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TABULAR SURFACES WITH BISHOP FRAME OF WEINGARTEN TYPES IN EUCLISIAN 3-SPACE

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Abstract. In this paper, we study a tubular surfaces with Bishop frame in Euclidean 3-space satisfying some equations in terms of the Gaussian curvature, the mean curvature,the second Gaussian curvature and second mean curvature.

*Key Words:*Tubular surface,Weingarten surfaces,Bishop frame,Gaussian curvature, Mean curvature, second Gaussian curvature,second mean curvature.
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1.Introduction

A surface M in Euclidean space E^3 or Minkowski space E_1^3 is called a Weingarten surface if there is some smooth relation $W(\kappa_1, \kappa_2) = 0$ between its two principal curvatures κ_1 and κ_2. In particular, if K and H denote respectively the Gauss curvature and the mean curvature of M, $W(\kappa_1, \kappa_2) = 0$ implies a relation $\Phi(K, H) = 0$. The simplest case of functions W or Φ is that they are linear, that is, $A\kappa_1 + B\kappa_2 = C$ or $AH + BK = C$, where A, B and C are constants with $A^2 + B^2 \neq 0$. Such surfaces are called linear Weingarten surfaces.The existence of a non-trivial functional relation $\Phi(K, H) = 0$ on a surface M parameterized by a patch $x(s, t)$ is equivalent to the vanishing of the corresponding Jacobian determinant, namely $\left|\frac{\partial(K,H)}{\partial(s,t)}\right| = 0$ [12].

When the constant $B = 0$, a linear Weingarten surface M reduces to a surface with constant Gaussian curvature. When the constant $A = 0$, a linear Weingarten surface M reduces to a surface with constant mean curvature. In such a sense, the linear Weingarten surfaces can be regarded as a natural generalization of surfaces with constant Gaussian curvature or with constant mean curvature [16].

The set of solutions of this equation is also called the curvature diagram of the surface.If the curvature diagram degenerates to exactly one point then the surface has two constant principal curvatures which is possible for only a piece of a plane, a sphere or a circular cylinder. If the curvature diagram is contained in one of the coordinate axes through the origin then the surface is developable. If the curvature diagram is contained in the main diagonal $\kappa_1 = \kappa_2$ then the surface is a piece of a plane or a sphere because every point is umbilic. The curvature diagram is contained in a straight line parallel to the diagonal $\kappa_1 = -\kappa_2$ if and only if the mean curvature is constant. It is contained in a standard hyperbola $\kappa_1 = \frac{c}{\kappa_2}$ if and only if the Gaussian curvature is constant [18].

If the second fundamental form II of a surface M in E_1^3 is non-degenerate, then it is regarded as a new pseudo-Riemannian metric.Therefore, the second Gaussian curvature K_{II} of non-degenerate second fundamental form II can be defined formally on the Riemannian or pseudo-Riemannian manifold (M, II) [16].

For a pair (X, Y), $X \neq Y$, of the curvatures K, H and K_{II} of M in E^3, if M satisfies $W(X, Y) = 0$ and $AX + BY = C$, then it said to be a (X, Y)-Weingarten surface and (X, Y)-linear Weingarten surface, respectively [16].

Several geometers [3, 4, 6, 11, 12, 13] have studied W-surfaces and LW-surfaces and obtained many interesting results.For study of these surfaces, W. Kühnel [11] and G.Stamou [13] investigate ruled (X, Y)-Weingarten surface in Euclidean 3-space E^3. Also,C.Baikoussis and Th. Koufogiorgos [1] studied helicoidal (H, K_{II})-Weingarten surfaces.F.Dillen and W. Kühnel [3] and F.Dillen and W.Sodsiri [4, 5] gave a classification of ruled (X, Y)-Weingarten surface in Minkowski 3-space E_1^3, where $X, Y \in \{K, H, K_{II}\}$. D. Koutroufiotis [10] and Th.Koufogiorgos and T. Hasanis [9] investigate closed ovaloid (X, Y)-linear Weingarten surface in E^3. D. W. Yoon [14] and D.E.Blair and Th.Koufogiorgos [2] classified ruled (X, Y)-linear Weingarten surface in E^3[16].D. W. Yoon [14] and J.S.Ro studied tubes in Euclidean 3-space satisfying some equation in terms of the Gaussian curvature, the mean curvature and the second Gaussian curvature.

Following the Jacobian equation and the linear equation with respect to the Gaussian curvature K, the mean curvature H and the second Gaussian curvature K_{II}, an interesting geometric question is raised: Classify all surfaces in Euclidean 3-space and Minkowski 3-space satisfying the conditions

$$\Phi(X, Y) = 0 \tag{1.1}$$

$$AX + BY = C \tag{1.2}$$

where $X, Y \in \{K, H, K_{II}\}$, $X \neq Y$.

In this paper, we would like to contribute the solution of the above question, by studying this question for tubes or tubular surfaces with Bishop frame in Euclidean 3-space E^3.

2.Preliminaries

We denote a surface M in E^3 by

$$M(s,t) = (m_1(s,t), m_2(s,t), m_3(s,t)).$$

Let U be the standard unit normal vector field on a surface M defined by

$$U = \frac{M_s \wedge M_t}{\|M_s \wedge M_t\|}$$

where $M_s = \frac{\partial M(s,t)}{\partial s}$ .Then the first fundamental form I and the second fundamental form II of a surface M are defined by, respectively

$$I = Eds^2 + 2Fdsdt + Gdt^2, II = eds^2 + 2fdsdt + gdt^2$$

where

$$\begin{array}{lll} E = \langle M_s, M_s \rangle & F = \langle M_s, M_t \rangle & G = \langle M_t, M_t \rangle \\ e = \langle M_{ss}, U \rangle & f = \langle M_{st}, U \rangle & g = \langle M_{tt}, U \rangle \end{array}$$

[16].On the other hand, the Gaussian curvature K and the mean curvature H are given by, respectively

$$K = \frac{eg - f^2}{EG - F^2}, H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)}.$$

From Brioschi's formula in a Minkowski 3-space, we are able to compute K_{II} of a surface by replacing the components of the first fundamental form E, F, G by the components of the second fundamental form e, f, g respectively in Brioschi's formula [16]. Consequently, the second Gaussian curvature K_{II} of a surface is defined by

$$K_{II} = \frac{1}{(|eg| - f^2)^2} \left\{ \begin{vmatrix} -\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_t & \frac{1}{2}g_s \\ \frac{1}{2}e_t & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix} \right\}.$$

[4,5].The principal curvatures of the surface $M(s,t)$ can be found by the following equation

$$k_1 = H + \sqrt{H^2 - K}, \quad k_2 = H - \sqrt{H^2 - K}.$$

The second mean curvature H_{II} of a surface M is defined by

$$H_{II} = H - \frac{1}{2\sqrt{|\det II|}} \sum_{i,j} \frac{\partial}{\partial u^i} \left(\sqrt{|\det II|} L^{ij} \frac{\partial}{\partial u^j} \left(\ln \sqrt{|K|} \right) \right) \qquad (2.4)$$

where u^i and u^i stand for "s" and "θ", respectively, and $\left(L^{ij}\right) = \left(L_{ij}\right)^{-1}$, where L_{ij} are the coefficients of second fundamental forms [4,5]

Definition 2.1. *κ_1, κ_2 be the principal curvatures of the surface $M(s,t)$ in E^3. The surface $M(s,t)$ has a curvature diagram if it is satisfies*

$$f(\kappa_1, \kappa_2) = a_1\kappa_1^2 + 2a_2\kappa_1\kappa_2 + a_3\kappa_2^2 + a_4\kappa_1 + a_5\kappa_2 + a_6 = 0 \tag{2.5}$$

for some constant $a_i (i = 1, 2, ..., 6)$.

Remark 2.1. It is well known that a minimal surface has vanishing second Gaussian curvature but that a surface with vanishing second Gaussian curvature need not be minimal [16].

3.Tubular Surfaces with Bishop Frame of Weingarten type in Euclidean 3-Space

Consider a framable space curve α.The Bishop and Frenet frames are defined by the following two systems of ordinary differential equations. All differentiations are with respect to s the arc-length parameter of the space curve α.

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \end{bmatrix}.$$

Let $\phi = \arg(k_1, k_2) = \arctan\frac{k_2}{k_1}$. Let ϕ' denote the derivative of ϕ with respect to arc-length. Then the relationships between frame parameters are

$$\kappa = \sqrt{k_1^2 + k_2^2}, \quad \tau = \phi' = \frac{k_1k_2' - k_1'k_2}{k_1^2 + k_2^2}$$

[19,20].A surface often associated with the Frenet frame is the cylinder.When the Frenet development is a constant point in the plane $\kappa\tau$-plane, α will be a helix lying on a cylinder
whose radius is determined by the curvature and angular velocity of the space curve's parametrization. There is an analogous feature for the Bishop frame [20].

Definition 3.1. *Let M be any relatively parallel field for a framable space curve α. A tubular surface of radius $\lambda > 0$ about α is the surface with parametrization*

$$M(s,\theta) = \alpha(s) + \lambda\left[N_1(s)\cos\theta + N_2(s)\sin\theta\right] \tag{3.1}$$

where $N_1(s), N_2(s)$ are first and second normals of the curve α [20] *.*

Theorem 3.1. *A tubular surface about unit-speed spacelike curve with timelike principal normal in Minkowski 3-space is a Weingarten surface. Let $(X,Y) \in \{(K,K_{II}),(H,K_{II})\}$ and let M a tubular surface defined by (3.1) with non-degenerate second fundamental form. If M is a (X,Y) -Weingarten surface, then the curvature of α is a non-zero constant. If α has non-zero constant torsion, then M is generated by a circular helix α in Minkowski 3-space.*

*Proof.*We have the natural frame $\{M_s, M_\theta\}$ of the surface M given by

$$\begin{aligned} M_s &= (1-\lambda k_1\cos\theta-\lambda k_2\sin\theta)\,T, \\ M_\theta &= -\lambda\sin\theta N_1+\lambda\cos\theta N_2. \end{aligned}$$

From which the components of the first fundamental form are

$$E=(1-\lambda k_1\cos\theta-\lambda k_2\sin\theta)^2, F=0, G=\lambda^2.$$

On the other hand, the unit normal vector field U is obtained by

$$U=-\left(N_1\cos\theta+N_2\sin\theta\right).$$

From this, the components of the second fundamental form of M are given by

$$\begin{aligned} e &= -\cos\theta\left(k_1-\lambda k_1^2\cos\theta-\lambda k_1k_2\sin\theta\right)-\sin\theta\left(k_2-\lambda k_2^2\sin\theta-\lambda k_1k_2\cos\theta\right), \\ f &= 0,\ g=\lambda. \end{aligned}$$

If the second fundamental form is non-degenerate, $eg-f^2\neq 0$. In this case, we can define formally the second Gaussian curvature K_{II} on M.On the other hand, the Gauss curvature K, the mean curvature H and the second Gaussian curvature K_{II} are given by, respectively

$$K=\frac{k_1\cos\theta+k_2\sin\theta}{\lambda(\lambda k_1\cos\theta+\lambda k_2\sin\theta-1)}, H=\frac{2\lambda\left(k_1\cos\theta+k_2\sin\theta\right)-1}{2\lambda(\lambda k_1\cos\theta+\lambda k_2\sin\theta-1)}, \tag{3.4}$$

$$K_{II}=\frac{\left\{\begin{array}{c} 4\lambda^2\left(k_2^2-2k_1k_2-k_1^2\right)\left(k_2^2+2k_1k_2-k_1^2\right)\cos^4\theta \\ +2\lambda k_1\left(8\lambda k_2\left(k_1^2-k_2^2\right)\sin\theta-3k_1^2+9k_2^2\right)\cos^3\theta \\ +\left(\begin{array}{c} -\left(18\lambda k_1^2k_2+6\lambda k_2^3\right)\sin\theta-8\lambda^2k_2^4 \\ +k_1^2+\left(24\lambda^2k_1^2-1\right)k_2^2 \end{array}\right)\cos^2\theta \\ +2k_1k_2\left(\left(8\lambda^2k_2^2+1\right)\sin\theta-9\lambda k_2\right)\cos\theta \\ +4\lambda^2k_2^4-6\lambda k_2^3\sin\theta+2k_2^2+k_1^2 \end{array}\right\}}{\left(k_1\cos\theta+k_2\sin\theta\right)\left\{\lambda\left(k_1\cos\theta+k_2\sin\theta\right)-1\right\}}. \tag{3.5}$$

Differentiating K and H with respect to s and θ, we get

$$K_s = -\frac{k_1' \cos\theta + k_2' \sin\theta}{\lambda\left(-1+\lambda k_1 \cos\theta + \lambda k_2 \sin\theta\right)^2}, \quad K_\theta = \frac{k_1 \sin\theta - k_2 \cos\theta}{\lambda\left(-1+\lambda k_1 \cos\theta + \lambda k_2 \sin\theta\right)^2}, \tag{3.6}$$

$$H_s = -\frac{k_1' \cos\theta + k_2' \sin\theta}{2\left(-1+\lambda k_1 \cos\theta + \lambda k_2 \sin\theta\right)^2}, \quad H_\theta = \frac{k_1 \sin\theta - k_2 \cos\theta}{2\left(-1+\lambda k_1 \cos\theta + \lambda k_2 \sin\theta\right)^2}. \tag{3.7}$$

We obtained $(K_{II})_s, (K_{II})_\theta$ and $H_{II}, (H_{II})_s, (H_{II})_\theta$ by using Maple.But the values of this calculations are so long, then we omited.Now, we investigate a tubular surface M in E^3 satisfying the Jacobi equation $\Phi(X,Y) = 0$.By using (3.6) and (3.7), M satisfies identically the Jacobi equation

$$\Phi(K,H) = K_s H_\theta - K_\theta H_s = 0.$$

Therefore, M is a Weingarten surface.We consider a tubular surface M with non-degenerate second fundamental form in E^3 satisfying the Jacobi equation

$$\Phi(K, K_{II}) = K_s \left(K_{II}\right)_\theta - K_\theta \left(K_{II}\right)_s = 0 \tag{3.8}$$

with respect to the Gaussian curvature K and the second Gaussian curvature K_{II}. Then, by (3.6) equation (3.8) becomes

$$\left(-k_1 k_2 k_2' \sin\theta + k_1 k_2 k_1' \cos\theta\right) = 0.$$

Since this polynomial is equal to zero for every θ, all its coefficients must be zero. Therefore, we conclude that $k_1 = k_2 = 0$.Thus $\kappa = 0$.We suppose that a tubular surface M with non-degenerate second fundamental form in E^3 is (H, K_{II})-Weingarten surface. Then it satisfies the equation

$$\Phi(H, K_{II}) = H_s \left(K_{II}\right)_\theta - H_\theta \left(K_{II}\right)_s = 0, \tag{3.9}$$

which implies

$$\left(k_1 k_2 k_2' \sin\theta - k_1 k_2 k_1' \cos\theta\right) = 0. \tag{3.10}$$

From (3.10) we can obtain $k_1 = k_2 = 0 (\kappa = 0)$.

Theorem 3.2. *Let M be a tubular surface satisfying the linear equation $AK + BH = C$. If $k_1 = k_2 = 0$, then it is an open part of a circular cylinder in Euclidean 3-space.Let $(X,Y) \in \{(K, K_{II}), (H, K_{II})\}$. If $k_1 = k_2 = 0$,then M is (X,Y)-linear Weingarten tubular surfaces in Euclidean 3-space.*

*Proof.*We study a tubular surface in E^3 satisfying a linear equation $AX + BY = C$. First of all, we suppose that a tubular surface M in E^3 is a linear Weingarten surface, that is, it satisfies the equation

$$AK + BH - C = \left(\sum_{i=0}^{6} g_i \cos^i \theta \right) = 0 \tag{3.12}$$

where the coefficients g_i are

$$\begin{aligned} g_0 &= A8k_2^4 \left(1 + \lambda^2 k_2^2 - 2\lambda k_2 \sin\theta\right) + B4k_2^3 \left(4\lambda k_2 + 2\lambda^3 k_2^2 - \left(5\lambda^2 k_2^2 + 1\right)\sin\theta\right) \\ &\quad + C8\lambda k_2^3 \left(\left(1 + 3\lambda^2 k_2^2\right)\sin\theta - 3\lambda k_2 - \lambda^3 k_2^3\right) \end{aligned}$$

$$\begin{aligned} g_1 &= A16k_1k_2^3 \left\{\left(2 + 3\lambda^2 k_2^2\right)\sin\theta - 5\lambda k_2\right\} + B4k_1k_2^2 \left\{-4k_2\left(4\lambda + 3\lambda^3 k_2^2\right)\sin\theta - 3 - 25\lambda^2 k_2^2\right\} \\ &\quad + C24\lambda k_1 \left\{-2\lambda k_2^3 \left(2 + \lambda^2 k_2^2\right)\sin\theta + k_2^2 + 5\lambda^2 k_2^4\right\} \end{aligned}$$

$$g_2 = \left\{ \begin{array}{c} 5\lambda^2 k_2^4 \left(-A - B\lambda + C\lambda^2\right)\left(k_2^2 - 5k_1^2\right) - 16k_2^2\left(A + 2B\lambda - 3C\lambda^2\right)\left(k_2^2 - 3k_1^2\right) \\ +k_2\left(4B - 8C\lambda\right)\left(k_2^2 - 3k_1^2\right)\sin\theta + 8\lambda k_2^3\left(4A + 5B\lambda - 6C\lambda^2\right)\left(k_2^2 - 5k_1^2\right)\sin\theta \end{array} \right\}$$

$$g_3 = \left\{ \begin{array}{c} 4k_1k_2\left(8A + 16B\lambda - 24C\lambda^2\right)\left(k_1^2 - k_2^2\right)\sin\theta + 32k_1k_2^3\lambda^2\left(A + B\lambda + C\lambda^2\right)\left(5k_1^2 - 3k_2^2\right)\sin\theta \\ +40\lambda k_1 k_2^2\left(4A + 5B\lambda - 6C\lambda^2\right)\left(k_2^2 - k_1^2\right) + k_1\left(4B\lambda - 8C\lambda\right)\left(3k_2^2 - k_1^2\right) \end{array} \right\}$$

$$g_4 = \left\{ \begin{array}{c} -4\lambda k_2\left(4A + 5B\lambda - 6C\lambda^2\right)\left(5k_1^4 + k_2^4 - 10k_1^2k_2^2\right)\sin\theta \\ +24\lambda^2 k_2^2\left(A + B\lambda - C\right)\left(5k_1^4 + k_2^4 - 10k_1^2k_2^2\right) \\ +\left(8A + 16B\lambda - 24C\lambda^2\right)\left(k_1^4 + k_2^4 - 6k_1^2k_2^2\right) \end{array} \right\}$$

$$g_5 = \left\{ \begin{array}{c} \left(A\lambda^2 + B\lambda^3 - C\lambda^4\right)\left(48k_1k_2^5 + 48k_1^5k_2 - 160k_1^3k_2^3\right)\sin\theta \\ -\left(16A\lambda + 20B\lambda^2 - 24C\lambda^3\right)\left(5k_1^4k_2 + k_1^5 - 10k_1^3k_2^2\right) \end{array} \right\}$$

$$g_6 = \left\{\left(A\lambda^2 + B\lambda^3 - C\lambda^4\right)\left(8\left(k_1^6 + k_2^6\right) + 120\left(k_1^2k_2^4 - k_1^4k_2^2\right)\right)\right\}.$$

Since this is an expression on the independent trigonometric term $\cos\theta$, all coefficients g_i must vanish.Then, we have $k_1 = k_2 = 0$ or $\kappa = 0$. Thus, M is an open part of a circular cylinder in Eculidean 3-space.Next, suppose that a tubular surface M with non-degenerate second fundamental form in E^3 satisfies the equation

$$AK + BK_{II} - C = \left(\sum_{i=0}^{8} g_i \cos^i \theta \right) = 0. \tag{3.13}$$

By (3.4) and (3.5), equation (3.13) becomes

$$\begin{aligned} g_0 &= A8k_2^4\left(1 + \lambda^2 k_2^2 - 2\lambda k_2 \sin\theta\right) \\ &\quad + B2\lambda k_2^2 \left\{ \begin{array}{c} 2k_2^4 + k_1^2 + \lambda^2 k_1^2 k_2^2 + 18\lambda^2 k_2^4 \\ +4\lambda^4 k_2^6 - \left(k_1^2 + 5k_2^2 + 7\lambda^2 k_2^4\right)2\lambda k_2 \sin\theta \end{array} \right\} \\ &\quad + C8\lambda k_2^3\left(\left(1 + 3\lambda^2 k_2^2\right)\sin\theta - 3\lambda k_2 - \lambda^3 k_2^3\right) \end{aligned}$$

$$
\begin{aligned}
g_1 &= A16k_1k_2^3\left\{\left(2+3\lambda^2k_2^2\right)\sin\theta-5\lambda k_2\right\}\\
&+B4\lambda k_1k_2\left\{\left(\begin{array}{c} k_1^2+3k_2^2+2\lambda^2k_1^2k_2^2\\ +53\lambda^2k_2^4+16\lambda^3k_2^6\end{array}\right)\sin\theta-23\lambda k_2^3-3\lambda k_1^2k_2-46\lambda^3k_2^5\right\}\\
&+C24\lambda k_1\left\{-2\lambda k_2^3\left(2+\lambda^2k_2^2\right)\sin\theta+k_2^2+5\lambda^2k_2^4\right\}
\end{aligned}
$$

$$
g_2=\left\{\begin{array}{c}
5\lambda^2k_2^4\left(-A+C\lambda^2\right)\left(k_2^2-5k_1^2\right)-16k_2^2\left(A-3C\lambda^2\right)\left(k_2^2-3k_1^2\right)\\
+2B\lambda\left\{\begin{array}{c}6k_1^2k_2^2+k_1^4-3k_2^4+16\lambda^4k_2^6\left(7k_1^2-k_2^2\right)\\ +\lambda^2k_2^2\left(-53k_2^4+6k_1^2+259k_1^2k_2^2\right)\end{array}\right\}\\
+k_2\left(-8C\lambda\right)\left(k_2^2-3k_1^2\right)\sin\theta+8\lambda k_2^3\left(4A-6C\lambda^2\right)\left(k_2^2-5k_1^2\right)\sin\theta\\
+\left(-588k_1^2k_2^5+84k_2^7\right)B\lambda^4\sin\theta+\left(-168k_1^2k_2^3+36k_2^5-12k_1^4k_2\right)B\lambda^2\sin\theta
\end{array}\right\}
$$

$$
g_3=\left\{\begin{array}{c}
4k_1k_2\left(8A+2B\lambda-24C\lambda^2\right)\left(k_1^2-k_2^2\right)\sin\theta\\
+32k_1k_2^3\lambda^2\left(A+C\lambda^2\right)\left(5k_1^2-3k_2^2\right)\sin\theta\\
+8k_1\left(85k_1^2k_2^3+k_1^4-52k_2^5\right)B\lambda^3\sin\theta\\
+64k_1k_2^5\left(7k_1^2-3k_2^2\right)B\lambda^5\sin\theta+40\lambda k_1k_2^2\left(4A-6C\lambda^2\right)\left(k_2^2-k_1^2\right)\\
+k_1\left(-8C\lambda\right)\left(3k_2^2-k_1^2\right)+196k_1k_2^4\left(3k_2^2-5k_1^2\right)B\lambda^4\\
+4k_1\left(43k_2^4-k_1^4-38k_1^2k_2^2\right)B\lambda^2
\end{array}\right\}
$$

$$
g_4=\left\{\begin{array}{c}
-4\lambda k_2\left(4A+4B\lambda-6C\lambda^2\right)\left(5k_1^4+k_2^4-10k_1^2k_2^2\right)\sin\theta\\
+28k_2^3\left(-3k_2^4+42k_1^2k_2^2-35k_1^4\right)B\lambda^4\sin\theta\\
+24\lambda^2k_2^2\left(A-C\right)\left(5k_1^4+k_2^4-10k_1^2k_2^2\right)\\
\left(8A+2B\lambda-24C\lambda^2\right)\left(k_1^4+k_2^4-6k_1^2k_2^2\right)\\
+16k_2^4\left(3k_2^4+35k_1^4-42k_1^2k_2^2\right)B\lambda^5\\
+2\left(k_1^6+250k_1^4k_2^2-515k_1^2k_2^4+52k_2^6\right)B\lambda^3
\end{array}\right\}
$$

$$
g_5=\left\{\begin{array}{c}
\left(A\lambda^2-C\lambda^4\right)\left(48k_1k_2^5+48k_1^5k_2-160k_1^3k_2^3\right)\sin\theta\\
+\left(192k_2^4+448k_1^4-896k_1^2k_2^2\right)k_1k_2^3B\lambda^5\sin\theta\\
+\left(208k_2^4+204k_1^4-680k_1^2k_2^2\right)k_1k_2B\lambda^3\sin\theta\\
-\left(16A\lambda+16B\lambda^2-24C\lambda^3\right)\left(5k_1^4k_2+k_1^5-10k_1^3k_2^2\right)
\end{array}\right\}
$$

$$
g_6=\left\{\begin{array}{c}
\left(A\lambda^2+\frac{17}{4}B\lambda^3-C\lambda^4\right)\left(8\left(k_1^6+k_2^6\right)+120\left(k_1^2k_2^4-k_1^4k_2^2\right)\right)\\
+\left(28\sin\theta-32\lambda k_2\right)\left(35k_1^4k_2^3-7k_1^6k_2-21k_1^2k_2^5+k_2^7\right)B\lambda^4
\end{array}\right\}
$$

$$
g_7=\left(\begin{array}{c}
\left(\left(k_1^6-k_2^6\right)-7k_1^2k_2\left(k_1^2-k_2^2\right)\right)16k_2\lambda\sin\theta\\
+7\left(21k_1^4k_2^2+7k_2^6-k_1^6-35k_1^2k_2^4\right)
\end{array}\right)4k_1B\lambda^4
$$

$$
g_8=\left(8\left(k_1^8+k_2^8\right)-224\left(k_1^6k_2^2+k_1^2k_2^6\right)+560k_1^4k_2^4\right)B\lambda^5.
$$

Since the identity holds for every θ, all the coefficients g_i must be zero. Therefore, we have $k_1=k_2=0$ or $\kappa=0$.In this case, the second fundamental form of M is degenerate.

Suppose that a tubular surface M with non-degenerate second fundamental form in E^3 satisfies the equation

$$AH + BK_{II} - C = \left(\sum_{i=0}^{8} g_i \cos^i \theta\right) = 0 \tag{3.14}$$

By (3.4), (3.5) and (3.14), we have

$$\begin{aligned} g_0 &= A4k_2^3\left(4\lambda k_2 + 2\lambda^3 k_2^2 - \left(5\lambda^2 k_2^2 + 1\right)\sin\theta\right) \\ &+ B2\lambda k_2^2\left\{2k_2^4 + k_1^2 + \lambda^2 k_1^2 k_2^2 + 18\lambda^2 k_2^4 + 4\lambda^4 k_2^6 - \left(k_1^2 + 5k_2^2 + 7\lambda^2 k_2^4\right)2\lambda k_2 \sin\theta\right\} \\ &+ C8\lambda k_2^3\left(\left(1 + 3\lambda^2 k_2^2\right)\sin\theta - 3\lambda k_2 - \lambda^3 k_2^3\right) \end{aligned}$$

$$\begin{aligned} g_1 &= A4k_1 k_2^2\left\{4k_2\left(4\lambda + 3\lambda^3 k_2^2\right)\sin\theta - 3 - 25\lambda^2 k_2^2\right\} \\ &+ B4\lambda k_1 k_2\left\{\left(\begin{array}{c} k_1^2 + 3k_2^2 + 2\lambda^2 k_1^2 k_2^2 \\ +53\lambda^2 k_2^4 + 16\lambda^3 k_2^6 \end{array}\right)\sin\theta - 23\lambda k_2^3 - 3\lambda k_1^2 k_2 - 46\lambda^3 k_2^5\right\} \\ &+ C24\lambda k_1\left\{-2\lambda k_2^3\left(2 + \lambda^2 k_2^2\right)\sin\theta + k_2^2 + 5\lambda^2 k_2^4\right\} \end{aligned}$$

$$g_2 = \left\{\begin{array}{c} 5\lambda^2 k_2^4\left(-A\lambda + C\lambda^2\right)\left(k_2^2 - 5k_1^2\right) - 16k_2^2\left(2A\lambda - 3C\lambda^2\right)\left(k_2^2 - 3k_1^2\right) \\ +2B\lambda\left\{6k_1^2 k_2^2 + k_1^4 - 3k_2^4 + 16\lambda^4 k_2^6\left(7k_1^2 - k_2^2\right) + \lambda^2 k_2^2\left(-53k_2^4 + 6k_1^2 + 259k_1^2 k_2^2\right)\right\} \\ +k_2\left(4A - 8C\lambda\right)\left(k_2^2 - 3k_1^2\right)\sin\theta + 8\lambda k_2^3\left(5A\lambda - 6C\lambda^2\right)\left(k_2^2 - 5k_1^2\right)\sin\theta \\ +\left(-588k_1^2 k_2^5 + 84k_2^7\right)B\lambda^4\sin\theta + \left(-168k_1^2 k_2^3 + 36k_2^5 - 12k_1^4 k_2\right)B\lambda^2\sin\theta \end{array}\right\}$$

$$g_3 = \left\{\begin{array}{c} 4k_1 k_2\left(16A\lambda + 2B\lambda - 24C\lambda^2\right)\left(k_1^2 - k_2^2\right)\sin\theta \\ +32k_1 k_2^3\lambda^2\left(A\lambda + C\lambda^2\right)\left(5k_1^2 - 3k_2^2\right)\sin\theta \\ +8k_1\left(85k_1^2 k_2^3 + k_1^4 - 52k_2^5\right)B\lambda^3\sin\theta \\ +64k_1 k_2^5\left(7k_1^2 - 3k_2^2\right)B\lambda^5\sin\theta + 40\lambda k_1 k_2^2\left(5A\lambda - 6C\lambda^2\right)\left(k_2^2 - k_1^2\right) \\ +k_1\left(4A\lambda - 8C\lambda\right)\left(3k_2^2 - k_1^2\right) + 196k_1 k_2^4\left(3k_2^2 - 5k_1^2\right)B\lambda^4 \\ +4k_1\left(43k_2^4 - k_1^4 - 38k_1^2 k_2^2\right)B\lambda^2 \end{array}\right\}$$

$$g_4 = \left\{\begin{array}{c} -4\lambda k_2\left(5A\lambda + 4B\lambda - 6C\lambda^2\right)\left(5k_1^4 + k_2^4 - 10k_1^2 k_2^2\right)\sin\theta \\ +28k_2^3\left(-3k_2^4 + 42k_1^2 k_2^2 - 35k_1^4\right)B\lambda^4\sin\theta \\ +24\lambda^2 k_2^2\left(A\lambda - C\right)\left(5k_1^4 + k_2^4 - 10k_1^2 k_2^2\right) \\ \left(16A\lambda + 2B\lambda - 24C\lambda^2\right)\left(k_1^4 + k_2^4 - 6k_1^2 k_2^2\right) \\ +16k_2^4\left(3k_2^4 + 35k_1^4 - 42k_1^2 k_2^2\right)B\lambda^5 \\ +2\left(k_1^6 + 250k_1^4 k_2^2 - 515k_1^2 k_2^4 + 52k_2^6\right)B\lambda^3 \end{array}\right\}$$

$$g_5 = \left\{\begin{array}{c} \left(A\lambda^3 - C\lambda^4\right)\left(48k_1 k_2^5 + 48k_1^5 k_2 - 160k_1^3 k_2^3\right)\sin\theta \\ +\left(192k_2^4 + 448k_1^4 - 896k_1^2 k_2^2\right)k_1 k_2^3 B\lambda^5\sin\theta \\ +\left(208k_2^4 + 204k_1^4 - 680k_1^2 k_2^2\right)k_1 k_2 B\lambda^3\sin\theta \\ -\left(20A\lambda^2 + 16B\lambda^2 - 24C\lambda^3\right)\left(5k_1^4 k_2 + k_1^5 - 10k_1^3 k_2^2\right) \end{array}\right\}$$

$$g_6=\left\{\begin{array}{c}\left(A\lambda^3+\frac{17}{4}B\lambda^3-C\lambda^4\right)\left(8\left(k_1^6+k_2^6\right)+120\left(k_1^2k_2^4-k_1^4k_2^2\right)\right)\\+\left(28\sin\theta-32\lambda k_2\right)\left(35k_1^4k_2^3-7k_1^6k_2-21k_1^2k_2^5+k_2^7\right)B\lambda^4\end{array}\right\}$$

$$g_7=\left(\begin{array}{c}\left(\left(k_1^6-k_2^6\right)-7k_1^2k_2\left(k_1^2-k_2^2\right)\right)16k_2\lambda\sin\theta\\+7\left(21k_1^4k_2^2+7k_2^6-k_1^6-35k_1^2k_2^4\right)\end{array}\right)4k_1B\lambda^4$$

$$g_8=\left(8\left(k_1^8+k_2^8\right)-224\left(k_1^6k_2^2+k_1^2k_2^6\right)+560k_1^4k_2^4\right)B\lambda^5$$

from which we can obtain $k_1=k_2=0$ or $\kappa=0$.

Theorem 3.3. *Let $(X,Y)\in\{(K,H_{II}),(H,H_{II})\}$. If $k_1=k_2=0$,then M is (X,Y)-linear Weingarten tubular surfaces in Euclidean 3-space.*

Theorem 3.4. *Let $(X,Y)\in\{(K,H,K_{II},H_{II})\}$. If $k_1=k_2=0$,then M is (X,Y)-linear Weingarten tubular surfaces in Euclidean 3-space.*

Theorem 3.5. *The tubular surface $M(s,\theta)$ is not umblical and minimal.*

*Proof.*Let κ_1,κ_2 be the principal curvatures of $M(s,\theta)$ in E^3 .The principal curvatures are found as follows;

$$\kappa_1=\frac{1}{\lambda},\kappa_2=\frac{k_1\cos\theta+k_2\sin\theta}{\lambda\left(k_1\cos\theta+k_2\sin\theta\right)-1}. \tag{3.15}$$

Since M has not a curvature diagram such that $\kappa_1-\kappa_2=0$ and $\kappa_1+\kappa_2=0$.Thus, M is not umblical and minimal.

Theorem 3.6. *If the Gauss curvature K of the surface $M(s,\theta)$ is zero, then the surface $M(s,\theta)$ is a circular cylinder or developable surface in Euclidean* 3*-space.*

*Proof.*By (3.15), equation (2.5) becomes

$$\begin{array}{c}\lambda^2\left(\kappa_1^2-\kappa_2^2\right)\left[\begin{array}{c}a_6\lambda^2+\lambda\left(a_4+a_5\right)\\+a_3+2a_2+a_1\end{array}\right]\cos^2\theta\\+\lambda\kappa_1\left\{\begin{array}{c}2\lambda\kappa_2\sin\theta\left[\begin{array}{c}\left(a_6\lambda^2+\lambda\left(a_4+a_5\right)\right)\\+a_3+2a_2+a_1\end{array}\right]\\-2a_6\lambda^2-\lambda\left(2a_4+a_5\right)-2a_2-2a_1\end{array}\right\}\cos\theta\\+\lambda^2\kappa_2^2\left[\begin{array}{c}a_6\lambda^2+\lambda\left(a_4+a_5\right)\\+a_3+2a_2+a_1\end{array}\right]\\-\lambda\kappa_2\sin\theta\left[\begin{array}{c}2a_6\lambda^2+\lambda\left(2a_4+a_5\right)\\+2a_2+2a_1\end{array}\right]\\+a_6\lambda^2+a_4\lambda+a_1=0.\end{array}$$

Since the identity holds for every θ, all the coefficients must be zero. Therefore, we have

$$\lambda^2\left(\kappa_1^2-\kappa_2^2\right)\left[a_6\lambda^2+\lambda\left(a_4+a_5\right)+a_3+2a_2+a_1\right]=0,$$

$$\lambda\kappa_1\left\{\begin{array}{c} 2\lambda\kappa_2\sin\theta\left[\begin{array}{c}\left(a_6\lambda^2+\lambda\left(a_4+a_5\right)\right)\\ +a_3+2a_2+a_1\end{array}\right]\\ -\left(2a_6\lambda^2-\lambda\left(2a_4+a_5\right)-2a_2-2a_1\right)\end{array}\right\}=0,$$

$$\left\{\begin{array}{c}\lambda^2\kappa_2^2\left[a_6\lambda^2+\lambda\left(a_4+a_5\right)+a_3+2a_2+a_1\right]\\ -\lambda\kappa_2\sin\theta\left[2a_6\lambda^2+\lambda\left(2a_4+a_5\right)+2a_2+2a_1\right]\\ +a_6\lambda^2+a_4\lambda+a_1\end{array}\right\}=0.$$

Thus, we get $a_6\lambda^2+a_4\lambda+a_1=0\ (\lambda\neq 0)$, $k_1=k_2=0$ and $\kappa=0$.The solution of equation $a_6\lambda^2+a_4\lambda+a_1=0$ respect to λ is

$$\lambda=\frac{-a_4\pm\sqrt{a_4^2-4a_6a_1}}{2a_6}. \tag{3.16}$$

The tubular surfaces which have the radius (3.16) and generated by the line are satisfy the equation $f(\kappa_1,\kappa_2)=0$.

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