lambda\int_\Omega b(x)f(u)vdx=0$$

for any $v\in W_0^{2,2}(\Omega)$.

Theorem 1.2. *Assume the hypotheses (1.6)-(1.7) and $(f1)$-$(f3)$ are fulfilled, then there exists $\overline{\mu}>0$, such that for any $0\le\mu<\overline{\mu}$ there exist an open interval $\Lambda\subset[0,\infty)$ and a constant $\delta_{\overline{\mu}}$, such that for every $\lambda\in\Lambda$, problem (1.5) has at least two non-trivial weak solutions in $W_0^{2,2}(\Omega)$, whose $W_0^{2,2}(\Omega)$-norms are less than $\delta_{\overline{\mu}}$.*

Theorem 1.2 will be proved by using a recent result on the existence of at least three critical points by G. Bonanno [3]. For the reader's convenience, we describe it as follows.

Lemma 1.3. *Let $(X,\|.\|)$ be a separable and reflexive real Banach space, $\mathcal{A},\mathcal{F}:X\to\mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $x_0\in X$ such that $\mathcal{A}(x_0)=\mathcal{F}(x_0)=0$, $\mathcal{A}(x)\ge 0$ for all $x\in X$ and there exist $x_1\in X$, $\rho>0$ such that* *(i)* $\rho < \mathcal{A}(x_1)$,

(ii) $\sup_{\{\mathcal{A}(x)<\rho\}} \mathcal{F}(x) < \rho \frac{\mathcal{F}(x_1)}{\mathcal{A}(x_1)}$.

Further, put

$$\overline{a} = \frac{\xi\rho}{\rho\frac{\mathcal{F}(x_1)}{\mathcal{A}(x_1)} - \sup_{\{\mathcal{A}(x)<\rho\}} \mathcal{F}(x)}, \text{ with } \xi > 1,$$

and assume that the functional $\mathcal{A} - \lambda\mathcal{F}$ is sequentially weakly lower semicontinuous, satisfies the Palais-Smale condition and

(iii) $\lim_{\|x\|\to\infty}[\mathcal{A}(x) - \lambda\mathcal{F}(x)] = +\infty$ *for every* $\lambda \in [0, \overline{a}]$.

Then, there exist an open interval $\Lambda \subset [0, \overline{a}]$ and a positive real number δ such that each $\lambda \in \Lambda$, the equation

$$D\mathcal{A}(u) - \lambda D\mathcal{F}(u) = 0$$

has at least three solutions in X whose $\|.\|$-norms are less than δ.

2. Proof of the main result

For each $\mu \in [0, \mu^\star)$, and $\lambda \in \mathbb{R}$, let us define the functional $J_{\mu,\lambda} : W_0^{2,2}(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} J_{\mu,\lambda}(u) &= \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \frac{\mu}{2}\int_\Omega \frac{a(x)}{|x|^4}|u|^2 dx - \lambda\int_\Omega b(x)F(u)dx \\ &= \mathcal{A}(u) - \lambda\mathcal{F}(u), \end{aligned} \tag{2.1}$$

where

$$\begin{aligned} \mathcal{A}(u) &= \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \frac{\mu}{2}\int_\Omega \frac{a(x)}{|x|^4}|u|^2 dx, \\ \mathcal{F}(u) &= \int_\Omega b(x)F(u)dx, \end{aligned} \tag{2.2}$$

for all $u \in W_0^{2,2}(\Omega)$. Then, by the Hardy inequality (1.2) and the hypothesis $(f1)$, we can show that $J_{\mu,\lambda}$ is well-defined and of C^1 class in $W_0^{2,2}(\Omega)$. Moreover, we have

$$DJ_{\mu,\lambda}(u)(v) = \int_\Omega \Delta u\Delta v dx - \mu\int_\Omega \frac{a(x)}{|x|^4}uv dx - \lambda\int_\Omega b(x)f(u)v dx$$

for all $v \in W_0^{2,2}(\Omega)$. Thus, weak solutions of problem (1.5) are exactly the critical points of the functional $J_{\mu,\lambda}$.

Lemma 2.1 *There exists $\overline{\mu} > 0$, such that for each $\mu \in [0, \overline{\mu})$, and $\lambda \in \mathbb{R}$, the functional $J_{\mu,\lambda}$ is sequentially weakly lower semi-continuous in $W_0^{2,2}(\Omega)$.*

Proof. Let $\{u_m\}$ be a sequence that converges weakly to u in $W_0^{2,2}(\Omega)$. Since $-A_0 \leq a(x) \leqq A_0$ for all $x \in \overline{\Omega}$, taking $\overline{\mu} = \frac{\mu^\star}{A_0}$, then for each $0 \leq \mu < \overline{\mu}$, using the same arguments as in the proof of [8, Theorem 3.2], we can obtain

$$\liminf_{m\to\infty} \int_\Omega \Big(|\Delta u_m|^2 dx - \mu \frac{a(x)}{|x|^4}|u_m|^2\Big) dx \geq \int_\Omega \Big(|\Delta u|^2 dx - \mu \frac{a(x)}{|x|^4}|u|^2\Big) dx. \tag{2.3}$$

On the other hand, by $(f1)$, there exists a constant $C > 0$, such that

$$|f(t)| \leq C(1 + |t|), \text{ for all } t \in \mathbb{R}.$$

Hence, using the the Hölder inequality, we have

$$\begin{aligned}
&\Big|\int_\Omega b(x)F(u_m)dx - \int_\Omega b(x)F(u)dx\Big| \\
&\leq \int_\Omega |b(x)||F(u_m) - F(u)|dx \\
&\leq \|b\|_{L^\infty(\Omega)} \int_\Omega |f(u + \theta_m(u_m - u))||u_m - u|dx \\
&\leq C\|b\|_{L^\infty(\Omega)} \int_\Omega (1 + |u + \theta_m(u_m - u)|)|u_m - u|dx \\
&\leq C\|b\|_{L^\infty(\Omega)} \Big[\Big(meas(\Omega)\Big)^{\frac{1}{2}} + \|u + \theta_m(u_m - u)\|_{L^2(\Omega)}\Big]\|u_m - u\|_{L^2(\Omega)}, \quad \theta_m \in (0, 1),
\end{aligned}$$

which shows that

$$\lim_{m\to\infty} \int_\Omega b(x)F(u_m)dx = \int_\Omega b(x)F(u)dx. \tag{2.4}$$

From relations (2.3) and (2.4), we conclude that

$$\liminf_{m\to\infty} J_{\mu,\lambda}(u_m) \geq J_{\mu,\lambda}(u)$$

and thus, $J_{\mu,\lambda}$ is sequentially weakly lower semi-continuous in $W_0^{2,2}(\Omega)$. □

Lemma 2.2. *For each $\mu \in [0, \overline{\mu})$, where $\overline{\mu}$ is given by Lemma 2.1 and $\lambda \in \mathbb{R}$, the functional $J_{\mu,\lambda}$ is coercive and satisfies the Palais-Smale condition.*

Proof. Let us fix $\lambda \in \mathbb{R}$, arbitrary. By $(f1)$, there exists $\delta = \delta(\lambda) > 0$, such that

$$|f(t)| \leq \Big(1 - \frac{\mu A_0}{\mu^\star}\Big)\frac{S_2^2}{1 + \|b\|_{L^\infty(\Omega)}}(1 + |\lambda|)^{-1}|t| \text{ for all } |t| > \delta.$$

Integrating the above inequality we have

$$|F(t)| \le \Big(1 - \frac{\mu A_0}{\mu^\star}\Big)\frac{S_2^2}{2(1+\|b\|_{L^\infty(\Omega)})}(1+|\lambda|)^{-1}|t|^2 + \max_{|s|\le\delta}|f(s)||t| \text{ for all } t \in \mathbb{R}.$$

Hence, since $-A_0 \le a(x) \le A_0$ for all $x \in \overline{\Omega}$ and (1.2), it follows from the continuous embeddings and the Hölder inequality that

$$\begin{aligned} J_{\mu,\lambda}(u) &= \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \frac{\mu}{2}\int_\Omega \frac{a(x)}{|x|^4}|u|^2 dx - \lambda\int_\Omega b(x)F(u)dx \\ &\ge \frac{1}{2}\int_\Omega |\Delta u|^2 dx - \frac{\mu A_0}{2}\int_\Omega \frac{|u|^2}{|x|^4}dx - |\lambda|\|b\|_{L^\infty(\Omega)}\int_\Omega |F(u)|dx \\ &\ge \frac{1}{2}\Big(1-\frac{\mu A_0}{\mu^\star}\Big)\int_\Omega |\Delta u|^2 dx - \frac{|\lambda|}{2(1+|\lambda|)}\Big(1-\frac{\mu A_0}{\mu^\star}\Big)S_2^2\int_\Omega |u|^2 dx \\ &\qquad - |\lambda|\|b\|_{L^\infty(\Omega)}\int_\Omega |u|dx \\ &\ge \frac{1}{2(1+|\lambda|)}\Big(1-\frac{\mu A_0}{\mu^\star}\Big)\|u\|_{2,2}^2 - \frac{|\lambda|\|b\|_{L^\infty(\Omega)}}{S_1}(meas(\Omega))^{\frac{1}{2}}\|u\|_{2,2}. \end{aligned}$$

Since $\overline{\mu} = \frac{\mu}{A_0} > 0$, we deduce that for each $\mu \in [0, \overline{\mu})$ and $\lambda \in \mathbb{R}$, the functional $J_{\mu,\lambda}$ is coercive.

Next, let $\{u_m\}$ be a sequence in $W_0^{2,2}(\Omega)$, such that

$$J_{\mu,\lambda}(u_m) \to c < \infty \text{ and } DJ_{\mu,\lambda}(u_m) \to 0 \text{ in } W^{-2,2}(\Omega) \text{ as } m \to \infty, \tag{2.5}$$

where $W^{-2,2}(\Omega)$ is the dual space of $W_0^{2,2}(\Omega)$.

Since $J_{\mu,\lambda}$ is coercive, the sequence $\{u_m\}$ is bounded in $W_0^{2,2}(\Omega)$. Then, there exist a subsequence of $\{u_m\}$, still denoted by $\{u_m\}$, that converges weakly to some $u \in W_0^{2,2}(\Omega)$ and $\{u_m\}$ converges strongly to u in $L^2(\Omega)$. We find that

$$\begin{aligned} \Big(1-\frac{\mu}{\overline{\mu}}\Big)\|u_m-u\|_{2,2}^2 &\le \|u_m-u\|_{2,2}^2 - \mu\int_\Omega a(x)\frac{|u_m-u|^2}{|x|^4}dx \\ &= DJ_{\mu,\lambda}(u_m)(u_m-u) + DJ_{\mu,\lambda}(u)(u-u_m) \\ &\qquad + \lambda\int_\Omega b(x)\Big(f(u_m)-f(u)\Big)(u_m-u)dx. \end{aligned} \tag{2.6}$$

Since $\{u_m\}$ converges weakly to u in $W_0^{2,2}(\Omega)$, $\|u_m-u\|_{2,2}$ is bounded. By (2.5), it implies that

$$\lim_{m\to\infty} DJ_{\mu,\lambda}(u_m)(u_m-u) = 0, \quad \lim_{m\to\infty} DJ_{\mu,\lambda}(u)(u-u_m) = 0. \tag{2.7}$$

On the other hand, by the Hölder inequality,

$$\begin{aligned}
&\Big|\int_\Omega b(x)\Big(f(u_m)-f(u)\Big)(u_m-u)dx\Big| \\
&\quad\le C\|b\|_{L^\infty(\Omega)}\int_\Omega (2+|u_m|+|u|)|u_m-u|dx \qquad (2.8)\\
&\quad\le C\|b\|_{L^\infty(\Omega)}\Big[2\Big(meas(\Omega)\Big)^{\frac{1}{2}}+\|u_m\|_{L^2(\Omega)}+\|u\|_{L^2(\Omega)}\Big]\|u_m-u\|_{L^2(\Omega)},
\end{aligned}$$

which approaches 0 as $m\to\infty$.

By (2.6), (2.7) and (2.8), the sequence $\{u_m\}$ converges strongly to u in $W_0^{2,2}(\Omega)$ and the functional $J_{\mu,\lambda}$ satisfies the Palais-Smale condition. □

Lemma 2.3. *For each $\mu\in[0,\overline{\mu})$ we have*

$$\lim_{\rho\to 0^+}\frac{\sup\{\mathcal{F}(u):\quad \mathcal{A}(u)<\rho\}}{\rho}=0,$$

where the functionals $\mathcal{A}$ and $\mathcal{F}$ are given by (2.2).

Proof. By $(f2)$, for an arbitrary small $\epsilon>0$, there exists $\delta>0$, such that

$$|f(t)|\le\frac{\epsilon}{2}\Big(1-\frac{\mu}{\overline{\mu}}\Big)\frac{S_2^2}{1+\|b\|_{L^\infty(\Omega)}}|t| \text{ for all } |t|<\delta,$$

where $\overline{\mu}$ is defined by Lemma 2.1. Combining the above inequality with the fact that

$$|f(t)|\le C(1+|t|) \text{ for all } t\in\mathbb{R}$$

we get

$$|F(t)|\le\frac{\epsilon}{4}\Big(1-\frac{\mu}{\overline{\mu}}\Big)\frac{S_2^2}{1+\|b\|_{L^\infty(\Omega)}}|t|^2+C_\delta|t|^q \qquad (2.9)$$

for all $t\in\mathbb{R}$, where $q\in\Big(2,\frac{2N}{N-4}\Big)$, and $C_\delta>0$ is a constant that does not depend on t.

Next, for each $\rho>0$, we define the sets

$$B_\rho^1=\Big\{u\in W_0^{2,2}(\Omega):\quad \mathcal{A}(u)<\rho\Big\}$$

and

$$B_\rho^2=\Big\{u\in W_0^{2,2}(\Omega):\quad \Big(1-\frac{\mu}{\overline{\mu}}\Big)\|u\|_{2,2}^2<2\rho\Big\}.$$

By (1.2), we have $B^1_\rho \subset B^2_\rho$. Moreover, using (2.9), it follows that for any $u \in B^2_\rho$,

$$\mathcal{F}(u) \leq \frac{\epsilon}{4}\Big(1-\frac{\mu}{\overline{\mu}}\Big)\|u\|^2_{2,2} + C_\delta S_q^{-q}\|u\|^q_{2,2}. \tag{2.10}$$

Since $0 \in B^1_\rho$ and $I(0)=0$, we have $0 \leq \sup_{u\in B^1_\rho} I(u)$. On the other hand, if $u \in B^2_\rho$, then

$$\|u\|_{2,2} \leq \Big(1-\frac{\mu}{\overline{\mu}}\Big)^{\frac{-1}{2}}(2\rho)^{\frac{1}{2}}.$$

Now, using (2.10), we deduce that

$$\begin{aligned} 0 \leq \frac{\sup_{u\in B^1_\rho}\mathcal{F}(u)}{\rho} &\leq \frac{\sup_{u\in B^2_\rho}\mathcal{F}(u)}{\rho} \\ &\leq \frac{\epsilon}{2} + C_\delta S_q^{-q}\Big(1-\frac{\mu}{\overline{\mu}}\Big)^{\frac{-q}{2}}(2\rho)^{\frac{q}{2}-1}. \end{aligned} \tag{2.11}$$

Since $q>2$, letting $\rho \to 0^+$, because $\epsilon > 0$ is arbitrary, we get the conclusion. □

Proof of Theorem 1.2. In order to prove Theorem 1.2, we shall apply Lemma 1.3 by choosing $X = W^{2,2}_0(\Omega)$ as well as $\mathcal{A}$ and $\mathcal{F}$ as in (2.2). Now, we shall check all assumptions of Lemma 1.3. Indeed, we have $\mathcal{A}(0)=\mathcal{F}(0)=0$ and since $-A_0 \leq a(x) \leq A_0$ for all $x \in \overline{\Omega}$, we deduce from (1.2) that for any $\mu < \overline{\mu}$, $\mathcal{A}(u) \geq 0$ for any $u \in W^{2,2}_0(\Omega)$.

Let $t_0 \in \mathbb{R}$ as in $(f3)$, i.e. $F(t_0) > 0$. For $\sigma \in (0,1)$, we define the function u_σ by

$$u_\sigma(x) = \begin{cases} 0, & \text{for } x \in \mathbb{R}^N \backslash B_{R_0}(0), \\ t_0, & \text{for } x \in B_{\sigma R_0}(0), \\ \frac{t_0}{2}\sin\Big[\frac{\pi}{(1-\sigma)R_0}\Big(\frac{1+\sigma}{2}R_0 - |x|\Big)\Big] + \frac{t_0}{2} & \text{for } x \in B_{R_0}(0)\backslash B_{\sigma R_0}(0), \end{cases}$$

where $B_r(0)$ denotes the N-dimensional open ball with center 0 and radius $r>0$, R_0 is given by (1.7), and $|.|$ denotes the usual Euclidean norm in $\mathbb{R}^N$. Since $u_\sigma \in C^1(\Omega)\cap C^2(\Omega\backslash\{x \in B_{R_0}(0) : |x| = \sigma R_0 \text{ and } |x| = R_0\})$ and $u_\sigma = |\nabla u_\sigma| = 0$ for all $|x| \geq R_0$ we have $u_\sigma \in W^{2,2}_0(\Omega)$ and $|u_\sigma(x)| \leq |t_0|$ for all $x \in \mathbb{R}^N$. From the definition of u_σ, a simple computation shows that

$$\begin{aligned} \mathcal{F}(u_\sigma) &= \int_{B_{\sigma R_0}(0)} b(x)F(u_\sigma)dx + \int_{B_{R_0}\backslash B_{\sigma R_0}(0)} b(x)F(u_\sigma)dx \\ &\geq \Big[b_{R_0}F(t_0)\sigma^N - \max_{|t|\leq R_0}|F(t)|(1-\sigma)^N\|b\|_{L^\infty(\Omega)}\Big]R_0^N\omega_N, \end{aligned}$$

where ω_N is the volume of the unit ball $B_1(0)$. If we choose $\sigma \in (0,1)$ close enough to 1, says σ_0, then the right-hand side of the last inequality becomes strictly positive. By Lemma 2.3, we can choose $\rho_0 \in (0,1)$ such that $\rho_0 < \mathcal{A}(u_{\sigma_0})$ and

$$\frac{\sup_{\mathcal{A}(u)<\rho_{\sigma_0}} \mathcal{F}(u)}{\rho_0} < \frac{\left[b_{R_0} F(t_0)\sigma_0^N - \max_{|t|\le R_0} |F(t)|(1-\sigma_0)^N \|b\|_{L^\infty(\Omega)}\right] R_0^N \omega_N}{2\mathcal{A}(u_{\sigma_0})} < \frac{\mathcal{F}(u_{\sigma_0})}{\mathcal{A}(u_{\sigma_0})}.$$

Now, in Lemma 1.3, we choose $x_0 = 0$, $x_1 = u_{\sigma_0}$, $\xi = 1 + \rho_0$ and

$$\overline{a} = a_\mu = \frac{1+\rho_0}{\frac{\mathcal{F}(u_{\delta_0})}{\mathcal{A}(u_{\delta_0})} - \frac{\sup_{\mathcal{A}(u)<\rho_{\sigma_0}} \mathcal{F}(u)}{\rho_0}} > 0.$$

For any $\mu \in [0, \overline{\mu})$, taking into account the above lemmas, all assumptions of Lemma 1.3 are verified. Then there exist an open interval $\Lambda_{\overline{\mu}} \subset [0, \overline{a}]$ and a number $\delta_{\overline{\mu}}$, such that for each $\lambda \in \Lambda_{\overline{\mu}}$, the equation $D\mathcal{A}(u) - \lambda D\mathcal{F}(u) = 0$ has at least three solutions in $W_0^{2,2}(\Omega)$ whose $W_0^{2,2}(\Omega)$-norms are less than $\delta_{\overline{\mu}}$. By $(f2)$, $f(0) = 0$, one of them may be the trivial one, so problem (1.5) has at least two non-trivial weak solutions with the required properties. □

Acknowledgments

This work was supported by Vietnam National Foundation for Science and Technology Development (NAFOSTED).

References

[1] A. E. Amrouss, F. Moradi and M. Moussaoui, *Existence of solutions for fourth-order PDEs with variable exponents*, Electronic J. Diff. Equ., **Vol. 2009**(153) (2009), 1-13.

[2] A. Ayoujil and A.R. El Amrouss, *On the spectrum of a fourth order elliptic equation with variable*, Nonlinear Anal., **71** (2009), 4916-4926.

[3] G. Bonanno, *Some remarks on a three critical points theorem*, Nonlinear Anal., **54** (2003), 651-665.

[4] Y. Deng and H. Pi, *Multiple solutions for p-harmonic type equations*, Nonlinear Anal., **71** (2009), 4952-4959.

[5] P. Drábek and M. Ôtani, *Global Bifurcation Results for the p-Biharmonic Operator*, Electronic J. Diff. Equ., **Vol. 2001**(48) (2001), 1-19.

[6] A. Kristály and C. Varga, *Multiple solutions for elliptic problems with singular and sublinear potentials*, Proc. Amer. Math. Soc., **135**(7) (2007), 2121-2126.

[7] L. Li and C.L. Tang, Three solutions for a Navier boundary value problem involving the p-biharmonic, *Nonlinear Anal.*, **72** (2010), 1339-1347.

[8] E. Montefusco, *Lower semicontinuity of functionals via concentration-compactness principle*, J. Math. Anal. Appl., **263** (2001), 264-276.

[9] M. Talbi and N. Tsouli, *On the spectrum of the weighted p-biharmonic operator with weight*, Mediterr. J. Math., **4** (2007), 73-86.

[10] Y. Wang and Y. Shen, *Multiple and sign-changing solutions for a class of semilinear biharmonic equation*, J. Diff. Equ., **246** (2009), 3109-3125.

[11] W. Wang and P. Zhao, *Non-uniformly nonlinear elliptic equations of p-biharmonic type*, J. Math. Anal. Appl., **348** (2008), 730-738.

[12] Y. Yao, Y. Shen and Z. Chen, *Biharmonic Equation and an Improved Hardy Inequality*, Acta Mathematicae Applicatae Sinica, **20**(3) (2004), 433-440.

[13] Y. Yao, R. Wang and Y. Shen, *Nontrivial solution for a class of semilinear biharmonic equation involving critical exponents*, Acta Mathematica Scientia, **27B**(3) (2007), 509-514.

Nguyen Thanh Chung
Department of Mathematics and Informatics
Quang Binh University
312 Ly Thuong Kiet, Dong Hoi, Quang Binh, Vietnam
email: *ntchung82@yahoo.com*