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HOMOMORPHISMS OF CERTAIN α-LIPSCHITZ OPERATOR ALGEBRAS

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Abstract. In a recent paper by A.A. Shokri, A. Ebadian and A.R. Medghalchi [7], a α-Lipschitz operator from a compact metric space X into a unital commutative Banach algebra B is defined. Let (X, d) be a compact metric space in $\mathbb{C}$, $0 < \alpha \leq 1$ and $(B, \| \, . \, \|)$ be a unital commutative Banach algebra. Let $L^\alpha(X, B)$ be the algebra of all bounded continuous operators $f : X \to B$ such that

$$p_\alpha(f) := \sup \left\{ \frac{\| f(x) - f(y) \|}{d^\alpha(x, y)} : x, y \in X, x \neq y \right\} < \infty \ .$$

Now in this work, we characterize homomorphisms of $L^\alpha(X, B)$.

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1. Introduction

Let (X, d) be a compact metric space with at least two elements in $\mathbb{C}$ and $(B, \| \, . \, \|)$ be a Banach space over the scaler field $\mathbb{F}$ $(= \mathbb{R}$ or $\mathbb{C})$. For a constant $0 < \alpha \leq 1$ and an operator $f : X \to B$, set

$$p_\alpha(f) := \sup_{s \neq t} \frac{\| f(t) - f(s) \|}{d^\alpha(s, t)} \ , \quad (s, t \in X),$$

which is called the Lipschitz constant of f. Define

$$L^\alpha(X, B) := \{ f : X \to B \quad : \quad p_\alpha(f) < \infty \} \, ,$$

and

$$l^\alpha(X, B) := \left\{ f : X \to B \quad : \quad \frac{\| f(t) - f(s) \|}{d^\alpha(s, t)} \to 0 \quad as \quad d(s, t) \to 0 \right\} .$$

The elements of $L^\alpha(X,B)$ and $l^\alpha(X,B)$ are called big and little α-Lipschitz operators, respectively [7]. Let $C(X,B)$ be the set of all continuous operators from X into B and for each $f \in C(X,B)$, define

$$\| f \|_\infty := \sup_{x\in X} \| f(x) \| .$$

For f, g in $C(X,B)$ and λ in $\mathbb{F}$, define

$$(f+g)(x) := f(x)+g(x), \quad (\lambda f)(x) := \lambda f(x), \quad (x \in X).$$

It is easy to see that $(C(X,B), \| \,.\, \|_\infty)$ becomes a Banach space over $\mathbb{F}$ and $L^\alpha(X,B)$ is a linear subspace of $C(X,B)$. For each element f of $L^\alpha(X,B)$, define

$$\| f \|_\alpha := \| f \|_\infty + p_\alpha(f).$$

When $(B, \| \,.\, \|)$ is a Banach space, Cao, Zhang and Xu [2] proved that $(L^\alpha(X,B), \| \,.\, \|_\alpha)$ is a Banach space over $\mathbb{F}$ and $l^\alpha(X,B)$ is a closed linear subspace of $(L^\alpha(X,B), \| \,.\, \|_\alpha)$, and when $(B, \| \,.\, \|)$ is a unital commutative Banach algebra, A.A. Shokri, A. Ebadian and A.R. Medghalchi [7] proved that $(L^\alpha(X,B), \| \,.\, \|_\alpha)$ is a Banach algebra over $\mathbb{F}$ under pointwise multiplication and $l^\alpha(X,B)$ is a closed linear subalgebra of $(L^\alpha(X,B), \| \,.\, \|_\alpha)$. Furthermore, Sherbert [5,6], Weaver [8,9], Jimenez-Vargas [3], Johnson [4], Cao, Zhang and Xu [2], Bade, Curtis and Dales [1] studied some properties of Lipschitz algebras.

Finally, in this paper, we will study the homomorphisms on the $L^\alpha(X,B)$.

2. Homomorphisms on the α-Lipschitz operator algebras

In this section, let us use (X,d) to denote a compact metric space in $\mathbb{C}$ which has at least two elements and $(B, \| \,.\, \|)$ to denote a unital commutative Banach algebra with unit $\mathbf{e}$ over the scalar field $\mathbb{F}(=\mathbb{R}$ or $\mathbb{C})$. Now, we characterize homomorphisms on the $L^\alpha(X,B)$ where $0<\alpha\le 1$.

Theorem 2.1. *Let (X,d_1) and (Y,d_2) be compact metric spaces in $\mathbb{C}$, $0<\alpha\le 1$ and $(B, \| \,.\, \|)$ be a unital commutative Banach algebra with unit $\mathbf{e}$. Then the map $T : L^\alpha(X,B) \to L^\alpha(Y,B)$ is a homomorphism if and only if there is a map $\varphi : Y \to X$ such that*

$$Tf = fo\varphi \quad (f \in L^\alpha(X,B)) ,$$

and there is a positive number M such that for every $x,y \in Y$

$$d_1(\varphi(x),\varphi(y)) \le M \; d_2^\alpha(x,y) .$$

Proof. Let $T : L^\alpha(X,B) \to L^\alpha(Y,B)$ be a homomorphism and $\Lambda \in B^*$ (B^* is dual space of B) be fixed (if $B = \mathbb{C}$ then $\Lambda = I$ is the identity map). Let $M_{L^\alpha(X,B)}$ be the maximal ideal space of $L^\alpha(X,B)$. For $x \in X$, define

$$\delta_x : L^\alpha(X,B) \to \mathbb{C}$$

$$\delta_x(f) := (\Lambda o f)(x) \ .$$

Then $\delta_x \in M_{L^\alpha(X,B)}$. We define

$$F_1 : X \longrightarrow M_{L^\alpha(X,B)}$$

$$F_1(x) = \delta_x \ ,$$

and

$$F_2 : Y \longrightarrow M_{L^\alpha(Y,B)}$$

$$F_2(y) = \delta_y \ ,$$

also

$$\psi : M_{L^\alpha(Y,B)} \longrightarrow M_{L^\alpha(X,B)}$$

$$h \longmapsto \psi(h)$$

where

$$\psi(h) : L^\alpha(X,B) \longrightarrow \mathbb{C}$$

$$\psi(h)(f) := h(Tf).$$

Set

$$\varphi := F_1^{-1} o \psi o F_2 \ .$$

Then φ is a map from Y into X such that $F_1 o \varphi = \psi o F_2$, and for every $y \in Y$ we have

$$(F_1 o \varphi)(y) = (\psi o F_2)(y) \quad or \quad F_1(\varphi(y)) = \psi(F_2(y)) \ .$$

Then $\delta_{\varphi(y)} = \psi(\delta_y)$. This implies that $\delta_{\varphi(y)}(f) = \psi(\delta_y)(f)$ for every $f \in L^\alpha(X,B)$ and $y \in Y$. Therefore $(\Lambda o f)(\varphi(y)) = \delta_y(Tf)$. Then

$$\Lambda\Big(f(\varphi(y))\Big) = (\Lambda o T f)(y) = \Lambda\Big((Tf)(y)\Big) \ .$$

Since Λ is arbitrary, $(f o \varphi)(y) = (Tf)(y) \quad (y \in Y)$. Therefore $Tf = f o \varphi$ for every $f \in L^\alpha(X,B)$. T is continuous, because if $\{f_n\}_{n\geq 1} \subset L^\alpha(X,B)$ be a sequence such that $f_n \to f$ ($f \in L^\alpha(X,B)$) then $f_n o \varphi \to f o \varphi$ and so $Tf_n \to Tf$.

Now we show that there is a positive number M such that for every $x, y \in Y$ we have

$$d_1(\varphi(x), \varphi(y)) \leq M d_2^\alpha(x,y) \ .$$

Let $s \in X$. Define $f_s : X \to B$ with $f_s(t) := d_1(t,s).\mathbf{e}$ $(t \in X)$. Then $f_s \in L^\alpha(X,B)$ and $\{f_s : s \in X\}$ is a bounded set in $L^\alpha(X,B)$. Since T is continuous, $\{Tf_s : s \in X\}$ is a bounded set in $L^\alpha(Y,B)$. Thus there is a positive number M such that for every $s \in X$, $\| Tf_s \|_\alpha \leq M$. Hence for every $s \in X$, $p_\alpha(Tf_s) \leq M$ and so for every $x, y \in Y$ such that $x \neq y$ we have

$$\frac{\| (Tf_s)(x) - (Tf_s)(y) \|}{d_2^\alpha(x,y)} \leq M \ .$$

With a simple calculation we have

$$\frac{d_1(\varphi(x),\varphi(y))}{d_2^\alpha(x,y)} \leq M$$

and so for every $x, y \in Y$ we have $d_1(\varphi(x),\varphi(y)) \leq M d_2^\alpha(x,y)$. So one half of the theorem is proved.

On the other hand, suppose that $T : L^\alpha(X,B) \to L^\alpha(Y,B)$ is a linear map and there is a map $\varphi : Y \to X$ such that

$$Tf = fo\varphi \ , \quad (f \in L^\alpha(X,B)) \ ,$$

and a positive number M such that for every $x, y \in Y$

$$d_1(\varphi(x),\varphi(y)) \leq M d_2^\alpha(x,y) \ .$$

Firstly, we show that for every $f \in L^\alpha(X,B)$, we have $fo\varphi \in L^\alpha(Y,B)$. Let $f \in L^\alpha(X,B)$. Then $fo\varphi \in C(Y,B)$ and

$$\begin{aligned} p_\alpha(fo\varphi) &= \sup_{x\neq y} \frac{\| (fo\varphi)(x) - (fo\varphi)(y) \|}{d_2^\alpha(x,y)} = \sup_{x\neq y} \frac{\| f(\varphi(x)) - f(\varphi(y)) \|}{d_2^\alpha(x,y)} \\ &= M \sup_{x\neq y} \frac{\| f(\varphi(x)) - f(\varphi(y)) \|}{M d_2^\alpha(x,y)} \\ &\leq M \sup_{x\neq y} \frac{\| f(\varphi(x)) - f(\varphi(y)) \|}{d_1(\varphi(x),\varphi(y))} = M p_1(f) < \infty. \end{aligned}$$

So $fo\varphi \in L^\alpha(Y,B)$. Now, if $f, g \in L^\alpha(X,B)$ is arbitrary, then

$$T(fg) = (fg)o\varphi = (fo\varphi)(go\varphi) = (Tf)(Tg).$$

Therefore T is a homomorphism. The proof is complete.

Corollary 2.2 *Let (X,d) be a compact metric space, and $(B, \| \ . \ \|)$ be a unital commutative Banach algebra. Then the map $T : L^\alpha(X,B) \to L^\alpha(X,B)$ is a non-zero automorphism if and only if there is a map $\varphi : X \to X$ such that for every*

$f \in L^\alpha(X, B)$, $Tf = fo\varphi$, and there is a positive number M such that for every $x, y \in X$

$$d(\varphi(x), \varphi(y)) \leq M d^\alpha(x, y) \ .$$

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