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CONVEXITY PROPERTIES FOR AN INTEGRAL OPERATOR

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Abstract. In this paper we obtain the order of convexity for an integral operator in the classes $\mathcal{B}(\mu,\alpha), \mathcal{N}(\beta), \beta-\mathcal{UCV}(\alpha)$ and $\beta-\mathrm{S}_p(\alpha)$.

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1. Introduction and Definitions

Let $\mathcal{U}=\{z:|z|<1\}$ the unit disk and $\mathcal{A}$ the class of all functions of the form:

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1}$$

which are analytic in $\mathcal{U}$ and satisfy the condition

$$f(0)=f'(0)-1=0.$$

We note by $\mathcal{S}$ the class of univalent and regular functions.
A functions $f(z)$ from the class $\mathcal{S}$ is said to be convex of order α if it satisfies

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha,\quad (z\in\mathcal{U})$$

We denote by $\mathcal{K}(\alpha)$ the subclass of $\mathcal{S}$ that consists all the functions that are convex of order α in $\mathcal{U}$.
The family $\mathcal{B}(\mu,\alpha), \mu\geq 0, 0\leq\alpha<1$ was defined by Frasin and Jahangari in [3]. This consists functions $f\in\mathcal{S}$ that satisfy the condition

$$\left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu}-1\right|<1-\alpha\quad (z\in\mathcal{U})$$

$\mathcal{B}(\mu,\alpha)$ is a comprehensive class of analytic functions that includes various new classes of analytic functions. Frasin and Darus in [2] introduce the special class $\mathcal{B}(2,\alpha)\equiv\mathcal{B}(\alpha)$.

Other classes of univalent analytic functions are $\mathcal{B}(1,\alpha) \equiv \mathcal{S}^*(\alpha)$ and $\mathcal{B}(0,\alpha) \equiv \mathcal{R}(\alpha)$.
$\mathcal{N}(\beta)$ is a subclass of $\mathcal{A}$ that consists all the functions $f(z)$, which satisfy the inequality

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) < \beta.$$

This class was studied by Uralegaddi et al. in [7] and Owa and Srivastava in [5].
M. Darus studied the classes $\beta - \mathcal{UCV}(\alpha)$ and $\beta - \mathrm{S}_p(\alpha)$ in [1].
We say that a function $f \in \mathcal{S}$ is in the class $\beta - \mathcal{UCV}(\alpha)$ that is the class of $\beta-$uniformly convex functions of order α if

$$\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}-\alpha\right) \geq \beta\left|\frac{zf''(z)}{f'(z)}-1\right|$$

for $-1 \leq \alpha \leq 1, \beta > 0$ and $z \in \mathcal{U}$.
A function $f \in \mathcal{S}$ is in the class $\beta - \mathrm{S}_p(\alpha)$ that is the class of $\beta-$starlike functions of order α if

$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}-\alpha\right) \geq \beta\left|\frac{zf'(z)}{f(z)}-1\right|$$

for $-1 \leq \alpha \leq 1, \beta > 0$ and $z \in \mathcal{U}$.
We consider the integral operator

$$K(z) = \int_0^z \prod_{i=1}^n \left(\frac{f_i(t)}{t}\right)^{\gamma_i} \cdot (g_i'(t))^{\eta_i} dt \tag{2}$$

The operator was developed from the operator introduced and studied by Pescar in [6]. To prove our main results we use the following lemma:

Lemma 0.1. *(General Schwarz-Lemma)[4] Let f the function regular in the disk $\mathcal{U}_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$, M fixed. If f has in $z = 0$ one zero with multiply $\geq m$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m, \ z \in \mathcal{U}_R \tag{3}$$

the equality (in the inequality (3) for $z \neq 0$) can hold only if $f(z) = e^{i\theta}\frac{M}{R^m}z^m$, where θ is constant.

2. Main results

Theorem 0.1. *Let $f_i, g_i \in \mathcal{A}$ with f_i belongs to the class $\mathcal{B}(\mu,\alpha), \mu \geq 0, 0 \leq \alpha < 1$ and g_i in the class $\mathcal{N}(\beta_i)$ for $i = \overline{1,n}$. If $M_i \geq 1, |f_i(z)| \leq M_i$ for $i = \overline{1,n}$, then the integral operator $K(z) \in \mathcal{K}(\delta)$, where*

$$\delta = 1 - \left(\sum_{i=1}^{n} |\gamma_i|((2-\alpha)M_i^{\mu-1} - 1) + \sum_{i=1}^{n} |\eta_i|(\beta_i - 1) \right)$$

and $\sum_{i=1}^{n} |\gamma_i|((2-\alpha)M_i^{\mu-1} - 1) + \sum_{i=1}^{n} |\eta_i|(\beta_i - 1) < 1, \alpha_i, \beta_i, \gamma_i, \eta_i \in \mathbb{C} - \{0\}$.

Proof. From (2) we obtain that

$$\frac{zK''(z)}{K'(z)} = \sum_{i=1}^{n} \left[\gamma_i \left(\frac{zf_i'(z)}{f_i(z)} - 1 \right) \right] + \sum_{i=1}^{n} \left[\eta_i \frac{zg_i''(z)}{g_i'(z)} \right] \tag{4}$$

Thus implies that:

$$\left| \frac{zK''(z)}{K'(z)} \right| \leq \sum_{i=1}^{n} \left(|\gamma_i| \left| \frac{zf_i'(z)}{f_i(z)} - 1 \right| \right) + \sum_{i=1}^{n} |\eta_i| \left| \frac{zg_i''(z)}{g_i'(z)} \right| \tag{5}$$

Since $|f_i(z)| \leq M_i$, applying General Schwarz-Lemma we obtain that $\left| \frac{f_i(z)}{z} \right| \leq M_i$, for $z \in \mathcal{U}, i = \overline{1,n}$.
Using the hypothesis $f_i \in \mathcal{B}(\mu,\alpha), g_i \in \mathcal{N}(\beta)$ and from (4) we obtain:

$$\begin{aligned}
\left| \frac{zK''(z)}{K'(z)} \right| &\leq \sum_{i=1}^{n} |\gamma_i| \left(\left| f_i'(z) \left(\frac{z}{f_i(z)} \right)^{\mu} \right| \left| \frac{f_i(z)}{z} \right|^{\mu-1} - 1 \right) + \sum_{i=1}^{n} |\eta_i| \left(\left| \frac{zg_i''(z)}{g_i'(z)} + 1 \right| - 1 \right) \\
&\leq \sum_{i=1}^{n} |\gamma_i| \left[\left(\left| f_i'(z) \left(\frac{z}{f_i(z)} \right)^{\mu} - 1 \right| + 1 \right) M_i^{\mu-1} - 1 \right] + \sum_{i=1}^{n} |\eta_i|(\beta_i - 1) \\
&\leq \sum_{i=1}^{n} |\gamma_i|[(2-\alpha)M_i^{\mu-1} - 1] + \sum_{i=1}^{n} |\eta_i|(\beta_i - 1) \\
&= 1 - \delta
\end{aligned}$$

By the above inequalities we obtain that $K(z) \in \mathcal{K}(\delta)$. □

For $n = 1$ in Theorem 0.1 we obtain:

Corollary 0.1. *Let $f, g \in \mathcal{A}$ with f in the class $\mathcal{B}(\mu,\alpha), \mu \geq 0, 0 \leq \alpha < 1$ and g in the class $\mathcal{N}(\beta)$. If $M \geq 1, |f(z)| \leq M$, then the integral operator $K(z) = \int\limits_0^z \left(\frac{f(t)}{t}\right)^\gamma \cdot (g'(t))^\eta \in \mathcal{K}(\phi) dt$, where*

$$\phi = 1 - \left(|\gamma|((2-\alpha)M^{\mu-1} - 1) + |\eta|(\beta - 1)\right)$$

and $|\gamma|((2-\alpha)M^{\mu-1} - 1) + |\eta|(\beta-1) < 1, \alpha, \beta \in \mathbb{C} - \{0\}$.

For $\mu = 0$ and $M_1 = M_2 = \cdots = M_n = M$ we obtain:

Corollary 0.2. *Let $f_i, g_i \in \mathcal{A}$ with f_i belongs to the class $\mathcal{R}(\alpha), 0 \leq \alpha < 1$ and g_i in the class $\mathcal{N}(\beta_i)$ for $i = \overline{1,n}$. If $M \geq 1, |f_i(z)| \leq M$ for $i = \overline{1,n}$, then the integral operator $K(z) \in \mathcal{K}(\delta)$, where*

$$\delta = 1 - \left(\sum_{i=1}^n |\gamma_i|((2-\alpha)\frac{1}{M} - 1) + \sum_{i=1}^n |\eta_i|(\beta_i - 1)\right)$$

and $\sum\limits_{i=1}^n |\gamma_i|((2-\alpha)\frac{1}{M} - 1) + \sum\limits_{i=1}^n |\eta_i|(\beta_i - 1) < 1, \alpha_i, \beta_i, \eta_i, \gamma_i \in \mathbb{C} - \{0\}$.

If we put $\mu = 1$ in Theorem 0.1 we obtain:

Corollary 0.3. *Let $f_i, g_i \in \mathcal{A}$ with f_i belongs to the class $\mathcal{S}^*(\alpha), 0 \leq \alpha < 1$ and g_i in the class $\mathcal{N}(\beta_i)$ for $i = \overline{1,n}$. If $|f_i(z)| \leq M (M \geq 1)$, then the integral operator $K(z) \in \mathcal{K}(\delta)$, unde*

$$\delta = 1 - \left(\sum_{i=1}^n |\gamma_i|(1-\alpha) + \sum_{i=1}^n \eta_i(\beta_i - 1)\right)$$

and $\sum\limits_{i=1}^n |\gamma_i|(1-\alpha) + \sum\limits_{i=1}^n \eta_i(\beta_i - 1) < 1, \alpha_i, \beta_i \in \mathbb{C} - \{0\}$.

Theorem 0.2. *If $f_i \in \beta_i - \mathrm{S}_p(\alpha_i), -1 \leq \alpha_i \leq 1, \beta_i > 0, \sum\limits_{i=1}^n \gamma_i \leq \frac{1}{2}$ and $g_i \in \beta_i - \mathcal{UCV}(\alpha_i), -1 \leq \alpha_i \leq 1, \beta_i > 0, \sum\limits_{i=1}^n \eta_i \leq \frac{1}{2}$ for $i = \overline{1,n}$, then $K(z) \in \mathcal{K}(\rho)$, where*

$$\rho = 1 + \sum_{i=1}^n (\alpha_i - 1)(\gamma_i + \eta_i).$$

Proof. Using (4) we have

$$\begin{aligned}\mathrm{Re}\frac{zK''(z)}{K'(z)} &= \mathrm{Re}\gamma_1\frac{zf_1'(z)}{f_1(z)} - \gamma_1 + \cdots + \mathrm{Re}\gamma_n\frac{zf_n'(z)}{f_n(z)} - \gamma_n + \mathrm{Re}\eta_1\frac{zg_1''(z)}{g_1'(z)} + \cdots + \mathrm{Re}\eta_n\frac{zg_n''(z)}{g_n'(z)} \\ &= \mathrm{Re}\gamma_1\left(\frac{zf_1'(z)}{f_1(z)} - \alpha_1\right) + (\gamma_1\alpha_1 - \gamma_1) + \cdots + \mathrm{Re}\gamma_n\left(\frac{zf_n''(z)}{f_n(z)} - \alpha_n\right) + \\ &+ (\gamma_n\alpha_n - \gamma_n) + \mathrm{Re}\eta_1\left(1 + \frac{zg_1''(z)}{g_1'(z)} - \alpha_1\right) - \eta_1 + \eta_1\alpha_1 + \cdots + \\ &+ \mathrm{Re}\eta_n\left(1 + \frac{zg_n''(z)}{g_n'(z)} - \alpha_n\right) - \eta_n + \eta_n\alpha_n\end{aligned} \tag{6}$$

Since $f_i \in \beta_i - \mathrm{S}_p(\alpha_i)$ and $g_i \in \beta_i - \mathcal{UCV}(\alpha_i)$ from (6) we obtain

$$\begin{aligned}\mathrm{Re}\frac{zK''(z)}{K'(z)} &\geq \gamma_1\beta_1\left|\frac{zf_1'(z)}{f_1(z)} - 1\right| + \cdots + \gamma_n\beta_n\left|\frac{zf_n'(z)}{f_n(z)}\right| + \sum_{i=1}^{n}\gamma_i(\alpha_i - 1) + \\ &+ \eta_1\beta_1\left|\frac{zg_1''(z)}{g_1'(z)} - 1\right| + \cdots + \eta_n\beta_n\left|\frac{zg_n''(z)}{g_n'(z)} - 1\right| + \sum_{i=1}^{n}\eta_i(\alpha_i - 1) \\ &\geq \sum_{i=1}^{n}(\alpha_i - 1)(\gamma_i + \eta_i)\end{aligned}$$

From the above relation we have

$$\mathrm{Re}\left(\frac{zK''(z)}{K'(z)} + 1\right) \geq 1 + \sum_{i=1}^{n}(\alpha_i - 1)(\gamma_i + \eta_i)$$

wich implies that $K(z) \in \mathcal{K}(\rho)$ □

If we put $n = 1$ in Theorem 0.2 we get:

Corollary 0.4. *If $f \in \beta - \mathrm{S}_p(\alpha), -1 \leq \alpha \leq 1, \beta > 0, \gamma \leq \frac{1}{2}$ and $g \in \beta - \mathcal{UCV}(\alpha), -1 \leq \alpha \leq 1, \beta > 0, \eta \leq \frac{1}{2}$, then $K(z) = \int_0^z \left(\frac{f(t)}{t}\right)^{\gamma} \cdot (g'(t))^{\eta}dt \in \mathcal{K}(\varphi)$, where*

$$\varphi = 1 + (\alpha - 1)(\gamma - \eta).$$

For $\gamma_1 = \cdots = \gamma_n = \gamma$ and $\eta_1 = \cdots = \eta_n = \eta$ we obtain:

Corollary 0.5. *If $f_i \in \beta_i - \mathrm{S}_p(\alpha_i), -1 \leq \alpha_i \leq 1, \beta_i > 0, \gamma \leq \frac{1}{2}$ and $g_i \in \beta_i - \mathcal{UCV}(\alpha_i), -1 \leq \alpha_i \leq 1, \beta_i > 0, \eta \leq \frac{1}{2}$ for $i = \overline{1, n}$, then $K(z) \in \mathcal{K}(\rho)$, where*

$$\rho = 1 + \sum_{i=1}^{n}(\alpha_i - 1)(\gamma - \eta).$$

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