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RADIUS OF STARLIKE AND PARTIAL SUM PROPERTY FOR HOLOMORPHIC FUNCTIONS DEFINED BY KOMATU OPERATOR

A. Tehranchi, A. Mousavi and N. Vezvaei

Abstract. In this paper we investigate some important properties of a holomorphic functions with negative coefficients by using Komatu operator. We provide necessary and sufficient conditions, radius of starlikeness, convexity and close-to-convexity for this class.

Key Words: Holomorphic, Convex, Starlike functions, Komatu operator.

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1. Introduction and Definitions

Let $\mathcal{B}$ denotes the class of functions analytic in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and let τ denotes the subclass of $\mathcal{B}$ consisting holomorphic functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

which are analytic in the unit disc Δ.

Definition 1.1. *The operator k_c^δ is the komatu operator ([2],[5]) defined by*

$$k_c^\delta = \int_0^1 \frac{(c+1)^\delta}{\Gamma(\delta)} t^c \left(\log \frac{1}{t}\right)^{\delta-1} \frac{f(tz)}{t} \mathrm{d}t.$$

By applying a simple calculation for $f \in \tau$ we get

$$k_c^\delta = z - \sum_{k=2}^{\infty} \left(\frac{c+1}{c+k}\right)^\delta a_k z^k. \tag{2}$$

From now on in this paper let

$$\xi_k(c,\delta) = \left(\frac{c+1}{c+k}\right)^\delta \Rightarrow k_x^\delta = z - \sum_{k=2}^{\infty} \xi_k(c,\delta) a_k z^k, \tag{3}$$

Definition 1.2. *A function $f(z)$ in τ is said to be in class of $\tau(\alpha,\beta,c,\delta)$ if*

$$Re\left\{\frac{k_c^\delta(f)}{z\left[k_c^\delta(f)\right]'}\right\} > \left|\frac{k_c^\delta(f)}{z\left[k_c^\delta(f)\right]'} - 1\right| + \beta, \tag{4}$$

where $0 \le \alpha < 1$, $0 \le \beta < 1$, $c \ge -1$ and $\delta > 0$.

Definition 1.3. *A function $f(z) \in \mathcal{B}$ is said to be convex of order $\mu (0 \le \mu < 1)$ if and only if $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \mu$, $z \in \Delta$ (see [4]).*

A function $f(z) \in \mathcal{B}$ is said to be starlike of order $\mu (0 \le \mu < 1)$ if and only if $Re\left\{1 + \frac{zf'(z)}{f'(z)}\right\} > \mu$, $z \in \Delta$ (see [1], [4]).

The family $\tau(\alpha,\beta,c,\delta)$ is a special interest for it contains many well-known as well as new classes of analytic univalent functions. This family is reviewed by Sh. Najafzadeh and A. Ebadian in [3], and also A. Tehranchi and S.R. Kulkarni in [5], [6].

2. A Necessary and Sufficient Conditions for f to belong to $\tau(\alpha,\beta,c,\delta)$

The following theorem gives a necessary and sufficient condition for a function to be in $\tau(\alpha,\beta,c,\delta)$. Before proving the theorem we need the following lemma.

Lemma 2.1. *Let $0 \le \alpha < 1$, $0 \le \beta < 1$ and $\gamma \in \mathbb{R}$. Then $Re(w) > \alpha|w-1| + \beta$ if and only if*

$$Re[w(1+\alpha e^{i\gamma}) - \alpha^{i\gamma}] > \beta. \tag{5}$$

Lemma 2.2. *Let $0 \leq \beta < 1$ and $w \in \mathbb{C}$. Then $Re(w) > \beta$ if and only if* $|w - (1+\beta)| < |w + (1-\beta)|$.

There is a mistake in the proof of Theorem 2.2 in [3], which is corrected as in the following:

Theorem 2.3. *Let $f \in \mathcal{B}$. Then $f(z) \in \tau(\alpha, \beta, c, \delta)$ if and only if*

$$\sum_{k=2}^{\infty} \frac{[(1+\alpha) - k(\alpha+\beta)]}{1-\beta} \xi_k(c,\delta) a_k < 1. \tag{6}$$

Proof. Let us assume that $f(z) \in \tau(\alpha, \beta, c, \delta)$. So by Lemma 2.1 and letting $w = \frac{k_c^\delta(f)}{z[k_c^\delta(f)]'}$ in (4) we obtain

$$Re[w(1+\alpha e^{i\gamma}) - \alpha e^{i\gamma}] > \beta.$$

So

$$Re\left[\frac{z - \sum_{k=1}^{\infty} \xi_k(c,\delta) a_k z^k}{z\left(1 - \sum_{k=2}^{\infty} k\xi_k(c,\delta) a_k z^{k-1}\right)}(1+\alpha e^{i\gamma}) - \alpha e^{i\gamma} - \beta\right] > 0$$

then

$$Re\left[\frac{1 - \beta - \sum_{k=2}^{\infty}(1-\beta k)\xi_k(c,\delta) a_k z^{k-1} - \alpha e^{i\gamma}\sum_{k=2}^{\infty}(1-k)\xi_k(c,\delta) a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} k\xi_k(c,\delta) a_k z^{k-1}}\right].$$

The above inequality must hold for all z in Δ. Letting $z = re^{-i\theta}$ where $0 \leq r < 1$ we obtain

$$Re\left[\frac{1 - \beta - \sum_{k=2}^{\infty}(1-\beta k) + \alpha e^{i\gamma}(1-k)\xi_k(c,\delta) a_k r^{k-1}}{1 - \sum_{k=2}^{\infty} k\xi_k(c,\delta) a_k r^{k-1}}\right] > 0.$$

By letting $r \to 1$ through half line $z = re^{-i\theta}$ and the mean value theorem we have

$$Re\left[(1-\beta) - \sum_{k=2}^{\infty}[(1-\beta k) + \alpha(1-k)]\xi_k(c,\delta) a_k r^{k-1}\right] > 0,$$

so we get

$$\sum_{k=2}^{\infty} \frac{[(1+\alpha)-k(\alpha+\beta)]}{1-\beta}\xi_k(c,\delta)a_k < 1.$$

Converesely, let (6) holds. We will show that (4) is satisfied and so $f(z) \in \tau(\alpha,\beta,c,\delta)$. By Lemma 2.2 it is enough to show that

$$|w(1+\alpha|w1|+\beta)| < |w+(1\alpha|w1|\beta)|,$$

If

$$\begin{aligned} R &= |w+1-\beta-\alpha|w-1|| \\ &= \frac{1}{|z[k_c^{\delta}(f)]'|}\left|2z-\beta z-\sum_{k=2}^{\infty}[1+(1-\beta)+\alpha-\alpha k]\xi_k(c,\delta)a_k z^k\right|. \end{aligned}$$

This implies that

$$R > \frac{|z|}{|z[k_c^{\delta}(f)]'|}\left[2-\beta-\sum_{k=2}^{\infty}[k+(1+\alpha)-k(\alpha+\beta)]\xi_k(c,\delta)a_k\right].$$

Similarly, if $L = |w-1-\beta-\alpha|w-1||$ we get

$$L < \frac{|z|}{|z[k_c^{\delta}(f)]'|}\left[\beta+\sum_{k=2}^{\infty}[-K+(1+\alpha)-k(\alpha+\beta)]\xi_k(c,\delta)a_k\right].$$

It is easy to verify that $R-L>0$ and so the proof is completed. □

Corollary 2.4. *Let $f \in \tau(\alpha,\beta,c,\delta)$ then*

$$a_k < \frac{1-\beta}{[(1+\alpha)-k(\alpha+\beta)]\xi_k(c,\delta)}, \quad n = 2,3,4,\cdots.$$

Theorem 2.5. *if $c_1 < c_2$, then $\tau(\alpha,\beta,c_2,\delta) \subset \tau(\alpha,\beta,c_1,\delta)$.*
Proof. Let $f(z) \in \tau(\alpha,\beta,c_2,\delta)$. Then we have

$$\sum_{k=2}^{\infty} \frac{[(1+\alpha)-k(\alpha+\beta)]}{1-\beta}\xi_k(c_2,\delta)a_k < 1.$$

But $\xi_k(c,\delta)$ is an increasing function of c, so $\xi_k(c_1,\delta) < \xi_k(c_2,\delta)$, and hence we have

$$\sum_{k=2}^{\infty}\frac{[(1+\alpha)-k(\alpha+\beta)]}{1-\beta}\xi_k(c_1,\delta)a_k < \sum_{n=4}^{\infty}\frac{[(1+\alpha)-k(\alpha+\beta)]}{1-\beta}\xi_k(c_2,\delta)a_k < 1,$$

therefore $f(z) \in \tau(\alpha,\beta,c_1,\delta)$.

Theorem 2.6. *(Growth Theorem) If $f(z) \in \tau(\alpha,\beta,c,\delta)$, then*

$$|z| - \frac{(1-\beta)}{(1-\beta)-(\alpha+\beta)}|z|^2 \le |k_c^{\delta}(f)| \le |z| + \frac{(1-\beta)}{(1-\beta)-(\alpha+\beta)}|z|^2. \qquad (7)$$

Proof. Let $f(z) \in \tau(\alpha,\beta,c,\delta)$. In view of Theorem 2.3 we have

$$\sum_{k=2}^{\infty} a_k\xi_k(c,\delta) < \frac{1-\beta}{(1-\beta)-(\alpha+\beta)}.$$

Therefore

$$\begin{aligned}|k_c^{\delta}(f)| &\le |z| + \sum_{k=2}^{\infty} a_k\xi_k(c,\delta)|z|^k \\ &\le |z| + |z|^2\sum_{k=2}^{\infty} a_k\xi_k(c,\delta) \\ &< |z| + \frac{1-\beta}{(1-\beta)-(\alpha+\beta)}|z|^2.\end{aligned}$$

and

$$\begin{aligned}|k_c^{\delta}(f)| &\ge |z| - \sum_{k=2}^{\infty} a_k\xi_k(c,\delta)|z|^k \\ &\ge |z| - |z|^2\sum_{k=2}^{\infty} a_k\xi_k(c,\delta) \\ &< |z| - \frac{1-\beta}{(1-\beta)-(\alpha+\beta)}|z|^2.\end{aligned}$$

3.Radius of Starlikeness, Convexity and Close-toconvex

In this section we will calculate Radius of Starlikeness, Convexity and Close-to-convexity for the class $\tau(\alpha,\beta,c,\delta)$.

Theorem 3.1. *Let $f \in \tau(\alpha, \beta, c, \delta)$. Then $f(z)$ is starlike of order $\mu (0 \leq \mu < 1)$ in $|z| < r = r_1(\alpha, \beta, c, \delta, \mu)$ where*

$$r_1(\alpha, \beta, c, \delta, \mu) = \inf_k \left[\frac{(1-\mu)[(1+\alpha) - k(\alpha+\beta)]}{(k-\mu)(1-\beta)} \xi_k(c, \delta) \right]^{\frac{1}{k-1}}. \tag{8}$$

Proof. For $0 \leq \mu < 1$ we need to show that $\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \mu$.
We have to show that

$$\begin{aligned} \left| \frac{zf'(z) - f(z)}{f(z)} \right| &= \left| \frac{-\sum_{k=2}^{\infty} a_k z^{k-1}(k-1)}{1 - \sum_{k=2}^{\infty} a_k z^{k-1}} \right| \\ &\leq \frac{\sum_{k=2}^{\infty} a_k |z|^{k-1}(k-1)}{1 - \sum_{k=2}^{\infty} a_k |z|^{k-1}} < 1 - \mu, \\ &\Rightarrow \sum_{k=2}^{\infty} a_k |z|^{k-1} \left(\frac{k-\mu}{1-\mu} \right) < 1. \end{aligned}$$

By Theorem 2.3, it is enough to consider

$$|z|^{k-1} < \frac{(1-\mu)[(1+\alpha) - k(\alpha+\beta)]}{(k-\mu)(1-\beta)} \xi_k(c, \delta).$$

This completes the proof. □

Theorem 3.2. *Let $f \in \tau(\alpha, \beta, c, \delta)$. Then $f(z)$ is convex of order $\mu (0 < \mu < 1)$ in $|z| < r = r_2(\alpha, \beta, c, \delta, \mu)$ where*

$$r_2(\alpha, \beta, c, \delta, \mu) = \inf_k \left[\frac{(1-\mu)[(1+\alpha) - k(\alpha+\beta)]}{k(k-\mu)(1-\beta)} \xi_k(c, \delta) \right]^{\frac{1}{k-1}}. \tag{9}$$

Proof. We show that $\left|\frac{zf''(z)}{f'(z)}\right| < 1-\mu$,

$$\text{i.e.} \quad \left|\frac{-\sum_{k=2}^{\infty} k(k-1)a_k z^{k-1}}{1-\sum_{k=2}^{\infty} k a_k z^{k-1}}\right| \leq \frac{\sum_{k=2}^{\infty} k(k-1)a_k |z|^{k-1}}{1-\sum_{k=2}^{\infty} k a_k |z|^{k-1}} < 1-\mu,$$

$$\Rightarrow \sum_{k=2}^{\infty} a_k |z|^{k-1} k \left(\frac{k-\mu}{1-\mu}\right) < 1.$$

By Theorem 2.3, it is enough letting

$$|z|^{k-1} \leq \frac{(1-\mu)[(1+\alpha)-k(\alpha+\beta)]}{k(k-\mu)(1-\beta)} \xi_k(c,\delta).$$

This completes the proof. □

Theorem 3.3. *if* $f(z) = z - \sum_{k=2}^{\infty} a_k z^k \in \tau(\alpha,\beta,c,\delta)$, *then* $f(z)$ *is close-to-convex of order* $\mu(0 \leq \mu < 1)$ *in* $|z| < r = r_3(\alpha,\beta,c,\delta,\mu)$ *where*

$$r_3(\alpha,\beta,c,\delta,\mu) = \inf_k \left[\frac{(1-\mu)[(1++\alpha)-k(\alpha+\beta)]}{k(1-\beta)} \xi_k(c,\delta)\right]^{\frac{1}{k-1}}. \tag{10}$$

Proof. We must show that $|f'(z)-1| \leq 1-\mu$ for $|z| < r = r_3(\alpha,\beta,c,\delta,\mu)$ when $r_3(\alpha,\beta,c,\delta,\mu)$ is given by (10). Now

$$|f'(z)-1| = \left|\sum_{k=2}^{\infty} k a_k z^{k-1}\right| \leq \sum_{n=2}^{\infty} k a_k |z|^{k-1} \leq 1-\mu$$
$$\Rightarrow \sum_{k=2}^{\infty} \frac{k a_k}{1-\mu} |z|^{k-1} < 1.$$

By Theorem 2.3, above inequality holds true if

$$|z|^{k-1} < \frac{(1-\mu)[(1+\alpha)-k(\alpha+\beta)]}{k(1-\beta)} \xi_k(c,\delta).$$

This completes the proof. □

4 Partial Sum property of $\tau(\alpha,\beta,c,\delta)$

Theorem 4.1. *The* $\tau(\alpha,\beta,c,\delta)$ *is convex set.*

Proof. Let $f(z)$ and $g(z)$ be the arbitrary elements of $\tau(\alpha, \beta, c, \delta$. Then for every $t(0 < t < 1)$ we show that $(1-t)f(z) + tg(z) \in \Omega(\alpha, \beta, \lambda)$. Thus, we have

$$(1-t)f(z) + tg(z) = z\sum_{k=2}^{\infty}[(1-t)a_k + tb_k]z^k$$

and hence

$$\sum_{k=2}^{\infty}\frac{[(1+\alpha)-k(\alpha+\beta)]}{1-\beta}\xi_k(c,\delta)[(1-t)a_k + tb_k] =$$

$$(1-t)\sum_{k=2}^{\infty}\frac{[(1+\alpha)-k(\alpha+\beta)]}{1-\beta}\xi_k(c,\delta)a_k + t\sum_{k=2}^{\infty}\frac{[(1+\alpha)-k(\alpha+\beta)]}{1-\beta}\xi_k(c,\delta)b_k < 1.$$

Corollary 4.2. *Suppose the $f(z)$ and $g(z)$ belong to $\tau(\alpha, \beta, c, \delta$. Then the function $h(z)$ defined by $h(z) = \frac{1}{2}(f(z) + g(z))$ also belongs to $\tau(\alpha, \beta, c, \delta)$.*

We say that g is subordinate of f denoted by $g \prec f$, if $g(z) = f(w(z))$, where w is an analytic Schwarz function with $w(0) = 0$, $|w(z)| \leq 1$.

Theorem 4.3. *Let $f(z) \in \tau(\alpha, \beta, c, \delta)$ and $g(z)$ be an arbitrary element of B, such that $g \prec f$, g is subordinate to f;*
and if

$$g_k = \frac{1}{k!}\left[\frac{d^k(f(w(z)))}{dz^k}\right]_{z=0}, \tag{11}$$

also if

$$\frac{\sum_{k=2}^{\infty}[(1+\alpha)-k(\alpha+\beta)]|g_k|}{|g_1|}\xi_k(c,\delta) < (1-\beta), \tag{12}$$

then $g \in \tau(\alpha, \beta, c, \delta)$.

Proof. Since $g \prec f$, by definition, there is an analytic function $w(z)$ such that $|w(z)| \leq |z|$ and $g(z) = f(w(z))$. But g is the composition of two analytic functions in the unit disk, therefore we can expand this function in terms of Taylor series at origin as below:

$$g(z) = \sum_{n=0}^{\infty} g_n z^n,$$

where gn is defined in (11). Hence

$$g_0 = \frac{f(w(0))}{0!} = 0, \quad g_1 = \frac{w'(0)f'(w(0))}{1!} = w'(0).$$

Therefore, we can write

$$g(z) = g_1(z) + \sum_{k=2}^{\infty} g_k z^k,$$

and

$$k_c^{\delta}(g(z)) = g_1(z) - \sum_{k=2}^{\infty} \xi_k(c,\delta) g_k z^k.$$

We must prove $g(z) \in \tau(\alpha, \beta, c, \delta)$ or

$$\sum_{k=2}^{\infty} \frac{[(1+\alpha) - k(\alpha+\beta)]}{(1-\beta)} \xi_k(c,\delta) g_k < 1.$$

By Theorem 2.3 we have

$$Re\left[(1-\beta)g_1 - \sum_{k=2}^{\infty}[(1-\beta k) + \alpha(1-k)]\xi_k(c,\delta) g_k r^{k-1}\right] > 0.$$

Letting $r \to 1$ and by (12) the last inequality is true and the result can be obtained.

□

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Abdolreza Tehranchi*, Ahmad Mousavi**
Department of Mathematics, Islamic Azad University, South Tehran Branch, Tehran, Iran.
* E-mail: *Tehranchi@azad.ac.ir, Tehranchiab@gmail.com*
**E-mail: *a_mousavi@azad.ac.ir moussavi.a@gmail.com*

Nazanin Vezvaei
Department of Pure Mathematics, Faculty of Mathematical Sciences, Tarbiat Modares University, P.O.Box:14115-134, Tehran, Iran.
E-mail: *nvezvaei@yahoo.com*