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THE MATRIX TRANSFORMATIONS ON DOUBLE SEQUENCE SPACE OF χ^2

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Abstract. Let χ^2 denote the space of all prime sense double gai sequences and Λ^2 the space of all prime sense double analytic sequences. First we show that the set $E = \left\{ s^{(mn)} : m, n = 1, 2, 3, \cdots \right\}$ is a determining set for χ^2. The set of all finite matrices transforming χ^2 into FK-space Y denoted by $\left(\chi^2 : Y\right)$. We characterize the classes $\left(\chi^2 : Y\right)$ when $Y = c_0^2, c^2, \chi^2, \ell^2, \Lambda^2$.

$\nearrow$	c_0^2	c^2	χ^2	ℓ^2	Λ^2
χ^2	Necessary and sufficient condition on the matrix are obtained				

But the approach to obtain these result in the present paper is by determining set for χ^2. First, we investigate a determining set for χ^2 and then we characterize the classes of matrix transformations involving χ^2 and other known sequence spaces.

Keywords : *Determining set, gai sequence, analytic sequence, double sequence*
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1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences respectively.
We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$ the set of positive integers. Then w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich[2]. Later on it was investigated by Hardy[3], Moricz[4], Moricz and Rhoades[5], Basarir and Solankan[1], Tripathy[6], Colak and Turkmenoglu[7], Turkmenoglu[8], and many others.

We need the following inequality in the sequel of the paper. For $a, b, \geq 0$ and $0 < p < 1$, we have

$$(a+b)^p \leq a^p + b^p \tag{1}$$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$is called convergent if and only if the double sequence. $(s_{mn})$is called convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij}(m,n = 1,2,3,...)$ (see[9]). A sequence $x = (x_{mn})$is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0$ as $m, n \to \infty$. The double gai sequences will be denoted by χ^2. Let $\phi = \{allfinitesequences\}$. Consider a double sequence $x = (x_{ij})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\Im_{ij}$ for all $m, n \in \mathbb{N}$,

$$\Im_{mn} = \begin{pmatrix} 0, & 0, & ...0, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0, & 0, & ...1, & -1, & ... \\ 0, & 0, & ...0, & 0, & ... \end{pmatrix}$$

with 1 in the $(m,n)^{th}$ position, -1 in the $(m+1, n+1)^{th}$ position and zero other wise. An FK-space(or a metric space)X is said to have AK property if $(\Im_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]} \to x$. An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous. If X is a sequence space, we give the following definitions:
(i)X'= the continuous dual of X;
(ii)$X^\alpha = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty,\, for\, each\, x \in X\right\}$
(iii)$X^\beta = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn}\, is\, convegent,\, foreach\, x \in X\right\}$
(iv)$X^\gamma = \left\{a = (a_{mn}) : \overset{sup}{m,n \geq 1} \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, foreachx \in X\right\}$;
(v)$let\, X\, beanFK - space \supset \phi;\, then\, X^f = \left\{f(\Im_{mn}) : f \in X'\right\}$;
(vi)$X^\Lambda = \left\{a = (a_{mn}) : sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty,\, foreach\, x \in X\right\}$;
$X^\alpha.X^\beta, X^\gamma$ are called $\alpha - (orK\ddot{o}the - Toeplitz)$dual of $X, \beta - (or\, generalized - K\ddot{o}the - Toeplitz)\, dual\, of\, X, \gamma -\, dual\, of\, X,\, \Lambda -\, dual\, of\, X\, respectively.$

2.Definitions and Preliminaries

Let w^2 denote the set of all complex double sequences. A sequence $x = (x_{mn})$ is said to be double analytic if $sup_{mn} |x_{mn}|^{1/m+n} < \infty$. The vector space of all prime sense double analytic sequences will be denoted by Λ^2. A sequence $x = (x_{mn})$ is called prime sense double entire sequence if $((m+n)! |x_{mn}|)^{1/m+n} \to 0 \, \text{as} \, m, n \to \infty$. The double gai sequences will be denoted by χ^2. The space Λ^2 is a metric space with the metric

$$d(x,y) = sup_{mn} \left\{ |x_{mn} - y_{mn}|^{1/m+n} : m, n : 1, 2, 3, ... \right\} \quad (2)$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in Λ^2. The space χ^2 is a metric space with the metric

$$d(x,y) = sup_{mn} \left\{ ((m+n)! |x_{mn} - y_{mn}|)^{1/m+n} : m, n : 1, 2, 3, ... \right\} \quad (3)$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ in χ^2.

Let X be an BK-space. Then $D = D(X) = \{x \in \phi : \|x\| \leq 1\}$ we do not assumethat $X \supset \phi$ (i.e)$D = \phi \bigcap (unit\, closed\, sphere\, in\, X)$

Let X be an BK space. A subset E of ϕ will be called a determining set for X if $D(X)$ is the absolutely convexhull of E. In respect of a metric space $(X, d), D = \{x \in \phi : d(x, 0) \leq 1\}$.

Given a sequence $x = \{x_{mn}\}$ and an four dimensional infinite matrix $A = \left(a_{mn}^{jk}\right), m, n, j, k = 1, 2, \cdots$ then $A-$ transform of x is the sequence $y = (y_{mn})$ when $y_{mn} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}^{jk} x_{mn} \, (j, k = 1, 2, \cdots)$. Whenever $\sum \sum a_{mn}^{jk} x_{mn}$ exists.

Let X and Y be FK-spaces. If $y \in Y$ whenever $x \in X$, then the class of all matrices A is denoted by $(X : Y)$.

3.Lemma

Let X be a BK-space and E is determining set for X. Let Y be an FK-space and A is an four dimensional infinite matrix. Suppose that either X has AK or A is row finite. Then $A \in (X : Y)$ if and only if (1) The columns of A belong to Y and (2) $A[E]$ is a bounded subset of Y.

4.Main Results

Theorem 1. *Let E be the set of all sequences in ϕ each of whose non-zero terms is*

$$\begin{pmatrix} 0, & 0, & ...0, & 0, & ... \\ 0, & 0, & ...0, & 0, & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0, & 0, & ...\frac{1}{(m+n)!}, & \frac{-1}{(m+n)!}, & 0... \\ 0, & 0, & ...0, & 0, & ... \end{pmatrix}$$

with $\frac{1}{(m+n)!}$, *in the* $(m,n)^{th}$, $\frac{-1}{(m+n)!}$, *in the* $(m+1,n+1)^{th}$ *position and zero other wise. Then* E *is determining set of* χ^2.

Proof. Step1. Recall that χ^2 is a metric space with the metric

$$d(x,y) = sup_{mn} \left\{ \left((m+n)! \left|x_{mn} - y_{mn}\right|\right)^{1/m+n} : m,n = 1,2,3,\cdots \right\}$$

Let A be the absolutely convex hull of E. Let $x \in A$. Then $x = \sum_{m=1}^{i} \sum_{n=1}^{j} t_{mn} s^{(mn)}$ with

$$\sum_{m,n=1}^{i,j} |t_{mn}| \leq 1. \tag{4}$$

and $s^{(mn)} \in E$.
Then $d(x,0) \leq |t_{11}| \, d\left(s^{(11)},0\right) + \cdots + |t_{ij}| \, d\left(s^{(ij)},0\right)$. But $d\left(s^{(mn)}\right) = 1$ for $m,n = 1,2,3,\cdots(i,j)$. Hence $d(x,0) \leq \sum_{m,n=1}^{i,j} |t_{mn}| \leq 1$ by using (4). Also $x \in \phi$. Hence $x \in D$. Thus

$$A \subset D \tag{5}$$

Step 2. Let $x \in D$

$\Rightarrow x \in \phi$ and $d(x,0) \leq 1$.

$$x = \begin{pmatrix} 2!x_{11}, & 3!x_{12}, & ..., & (1+n)!x_{1n}, & ... \\ 3!x_{21}, & 4!x_{22}, & ..., & (2+n)!x_{2n}, & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ (m+1)!x_{m1}, & (m+2)!x_{m2}, & ..., & (m+n)!x_{mn}, & ... \\ 0, & 0, & ...0, & 0, & ... \end{pmatrix} \text{ and}$$

$$sup \begin{pmatrix} (2!\,|x_{11}|)^{1/2}, & (3!\,|x_{12}|)^{1/3}, & ..., & ((1+n)!\,|x_{1n}|)^{1/1+n}, & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ ((m+1)!\,|x_{m1}|)^{1/m+1}, & ((m+2)!\,|x_{m2}|)^{1/m+2}, & ..., & ((m+n)!\,|x_{mn}|)^{1/m+n}, & ... \\ 0, & 0, & ...0, & 0, & ... \end{pmatrix} \quad (6)$$

Case (i). Suppose that $2!\,|x_{11}| \geq \cdots \geq (m+n)!\,|x_{mn}|$.

Let $\xi_{mn} = Sgn\,((m+n)!x_{mn}) = \frac{(m+n)!|x_{mn}|}{(m+n)!x_{mn}}$ for $m, n = 1, 2, \cdots (i, j)$

$$S_{k\ell} = \begin{pmatrix} \xi_{11}, & \xi_{12}, & ..., & \xi_{1\ell}, & ... \\ \xi_{21}, & \xi_{22}, & ..., & \xi_{2\ell}, & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ \xi_{k1}, & \xi_{k2}, & ..., & \xi_{k\ell}, & ... \\ 0, & 0, & ...0, & 0, & ... \end{pmatrix} \text{ for } k, \ell = 1, 2, 3, \cdots (i, j)$$

Then $s_{k\ell} \in E$ for $k, \ell = 1, 2, 3, \cdots (i, j)$. Also

$x = (|2!x_{11} - 3!x_{12}| - |3!x_{21} - 4!x_{22}|)\, S_{11} + \cdots +$
$(|(m+n)!x_{mn} - (m+n+1)!x_{mn+1}| - |(m+n+1)!x_{m+1n} - (m+n+2)!x_{m+1n+1}|)\, S_{mn}$

$= t_{11}S_{11} + \cdots + t_{mn}S_{mn}$. so that

$t_{11} + \cdots + t_{mn} = |2!x_{11} - 3!x_{12}| - |(m+n+1)!x_{m+1n} - (m+n+2)!x_{m+1n+1}|$

$= |2!x_{11} - 3!x_{12}|$ because $|(m+n+1)!x_{m+1n} - (m+n+2)!x_{m+1n+1}| =$
0

≤ 1 by using (6)

Hence $x \in A$. Thus $D \subset A$.

Case (ii). Let y be x and let $2!\,|y_{11}| \geq \cdots \geq (m+n)!\,|y_{mn}|$.

Express y as a member of A as in case(i). Since E is invariant under permutation of the terms of its members, so is A. Hence $x \in A$. Thus $D \subset A$. Therefore in both cases

$$D \subset A \tag{7}$$

From (5) and (7) $A = D$. Consequently E is a determining set for χ^2. This completes the proof.

Proposition 2. *χ^2 has AK*

Proof. Let $x = (x_{mn}) \in \chi^2$ and take $x^{[mn]} = \sum_{i,j=1}^{m,n} x_{ij}\Im_{ij}$ for all $m, n \in \mathbb{N}$.
Hence $d\left(x, x^{[rs]}\right) = sup_{mn}\left\{\left((m+n)!\,|x_{mn}|\right)^{1/m+n} : m \geq r+1, n \geq s+1\right\}$
$\rightarrow 0$ as $m, n \rightarrow \infty$
Therefore, $x^{[rs]} \rightarrow x$ as $r, s \rightarrow \infty$ in χ^2. Thus χ^2 has AK. This completes proof.

Proposition 3. *An infinite matrix $A = \left(a_{mn}^{jk}\right)$ is in the class*

$$A \in \left(\chi^2 : c_0^2\right) \Leftrightarrow lim_{n,k\rightarrow\infty}\left(a_{mn}^{jk}\right) = 0 \tag{8}$$

$$\Leftrightarrow sup_{mn}\left|a_{m1}^{j1} + \cdots + a_{mn}^{jk}\right| < \infty. \tag{9}$$

Proof. In Lemma 1. Take $X = \chi^2$. has AK property take $Y = \left(c_0^2\right)$ be an FK-space. Further more χ^2 is a determining set E (as in given Theorem 1). Also $A[E] = A\left(s^{(mn)}\right) = \left\{\left(a_{m1}^{j1} + \cdots + a_{mn}^{jk}\right)\right\}$. Again by Lemma 1. $A \in \left(\chi^2 : c_0^2\right)$ if and only if (i)The columns of A belong to c_0^2 and (ii)$A\left(s^{(mn)}\right)$ is a bounded subset χ^2. But the condition
(i)$\Leftrightarrow \left\{a_{mn}^{jk} : j, k = 1, 2, \cdots\right\}$ is exits for all m, n.
(ii)$\Leftrightarrow sup_{mn}\left|a_{m1}^{j1} + \cdots + a_{mn}^{jk}\right| < \infty$.
Hence we conclude that $A \in \left(\chi^2 : c_0^2\right) \Leftrightarrow$ conditions (8) and (9) are satisfied.
The following proofs are similar. Hence we omit the proof.

Proposition 4. *An infinite matrix $A = \left(a_{mn}^{jk}\right)$ is in the class*

$$A \in \left(\chi^2 : c^2\right) \Leftrightarrow lim_{n,k\rightarrow\infty}\left(a_{mn}^{jk}\right) exists(m, j = 1, 2, 3, ...) \tag{10}$$

$$\Leftrightarrow sup_{mn} \left| a^{j1}_{m1} + \cdots + a^{jk}_{mn} \right| < \infty. \tag{11}$$

Proposition 5. *An infinite matrix* $A = \left(a^{jk}_{mn} \right)$ *is in the class*

$$A \in \left(\chi^2 : \chi^2 \right) \Leftrightarrow sup_{mn} \left(\frac{1}{(m+n)!} \left| a^{j1}_{m1} + \cdots + a^{jk}_{mn} \right| \right)^{1/m+n} < \infty. \tag{12}$$

$$\Leftrightarrow lim_{n,k \to \infty} \left(\frac{1}{(m+n)!} \left| a^{jk}_{mn} \right| \right)^{1/m+n} = 0, for \quad m, j = 1, 2, 3, ... \tag{13}$$

$$\Leftrightarrow d \left(a^{j1}_{m1}, a^{j2}_{m2}, \cdots, a^{jk}_{mn} \right) is \quad bounded \tag{14}$$

for each metric d *on* χ^2 *and for all* m, n.

Proposition 6. *An infinite matrix* $A = \left(a^{jk}_{mn} \right)$ *is in the class*

$$A \in \left(\chi^2 : \ell^2 \right) \Leftrightarrow \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left| a^{jk}_{mn} \right| converges (j, k = 1, 2, 3, ...) \tag{15}$$

$$\Leftrightarrow sum_{mn} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left| a^{jk}_{mn} \right| < \infty \tag{16}$$

Proposition 7 *An infinite matrix* $A = \left(a^{jk}_{mn} \right)$ *is in the class*

$$A \in \left(\chi^2 : \Lambda^2 \right) \Leftrightarrow sup_{mn} \left(\left| \sum_{\gamma=1}^{n} \sum_{\mu=1}^{k} a^{j\mu}_{m\gamma} \right|^{1/m+n} \right) < \infty \tag{17}$$

$$\Leftrightarrow d \left(a^{j1}_{m1}, a^{j2}_{m2}, \cdots a^{jk}_{mn} \right) isbounded \tag{18}$$

for each metric d *on* Λ^2 *and for all* m, n.

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