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NON-METRIC CONNECTION ON GENERALISED WEAKLY SYMMETRIC MANIFOLDS

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Abstract. The paper aims to study the behaviour of a generalised weakly symmetric manifolds together with a special type of non-metric connection. We investigated the conditions under which a generalised weakly symmetric manifolds under non metric connection reduces to one under metric connection. We have further explored a special conformally flat space under the non-metric connection and draw the inference that it is a subprojective manifold and can be isometrically immersed in Euclidean space as a hyper surface under certain condition.

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1. Introduction

Let $\widetilde{\nabla}$ and ∇ respectively be semi-symmetric non-metric connection and Levi-Civita connection on (M^n, g). Denote $\widetilde{R}$, $\widetilde{S}$, R, S, as the curvature tensor and Ricci tensor with respect to $\widetilde{\nabla}$ and ∇ respectively.

Recently Baishya [11] inaugurated the thought of generalized weakly symmetric manifold, denoted by $(GWS)_n$. A Riemannian manifold is assumed to be $(GWS)_n$, if R accepts the equation:

$$\begin{aligned}
& (\nabla_X R)(Y,Z,U,V) \\
= \; & a_1(X)R(Y,Z,U,V) + b_1(Y)R(X,Z,U,V) \\
& +b_1(Z)R(Y,X,U,V) + d_1(U)R(Y,Z,X,V) \\
& +d_1(V)R(Y,Z,U,X) + a_2(X)G(Y,Z,U,V) \\
& +b_2(Y)G(X,Z,U,V) + b_2(Z)G(Y,X,U,V) \\
& +d_2(U)G(Y,Z,X,V) + d_2(V)G(Y,Z,U,X), \qquad (1)
\end{aligned}$$

for all vector fields $X, Y, Z, U, V \in \chi(M^n)$, and a_i, b_i, d_i are 1-forms defined by $a_i(X) = g(X, X_{ai}), b_i(X) = g(X, X_{bi}), d_i(X) = g(X, X_{di}), i = 1, 2$ which are not simultaneously zero and $G = \frac{1}{2} g \wedge g$, $\wedge$ being the Kulkarni–Nomizu product. We consider $(a_1, b_1, d_1, a_2, b_2, d_2)$ as a solution of $(GWS)_n$. The charm of $(GWS)_n$ is that it has the taste of the followings:

(i) locally symmetric space for $(0, 0, 0, 0, 0, 0)$,
(ii) generalized recurrent space for $(h_1, 0, 0, h_2, 0, 0)$,
(iii) recurrent space for $(h_1, 0, 0, 0, 0, 0)$,
(iv) generalized pseudo symmetric space for $(2h_1, h_1, h_1, 2h_2, h_2, h_2)$,
(v) almost generalized pseudo symmetric space for $(k_1 + h_1, h_1, h_1, k_{2+}h_2, h_2, h_2)$,
(vi) almost pseudo symmetric space for $(k_1 + h_1, h_1, h_1, 0, 0, 0)$,
(vii) almost quasi pseudo symmetric space for $(k_1 + h_1, -h_1, -h_1, 0, 0, 0)$
(viii) quasi pseudo symmetric space for $(2h_1, -h_1, -h_1, 0, 0, 0)$
(ix) pseudo symmetric space for $(2h_1, h_1, h_1, 0, 0, 0)$,
(x) semi-pseudo symmetric space for $(0, h_1, h_1, 0, 0, 0)$,
(xi) generalized semi-pseudo symmetric for $(0, h_1, h_1, 0, h_2, h_2)$,
(xii) weakly symmetric space for $(h_1, j_1, k_1, 0, 0, 0)$.

Also, a non-flat Riemannian manifold $(M^n, g)(n > 2)$ is called generalized weakly Ricci symmetric if its Ricci tensor S of type $(0, 2)$ is not identically zero and if it satisfies the condition:

$$\begin{aligned}(\nabla_X S)(Y, Z) &= a_1(X)S(Y, Z) + b_1(Y)S(X, Z) + d_1(Z)S(Y, X) \\ &\quad + a_2(X)g(Y, Z) + b_2(Y)g(X, Z) + d_2(Z)g(Y, X) \qquad (2)\end{aligned}$$

where $a_i, b_i, d_i, (i = 1, 2)$ are non-zero 1-form as mentioned above. We shall denote such a manifold by $(GWRS)_n$.

Now using the symmetric property, precisely,

$$(\nabla_X R)(Y, Z, U, V) = (\nabla_X R)(U, V, Y, Z) \qquad (3)$$

we can easily infer from (1) that

$$\begin{aligned}(\nabla_X R)(Y, Z, U, V) &= A_1(X)R(Y, Z, U, V) + B_1(Y)R(X, Z, U, V) \\ &\quad + B_1(Z)R(Y, X, U, V) + B_1(U)R(Y, Z, X, V) \\ &\quad + B_1(V)R(Y, Z, U, X) + A_2(X)G(Y, Z, U, V) \\ &\quad + B_2(Y)G(X, Z, U, V) + B_2(Z)G(Y, X, U, V) \\ &\quad + B_2(U)G(Y, Z, X, V) + B_2(V)G(Y, Z, U, X) \qquad (4)\end{aligned}$$

where $A_i(X) = a_i(X) = g(X, X_{Ai}), B_i(X) = \left(\frac{b_i+d_i}{2}\right)(X) = g(X, B_i), i = 1, 2.$

The present paper deals with generalized weakly symmetric manifolds $(GWS)_n (n > 3)$ admitting a type of semi-symmetric non-metric connection $\widetilde{\nabla}$ whose torsion tensor T is given by

$$T(X,Y) = A_1(Y)X - A_1(X)Y \tag{5}$$

and whose curvature tensor $\widetilde{R}$ and the torsion tensor T is given by

$$\widetilde{R}(X,Y)Z = 0 \tag{6}$$

and

$$\begin{aligned}(\widetilde{\nabla}_X T)(Y,Z) &= 2A_1(X)T(Y,Z) + A_1(Y)T(X,Z) + A_1(Z)T(X,Y) \\ &\quad +2B_1(X)T(Y,Z) + B_1(Y)T(X,Z) + B_1(Z)T(X,Y)\end{aligned} \tag{7}$$

where $\widetilde{\nabla}$ is defined in (11).

We presented our study as follows: Section 2 commences with some basic properties of semi-symmetric non-metric connection. Here, we investigated the conditions under which $[(GWS)_n, \widetilde{\nabla}]$ reduces to $[(GWS)_n, \nabla]$. Following that the study of a semi-symmetric non-metric connection having torsion tensor T, given by (5) and satisfying (6) and (7) is made and in which we concluded that every $(GWS)_n$ with certain 1-forms transforms to $(GWRS)_n$. In this section we have further shown that the manifold under discussion is of constant scalar curvature and its associated 1-forms are closed. Next in section 3, we have explored through a special conformally flat $(GWS)_n (n > 3)$ space having a special type of semi-symmetric non-metric connection and draw the inference that it is a subprojective manifold and can be isometrically immersed in Euclidean space as a hypersurface under certain condition.

2. $(GWS)_n$ with special type of semi-symmetric non-metric connection

Consider the symmetric endomorphism L of the tangent space at each point of $(GWS)_n$ corresponding to the Ricci tensor, that is,

$$g(LX,Y) = S(X,Y). \tag{8}$$

If we put $Y = V = e_i$ in (1) where $\{e_i\}(1 \le i \le n)$ is an orthonormal basis of the tangent space at each point of the manifold and summing over $i(1 \le i \le n)$ we obtain:

$$\begin{aligned}(\nabla_X S)(Z,U) &= A_1(X)S(Z,U) + B_1(Z)S(X,U) + B_1(U)S(Z,X) \\ &\quad +[(n-1)A_2(X) + 2B_2(X)]g(Z,U) \\ &\quad +(n-2)B_2(Z)g(X,U) + (n-2)B_2(U)g(Z,X) \\ &\quad +B_1(R(X,Z)U) + B_1(R(X,U)Z).\end{aligned} \tag{9}$$

Next, contracting (9) with respect to Z and U, we have

$$X(r) = A_1(X)r + 4S(X, X_{B1}) + n(n-1)[A_2(X) + 4B_2(X)]. \tag{10}$$

Thus in a generalized weakly symmetric manifolds $(GWS)_n(n > 3)$ the 1-forms are related by (10).

A semi-symmetric non-metric connection $\widetilde{\nabla}$ is defined by Agashe Chafle [[2]] as:

$$\widetilde{\nabla}_Y Z = \nabla_Y Z + A_1(Z)Y \tag{11}$$

for all vector fields X, Y.

Let us denote the curvature tensors with respect to the connection $\widetilde{\nabla}$ and ∇ by $\widetilde{R}$ and R respectively. Then by (11) we have

$$\widetilde{R}(Y, Z, U, V) = R(Y, Z, U, V) + \alpha(Y, U)g(Z, V) - \alpha(Z, U)g(Y, V) \tag{12}$$

where α is a tensor field of type $(0, 2)$ given by

$$\alpha(Y, Z) = (\nabla_Y A_1)(Z) - A_1(Y)A_1(Z). \tag{13}$$

Also by (11) we have

$$(\widetilde{\nabla}_Y A_1)(Z) = (\nabla_Y A_1)(Z) - A_1(Z)A_1(Y). \tag{14}$$

Hence it follows that

$$\alpha(Y, Z) = (\widetilde{\nabla}_Y A_1)(Z). \tag{15}$$

Now from (4) we have for $(GWS)_n$

$$\begin{aligned} &\left(\widetilde{\nabla}_X \widetilde{R}\right)(Y, Z, U, V) \\ &= A_1(X)\widetilde{R}(Y, Z, U, V) + B_1(Y)\widetilde{R}(X, Z, U, V) \\ &+B_1(Z)\widetilde{R}(Y, X, U, V) + B_1(U)\widetilde{R}(Y, Z, X, V) \\ &+B_1(V)\widetilde{R}(Y, Z, U, X) + A_2(X)G(Y, Z, U, V) \\ &+B_2(Y)G(X, Z, U, V) + B_2(Z)G(Y, X, U, V) \\ &+B_2(U)G(Y, Z, X, V) + B_2(V)G(Y, Z, U, X). \end{aligned} \tag{16}$$

Now using (11) we get

$$\widetilde{R}(Y, Z)U = R(Y, Z)U + \alpha(Y, U)Z - \alpha(Z, U)Y. \tag{17}$$

Therefore using (17) in (16) we get

$$\begin{aligned}\left(\widetilde{\nabla}_X\widetilde{R}\right)(Y,Z,U,V) &= (\nabla_X R)(Y,Z,U,V)\\ &+A_1(X)[\alpha(Y,U)g(Z,V)-\alpha(Z,U)g(Y,V)]\\ &+B_1(Y)[\alpha(X,U)g(Z,V)-\alpha(Z,U)g(X,V)]\\ &+B_1(Z)[\alpha(Y,U)g(X,V)-\alpha(X,U)g(Y,V)]\\ &+B_1(U)[\alpha(Y,X)g(Z,V)-\alpha(Z,X)g(Y,V)]\\ &+B_1(V)[\alpha(Y,U)g(Z,X)-\alpha(Z,U)g(Y,X)]. \qquad (18)\end{aligned}$$

Proposition 1. *If the vector field associated to the 1-form A_1 is reccurent, then $\alpha(Y,U)=0$.*

Theorem 1. *If the 1-forms A_1, B_1, D_1 satisfies the relation*

$$A_1(X_{A1})+3B_1(X_{A1})=0 \qquad (19)$$

where X_{A1} is a vector field associated to the 1-form A_1, then $[(GWS)_n,\widetilde{\nabla}]$ reduces to $[(GWS)_n,\nabla]$.

Proof. From 18, $[(GWS)_n,\widetilde{\nabla}]$ and $[(GWS)_n,\nabla]$ will be equivalent if the following relation holds:

$$\begin{aligned}&A_1(X)[\alpha(Y,U)g(Z,V)-\alpha(Z,U)g(Y,V)]\\ &+B_1(Y)[\alpha(X,U)g(Z,V)-\alpha(Z,U)g(X,V)]\\ &+B_1(Z)[\alpha(Y,U)g(X,V)-\alpha(X,U)g(Y,V)]\\ &+B_1(U)[\alpha(Y,X)g(Z,V)-\alpha(Z,X)g(Y,V)]\\ &+B_1(V)[\alpha(Y,U)g(Z,X)-\alpha(Z,U)g(Y,X)]\\ =\ &0. \qquad (20)\end{aligned}$$

Substituing $X=Z=U=V=\gamma_1$ in 20 we have

$$[A_1(X_{A1})+3B_1(X_{A1})][\alpha(Y,X)g(X,X)-\alpha(X,X)g(Y,X)]=0. \qquad (21)$$

Thus it is clear from 21 that $[(GWS)_n,\widetilde{\nabla}]$ and $[(GWS)_n,\nabla]$ are equivalent if (19) holds.

Next, contracting (16) and (17) we have

$$\begin{aligned}(\widetilde{\nabla}_X\widetilde{S})(Z,U) &= A_1(X)\widetilde{S}(Z,U)+B_1(\widetilde{R}(X,Z)U)\\ &+B_1(Z)\widetilde{S}(X,U)+B_1(U)\widetilde{S}(Z,X)\\ &-B_1(\widetilde{R}(U,X)Z)+A_2(X)(n-1)g(Z,U)\\ &+B_2(X)g(Z,U)+(n-2)B_2(Z)g(X,U)\\ &+B_2(X)g(Z,U)+(n-2)B_2(U)g(Z,X). \qquad (22)\end{aligned}$$

$$\widetilde{S}(Y,Z) = S(Y,Z) - (n-1)\alpha(Y,Z). \tag{23}$$

Substituting $Z = U = X_{A1}$ we obtain

$$\begin{aligned}(\widetilde{\nabla}_X \widetilde{S})(X_{A1}, X_{A1}) \;=\;& A_1(X)\widetilde{S}(X_{A1}, X_{A1}) + 2B_1(\widetilde{R}(X, X_{A1})X_{A1}) \\ &+2B_1(X_{A1})\widetilde{S}(X, X_{A1}) + A_2(X)(n-1)g(X_{A1}, X_{A1}) \\ &+2B_2(X)g(X_{A1}, X_{A1}) + 2(n-2)B_2(X_{A1})g(X, X_{A1}) \end{aligned} \tag{24}$$

Now using (17) and (23) we have

$$\begin{aligned}(\widetilde{\nabla}_X \widetilde{S})(X_{A1}, X_{A1}) \;=\;& A_1(X)[S(X_{A1}, X_{A1}) - (n-1)\alpha(X_{A1}, X_{A1})] \\ &+2B_1[R(X, X_{A1})X_{A1} + \alpha(X, X_{A1})X_{A1} - \alpha(X_{A1}, X_{A1})X] \\ &+2B_1(X_{A1})[S(X, X_{A1}) - (n-1)\alpha(X, X_{A1})] \\ &+[A_2(X)(n-1) + 2B_2(X)]g(X_{A1}, X_{A1}) \\ &+2(n-2)B_2(X_{A1})g(X, X_{A1}) \end{aligned} \tag{25}$$

which by using (9) reduced to

$$\begin{aligned}(\widetilde{\nabla}_X \widetilde{S})(X_{A1}, X_{A1}) \;=\;& (\nabla_X S)(X_{A1}, X_{A1}) - [(n-1)A_1(X) + 2B_1(X)]\alpha(X_{A1}, X_{A1}) \\ &-2(n-2)B_1(X_{A1})\alpha(X, X_{A1}). \end{aligned} \tag{26}$$

Theorem 2. *In $(GWS)_n (n > 3)$ with 1-forms $(A_1, B_1, B_1, A_2, B_2, B_2)$ the covariant derivative with respect to the non-metric connection $\widetilde{\nabla}$ of the associated Ricci tensor $\widetilde{S}$ satisfy the relation (26).*

The subject of this section is about a Riemannian manifold admitting a semi-symmetric non-metric connection whose torsion tensor T is given by (5) and whose curvature tensor $\widetilde{R}$ and the torsion tensor T satisfy (6) and (7) respectively. Then, from (5), we get by contracting over X

$$(C_1^1 T)(Y) = (n-1)A_1(Y). \tag{27}$$

From (27), it follows that

$$(\widetilde{\nabla}_X C_1^1 T)(Y) = (n-1)(\widetilde{\nabla}_X A_1)(Y). \tag{28}$$

Now contracting (7) and using (27) we have:

$$(\widetilde{\nabla}_X C_1^1 T)(Z) = (n-1)A_1(X)A_1(Z) + (2n-3)A_1(Z)B_1(X) - (n-2)A_1(X)B_1(Z). \tag{29}$$

Now using (28) in (29) we get

$$(\widetilde{\nabla}_X A_1)(Z) = A_1(X)A_1(Z) + \frac{(2n-3)}{n-1}A_1(Z)B_1(X) - \frac{(n-2)}{n-1}A_1(X)B_1(Z). \tag{30}$$

Hence from (15) it follows that

$$\alpha(X,Z) = A_1(X)A_1(Z) + \frac{(2n-3)}{n-1}A_1(Z)B_1(X) - \frac{(n-2)}{n-1}A_1(X)B_1(Z). \tag{31}$$

Now from (12) we have

$$\widetilde{R}(X,Y)Z = R(X,Y)Z + \alpha(X,Z)Y - \alpha(Y,Z)X. \tag{32}$$

Since $\widetilde{R}(X,Y)Z = 0$ so from (32) we get

$$R(X,Y)Z = \alpha(Y,Z)X - \alpha(X,Z)Y. \tag{33}$$

On account of the relation (31) we can write (33) as

$$\begin{aligned} R(X,Y)Z &= [A_1(Y)A_1(Z) + \frac{(2n-3)}{n-1}A_1(Z)B_1(Y) - \frac{(n-2)}{n-1}A_1(Y)B_1(Z)]X \\ &\quad -[A_1(X)A_1(Z) + \frac{(2n-3)}{n-1}A_1(Z)B_1(X) - \frac{(n-2)}{n-1}A_1(X)B_1(Z)]Y. \end{aligned} \tag{34}$$

Now contracting (34) we obtain

$$S(Y,Z) = -(n-1)A_1(Y)A_1(Z) - (2n-3)A_1(Z)B_1(Y) + (n-2)A_1(Y)B_1(Z). \tag{35}$$

Again contracting (35), we get a scalar curvature as

$$\begin{aligned} r &= -(n-1)A_1(X_{A1}) - (2n-3)A_1(X_{B1}) + (n-2)A_1(X_{B1}) \\ &= -(n-1)A_1(X_{A1} + X_{B1}) \end{aligned} \tag{36}$$

where γ_1 and β_1 are vector fields defined by (4)
Now using the symmetric property of Ricci tensor S we have from (35) we have

$$S(Y,Z) - S(Z,Y) = 0$$

$$(n-1)[A_1(Y)B_1(Z) - A_1(Z)B_1(Y)] = 0$$

$$A_1(Y)B_1(Z) = A_1(Z)B_1(Y). \tag{37}$$

Therefore it follows that

$$A_1(X) = \sigma B_1(X) \tag{38}$$

where σ is a non-zero scalar function.By (38), (34) can be written as

$$\begin{aligned} R(X,Y)Z &= [\sigma^2 B_1(Y)B_1(Z) + \frac{(2n-3)}{n-1}\sigma B_1(Z)B_1(Y) - \frac{(n-2)}{n-1}\sigma B_1(Y)B_1(Z)]X \\ &\quad -[\sigma^2 B_1(X)B_1(Z) + \frac{(2n-3)}{n-1}\sigma B_1(Z)B_1(X) \\ &\quad -\frac{(n-2)}{n-1}\sigma B_1(X)B_1(Z)]Y. \end{aligned} \tag{39}$$

Hence, we have

$$\begin{aligned} R(X,Y,Z,U) &= [\sigma^2 B_1(Y)B_1(Z) + \frac{(2n-3)}{n-1}\sigma B_1(Z)B_1(Y) - \frac{(n-2)}{n-1}\sigma B_1(Y)B_1(Z)]g(X,U) \\ &\quad -[\sigma^2 B_1(X)B_1(Z) + \frac{(2n-3)}{n-1}\sigma B_1(Z)B_1(X) - \frac{(n-2)}{n-1}\sigma B_1(X)B_1(Z)]g(Y,U). \end{aligned}$$

Putting $U = X_{B1}$ we have

$$\begin{aligned} R(X,Y,Z,X_{B1}) &= \sigma B_1(X)B_1(Y)B_1(Z)[\sigma + \frac{(2n-3)}{n-1} - \frac{n-2}{n-1}] \\ &\quad -\sigma B_1(X)B_1(Y)B_1(Z)[\sigma + \frac{(2n-3)}{n-1} - \frac{n-2}{n-1}] \\ &= 0. \end{aligned} \tag{40}$$

Hence we get from (40) we get

$$B_1(R(X,Y)Z) = 0. \tag{41}$$

Substituing (41) in (9) we get

$$\begin{aligned} (\nabla_X S)(Z,U) &= A_1(X)S(Z,U) + B_1(Z)S(X,U) + B_1(U)S(Z,X) \\ &\quad +[(n-1)A_2(X) + 2B_2(X)]g(Z,U) \\ &\quad +(n-2)B_2(Z)g(X,U) + (n-2)B_2(U)g(Z,X) \\ &= A_1(X)S(Z,U) + B_1(Z)S(X,U) + B_1(U)S(Z,X) \\ &\quad +A^2(X)g(Z,U) + B^2(Z)g(X,U) + B^2(U)g(Z,X) \end{aligned} \tag{42}$$

where $A^2(X) = (n-1)A_2(X) + 2B_2(X), B^2(Z) = (n-2)B_2(Z), B^2(U) = (n-2)B_2(U)$.

Theorem 3. *Every $(GWS)_n$ with $1-forms$ (A_1, B_1, A_2, B_2) reduces to $(GWRS)_n$ with $1-forms$ $(A_1, B_1, D_1, A^2, B^2, D^2)$ under the non metric connection $\widetilde{\nabla}$ given by (11) and the curvature tensor $\widetilde{R}$ and torsion tensor T are given by (6) and (7) respectively.*

Also due to (38) we have

$$B_1(X) = \psi A_1(X). \tag{43}$$

Again using (43) in (30) we get

$$\begin{aligned} (\widetilde{\nabla}_X A_1)(Z) &= A_1(X)A_1(Z) + \frac{(2n-3)}{n-1}A_1(Z)B_1(X) - \frac{(n-2)}{n-1}A_1(X)B_1(Z) \\ &= [1 + \frac{2n-3}{n-1}\psi - \frac{n-2}{n-1}\psi]A_1(X)A_1(Z) \\ &= [1+\psi]A_1(X)A_1(Z). \end{aligned} \tag{44}$$

Hence from (44) it is observed that the associated 1-form A_1 is closed and thus it also follows that the 1-form B_1 is closed. Agashe and Chafle [2] proved that if a Riemannian manifold $(M^n, g)(n > 3)$ admits a semi symmetric non-metric connection whose curvature tensor vanishes, then the manifold is projectively flat and hence a manifold of constant scalar curvature r is non-zero. Moreover as A_1 is non-zero so from (36), the scalar curvature r is non-zero. Assembling all these, we state the following theorem:

Theorem 4. *If a $(GWS)_n(n > 3)$ admits a semi-symmetric non-metric connection whose torsion tensor T is given by (5) and whose curvature tensor $\widetilde{R}$ and torsion tensor T satisfy (6) and (7) respectively, then the manifold is of constant scalar curvature whose associated 1-forms A_1, B_1 are closed.*

Again from (17), using Bianchi's 1st identity we can write

$$\begin{aligned}
& \widetilde{R}(Y,Z)U + \widetilde{R}(Z,U)Y + \widetilde{R}(U,Y)Z \\
= & [\alpha(Y,U)Z - \alpha(Z,U)Y] + [\alpha(Z,Y)U - \alpha(U,Y)Z] + [\alpha(U,Z)Y - \alpha(Y,Z)U] \\
= & [\alpha(Y,U)Z - \alpha(U,Y)Z] + [\alpha(Z,Y)U - \alpha(Y,Z)U] + [\alpha(U,Z)Y - \alpha(Z,U)Y] \\
= & dA_1(Y,U)Z + dA_1(Z,Y)U + dA_1(U,Z)Y. \qquad (45)
\end{aligned}$$

Now since by the above theorem A_1 is closed so $dA_1 = 0$ and thus we arrive at the following corollary

Corollary 5. *If a $(GWS)_n(n > 3)$ admits a semi-symmetric non-metric connection whose torsion tensor T is given by (5) and whose curvature tensor $\widetilde{R}$ and torsion tensor T satisfy (6) and (7) respectively, then the curvature tensor $\widetilde{R}$ satisfy Bianchi's 1st identity i.e. $\widetilde{R}(Y,Z)U + \widetilde{R}(Z,U)Y + \widetilde{R}(U,Y)Z = 0$.*

3. Special conformally flat $(GWS)_n(n > 3)$ and the case of a special type of semi-symmetric non-metric connection

The notion of a special conformally flat manifold which generalises the concept subprojective manifold was studied by Chen and Yano [[3]]. A conformally flat manifold is called a special conformally flat manifold if the tensor K of type $(0, 2)$ defined by

$$K(X,Y) = -\frac{1}{n-2}S(X,Y) + \frac{r}{2(n-1)(n-2)}g(X,Y) \qquad (46)$$

can be expresssed in the form

$$K(X,Y) = -\frac{a^2}{2}g(X,Y) + b(\nabla_X a)(\nabla_Y a). \qquad (47)$$

where a and b are two scalars such that a is positive. Moreover, if b is a function a then the special conformally flat manifold is called a subprojective manifold [[4]].

Let us consider a $(GWS)_n (n > 3)$ admitting a semi-symmetric non-metric connection whose torsion tensor T is given by (5) and whose curvature tensor $\widetilde{R}$ and torsion tensor T satisfy (6) and (7) respectively.

Using (35) and (43) in (46) we have

$$K(X,Y) = \frac{r}{2(n-1)(n-2)} g(X,Y) + \frac{n-1}{n-2}(1+\psi)A_1(X)A_1(Y). \tag{48}$$

Thus, we have

$$a^2 = -\frac{r}{(n-1)(n-2)}. \tag{49}$$

Since, $r \neq 0$ (36), so a^2 is positive if $r < 0$.

Using (4) and (8) in (10) we find

$$X(r) = rA_1(X) + 4B_1(LX_{B1}) + n(n-1)[A_2(X) + 4B_2(X)]. \tag{50}$$

Again using (41) in (50) we get

$$X(r) = rA_1(X) + n(n-1)[A_2(X) + 4B_2(X)]. \tag{51}$$

Now taking the covariant derivative on both sides of (49) with respect to X and using (51) we have

$$\begin{aligned} \nabla_X a &= -\frac{rA_1(X) + n(n-1)[A_2(X) + 4B_2(X)]}{2a(n-1)(n-2)} \\ &= -\frac{rA_1(X)}{2a(n-1)(n-2)} - \frac{n}{2a(n-2)}[A_2(X) + 4B_2(X)] \end{aligned} \tag{52}$$

which yields

$$A_1(X) = -\frac{2a(n-1)(n-2)}{r}\nabla_X a. \tag{53}$$

for $A_2(X) + 4B_2(X) = 0$.Thus due to (49) and (53), we can express (48) as

$$K(X,Y) = -\frac{a^2}{2} g(X,Y) + \frac{4a^2(n-1)^3(n-2)}{r^2}(1+\psi)(\nabla_X a)(\nabla_Y a). \tag{54}$$

Setting

$$b = \frac{4a^2(n-1)^3(n-2)}{r^2}(1+\psi) \tag{55}$$

in (54) we see that (48) can be written as (47).

Thus we can conclude that the $(GWS)_n (n > 3)$ under the above mentioned conditions is a special conformally flat manifold provided $A_2(X) + 4B_2(X) = 0$. Also by (55), b being a function of a, the considered manifold is a subprojective manifold provided $A_2(X) + 4B_2(X) = 0$. Hence, we state the following theorem.

Theorem 6. *If a $(GWS)_n (n > 3)$ admits a semi-symmetric non-metric connection whose torsion tensor T is given by (5) and whose curvature tensor $\widetilde{R}$ and torsion tensor T satisfy (6) and (7) respectively, then the manifold is a subprojective manifold provided $A_2(X) + 4B_2(X) = 0$.*

Again, we recall the [Corollary 1] of [3] which states that every simply connected subprojective space can be isometrically immersed in a Euclidean space as a hypersurface. Thus, using this corollary, we conclude the following theorem.

Theorem 7. *Let $(GWS)_n(n > 3)$ be a simply connected space bearing a semi-symmetric non-metric connection with torsion tensor T given by (5). If for such space the curvature tensor $\widetilde{R}$ and torsion tensor T satisfy (6) and (7) respectively, then it can be isometrically immersed in Euclidean space as a hypersurface, provided $A_2(X) + 4B_2(X) = 0$.*

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