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ON A NEW CLASS OF VOLTERRA-FREDHOLM FRACTIONAL INTEGRAL EQUATIONS

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Abstract. This manuscript's main objective is to examine the uniqueness and estimates of the Atangana-Baleanu (AB) fractional mixed Volterra-Fredholm integral equations under various types of contraction conditions have been investigated in the context of special spaces. An interesting example is given to demonstrate the rationality and superiority of obtained results.

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1. Introduction

Recently, many scientists have applied fractional derivatives with different types of definitions, such as Atangana-Baleanu fractional integral [6], Caputo fractional derivative [8], and Caputo-Fabrizio fractional derivative [9], to many real-world problems and pointed-out the powerfulness of using such noninteger-order and nonlocal kernels to numerically solve different types of integral equations and to describe the dynamics and properties of these problems; see, for example, [1, 2, 7, 26, 31].

Gronwall inequality plays very important role in studying the various properties such as estimates of solution, continuous dependence and others of differential equation. Recently in [5, 10, 12, 33] the authors have obtained the fractional Gronwall inequality using various fractional definition.

One of these problems is studying numerical solution of the nonlinear Volterra-Fredholm integral equations by involving the well-known Atangana-Baleanu fractional derivative. Note that nonlinear Volterra-Fredholm integral equations appear in many applications in different disciplines such as neural networks [21], the pulses of sound re-flections [27], and mathematical physics such as Lane-Emden-type equations [32]. The existence, uniqueness and numerical solution have been obtained in [3, 5, 11, 13, 14, 15, 16, 17, 22, 23, 24, 25, 29].

Motivated by the previous efforts, we investigate the uniqueness, estimates and continuous dependent of the solution of a new kind of nonlinear Volterra-Fredholm fractional equations involving Caputo on Atangana-Baleanu fractional derivatives of different orders which has the form:

$$x(t) = \mathfrak{f}(t, x(t), {}_a^t I_{AB}^{\alpha} k(t, \tau, x(\tau)), {}_a^b I_{AB}^{\alpha} h(t, \tau, x(\tau))), \tag{1}$$

for $0 < \alpha < 1$, where $I = [a, b], k, h \in C(I \times I \times \mathbb{R}, \mathbb{R})$ and $\mathfrak{f} \in C(I \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

The main purpose of this work is to establish some fundamental properties of solutions of (1). The well known Banach fixed point theorem with Bielecki type norm and Gronwall type inequality given in [28] is used for presenting the results.

2. Preliminaries

Here we introduce some notations, main definitions, and theorems which are crucial in what follows [4, 5, 18, 19, 20].

The left Riemann-Liouville fractional integral of order α for $\alpha > 0$ is defined as [30]:

$$\left({}_a I^{\alpha} x\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} x(s) ds.$$

Definition 1. *[4] Let $x \in H^1(a, b), a < b$ and α in $[0, 1]$. The Caputo Atangana-Baleanu (ABC) fractional derivative of x of order α is defined by*

$$\left(\begin{matrix} ABC \\ a \end{matrix} D^{\alpha} x \right)(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t x'(s) E_{\alpha} \left[-\alpha \frac{(t-s)^{\alpha}}{1-\alpha} \right] ds,$$

where E_{α} is the Mittag-Leffler function defined by $E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\ n\ \alpha+1)}$ and $B(\alpha)$ is a normalizing positive function satisfying $B(0) = B(1) = 1$.

The Riemann Atangana-Baleanu fractional derivative of x of order α is defined by

$$\left(\begin{matrix} ABR \\ a \end{matrix} D^{\alpha} x \right)(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t x(s) E_{\alpha} \left[-\alpha \frac{(t-s)^{\alpha}}{1-\alpha} \right] ds.$$

The associated fractional integral is defined by:

$$\left({}_a^{AB} I^{\alpha} x\right)(t) = \frac{1-\alpha}{B(\alpha)} x(t) + \frac{\alpha}{B(\alpha)} \left(\left({}_a I^{\alpha} x\right)(t) \right),$$

where ${}_a I^{\alpha}$ is the left Riemann-Liouville fractional integral.

Now we construct the appropriate metric space. Let $\xi > 0$ be a constant and consider the space of all continuous function $C(I, \mathbb{R})$ where $I = [a, b]$. We denote this special space by $C_{\xi,\alpha}(I, \mathbb{R})$

$$d_{\xi,\alpha}(u, v) = \mathrm{Sup}_{t\in I} \frac{|u(t) - v(t)|}{E_\alpha\left(\xi(t-a)^\alpha\right)},$$

with norm defined by

$$|u|_{\xi,\alpha} = \mathrm{Sup}_{t\in I} \frac{|u(t)|}{E_\alpha\left(\xi(t-a)^\alpha\right)}$$

where $E_\alpha : \mathbb{R} \to \mathbb{R}$ is a one parameter Mittag-Leffler function. The above definitions $d_{\xi,\alpha}$ and $|\cdot|_{\xi,\alpha}$ are the variants of Bielecki's metric and norm.

The Gronwall inequality in the frame of fractional integrals associated with the A-B fractional derivative is given in [5, 28] as follows:

Theorem 1. *[28] Suppose that* $\alpha > 0, c(t)\left(1 - \frac{1-\alpha}{B(\alpha)}d(t)\right)^{-1}$ *is nonnegative, nondecreasing and locally integrable function on* $[a, b)$, $\frac{\alpha d(t)}{B(\alpha)}\left(1 - \frac{1-\alpha}{B(\alpha)}d(t)\right)^{-1}$ *is nonnegative and bounded on* $[a, b)$ *and* $x(t)$ *is nonnegative and locally integrable on* $[a, b)$ *with*

$$x(t) \le c(t) + d(t)\left({}^{AB}_{a}I^\alpha x\right)(t).$$

Then

$$x(t) \le \frac{c(t)B(\alpha)}{B(\alpha) - (1-\alpha)d(t)} E_\alpha\left(\frac{\alpha d(t)(t-a)^\alpha}{B(\alpha) - (1-\alpha)d(t)}\right).$$

3. Main results

3.1. Uniqueness of Solution

In the first part of this section, we will use the Banach fixed point theorem combined with the obtained Gronwall's inequality to prove the uniqueness of solution of the equation (1).

Theorem 2. *Let* $P > 0, Q \ge 0, \xi > 1$ *be constants. Suppose that the functions* f, k *in* (1) *satisfy the conditions*

$$|\mathfrak{f}(t, u, v, y) - \mathfrak{f}(t, \bar{u}, \bar{v}, \bar{y})| \le Q[|u - \bar{u}| + |v - \bar{v}| + |y - \bar{y}|] \tag{2}$$

$$|k(t, s, u) - k(t, s, \bar{u})| \le P_1|u - \bar{u}|, \ |h(t, s, u) - h(t, s, \bar{u})| \le P_2|u - \bar{u}| \tag{3}$$

and

$$m_1 = \sup_{t\in I} \frac{1}{E_\alpha\left(\xi(t-a)^\alpha\right)} \left|\mathfrak{f}\left(t, 0, {}^{t}_{a}I^\alpha_{AB}k(t, s, 0), {}^{b}_{a}I^\alpha_{AB}h(t, s, 0)\right)\right| < \infty. \tag{4}$$

If $Q\left(1+\frac{P}{\xi}\right)<1$ *then the integral equation* (1) *has a unique solution* $x\in C_{\xi,\alpha}(I,\mathbb{R})$

Proof. Consider the equivalent formulation of (1) we have

$$\begin{aligned}x(t)=\mathfrak{f}\left(t,x(t),{}_{a}^{t}I_{AB}^{\alpha}k(t,\tau,x(\tau)),{}_{a}^{b}I_{AB}^{\alpha}h(t,\tau,x(\tau))\right)-\mathfrak{f}\left(t,0,{}_{a}^{t}I_{AB}^{\alpha}k(t,\tau,0),{}_{a}^{b}I_{AB}^{\alpha}h(t,\tau,0)\right)\\ +\mathfrak{f}\left(t,0,{}_{a}^{t}I_{AB}^{\alpha}k(t,\tau,0),{}_{a}^{b}I_{AB}^{\alpha}h(t,\tau,0)\right),\end{aligned}\tag{5}$$

for $t\in I$. We shall show that (5) has unique solution and thus (1) must also have unique solution. Let $x\in C_{\xi,\alpha}(I,\mathbb{R})$ and define the operator $\mathfrak{T}$ by

$$\begin{aligned}(\mathfrak{T}x)(t)=\mathfrak{f}\left(t,x(t),{}_{a}^{t}I_{AB}^{\alpha}k(t,\tau,x(\tau)),{}_{a}^{b}I_{AB}^{\alpha}h(t,\tau,x(\tau))\right)-\mathfrak{f}\left(t,0,{}_{a}^{t}I_{AB}^{\alpha}k(t,\tau,0),{}_{a}^{b}I_{AB}^{\alpha}h(t,\tau,0)\right)\\ +\mathfrak{f}\left(t,0,{}_{a}^{t}I_{AB}^{\alpha}k(t,\tau,0),{}_{a}^{b}I_{AB}^{\alpha}h(t,\tau,0)\right).\end{aligned}\tag{6}$$

Now we show that $\mathfrak{T}$ maps $C_{\xi,\alpha}(I,\mathbb{R})$ into itself. We have

$$\begin{aligned}|\mathfrak{T}x|_{\xi,\alpha}&=\mathrm{Sup}_{t\in I}\,\frac{|(\mathfrak{T}x)(t)|}{E_\alpha\left(\xi(t-a)^\alpha\right)}\\
&\le\sup_{t\in I}\frac{1}{E_\alpha\left(\xi(t-a)^\alpha\right)}\Big|\mathfrak{f}\left(t,x(t),{}_{a}^{t}I_{AB}^{\alpha}k(t,\tau,x(\tau)),{}_{a}^{b}I_{AB}^{\alpha}h(t,\tau,x(\tau))\right)\\
&-\mathfrak{f}\left(t,0,{}_{a}^{t}I_{AB}^{\alpha}k(t,\tau,0),{}_{a}^{b}I_{AB}^{\alpha}h(t,\tau,0)\right)\Big|\\
&+\sup_{t\in I}\frac{1}{E_\alpha\left(\xi(t-a)^\alpha\right)}\Big|\mathfrak{f}\left(t,0,{}_{a}^{t}I_{AB}^{\alpha}k(t,\tau,0),{}_{a}^{b}I_{AB}^{\alpha}h(t,\tau,0)\right)\Big|\\
&\le m_1+\sup_{t\in I}\frac{1}{E_\alpha\left(\xi(t-a)^\alpha\right)}Q\left[|x(t)|+{}_{a}^{t}I_{AB}^{\alpha}P_1|x(\tau)|+{}_{a}^{b}I_{AB}^{\alpha}P_2|x(\tau)|\right]\\
&=m_1+Q\Big[\sup_{t\in I}\frac{|x(t)|}{E_\alpha\left(\xi(t-a)^\alpha\right)}+P_1\sup_{t\in I}\frac{1}{E_\alpha\left(\xi(t-a)^\alpha\right)}{}_{a}^{t}I_{AB}^{\alpha}E_\alpha\left(\xi(t-a)^\alpha\right)\frac{|x(\tau)|}{E_\alpha\left(\xi(t-a)^\alpha\right)}\\
&+P_2\sup_{t\in I}\frac{1}{E_\alpha\left(\xi(t-a)^\alpha\right)}{}_{a}^{b}I_{AB}^{\alpha}E_\alpha\left(\xi(t-a)^\alpha\right)\frac{|x(\tau)|}{E_\alpha\left(\xi(t-a)^\alpha\right)}\Big]\\
&\le m_1+Q\Big[|x|_{\xi,\alpha}+P_1|x|_{\xi,\alpha}\sup_{t\in I}\frac{1}{E_\alpha\left(\xi(t-a)^\alpha\right)^a}{}_{a}^{t}I_{AB}^{\alpha}E_\alpha\left(\xi(t-a)^\alpha\right)\\
&+P_2|x|_{\xi,\alpha}\sup_{t\in I}\frac{1}{E_\alpha\left(\xi(t-a)^\alpha\right)^a}{}_{a}^{b}I_{AB}^{\alpha}E_\alpha\left(\xi(t-a)^\alpha\right)\Big]\\
&\le m_1+Q\Big[|x|_{\xi,\alpha}+P_1|x|_{\xi,\alpha}\sup_{t\in I}\frac{1}{E_\alpha\left(\xi(t-a)^\alpha\right)}+P_2|x|_{\xi,\alpha}\sup_{t\in I}\frac{1}{E_\alpha\left(\xi(t-a)^\alpha\right)}\\
&\left(\frac{(1-\alpha)}{B(\alpha)}E_\alpha\left(\xi(t-a)^\alpha\right)+\frac{\alpha}{B(\alpha)}{}_{a}^{t}I_{AB}^{\alpha}E_\alpha\left(\xi(t-a)^\alpha\right)+\frac{\alpha}{B(\alpha)}{}_{a}^{b}I_{AB}^{\alpha}E_\alpha\left(\xi(t-a)^\alpha\right)\right)\Big]\end{aligned}$$

$$\leq m_1 + Q\left[|x|_{\xi,\alpha} + P_1|x|_{\xi,\alpha}\sup_{t\in I}\frac{1}{E_\alpha\left(\xi(t-a)^\alpha\right)} + P_2|x|_{\xi,\alpha}\sup_{t\in I}\frac{1}{E_\alpha\left(\xi(t-a)^\alpha\right)}\right.$$
$$\left.\left(\frac{(1-\alpha)}{B(\alpha)}E_\alpha\left(\xi(t-a)^\alpha\right) + \frac{2\alpha}{B(\alpha)}\frac{E_\alpha\left(\xi(t-a)^\alpha\right)-1}{\xi}\right)\right] \tag{7}$$
$$\leq m_1 + Q|x|_{\xi,\alpha}\left[1 + [P_1\frac{1}{B(\alpha)} + P_2\frac{1}{B(\alpha)}]\left((1-\alpha) + \frac{2\alpha}{\xi}\right)\right]$$

Now we show that the operator $\mathfrak{T}$ is a contraction map. Let $v, w \in C_{\xi,\alpha}(I,\mathbb{R})$, from the hypotheses we have

$$d_{\xi,\alpha}(\mathfrak{T}v,\mathfrak{T}w)$$
$$= \sup_{t\in I}\frac{|(\mathfrak{T}v)(t)-(\mathfrak{T}w)(t)|}{E_\alpha\left(\xi(t-a)^\alpha\right)}$$
$$= \sup_{t\in I}\frac{1}{E_\alpha\left(\xi(t-a)^\alpha\right)}\mid f\left(t, v(t), {}_a^tI_{AB}^\alpha k(t,\tau,v(\tau)), {}_a^bI_{AB}^\alpha h(t,\tau,v(\tau))\right)$$
$$-f\left(t, w(t), {}_a^tI_{AB}^\alpha k(t,\tau,w(\tau)), {}_a^bI_{AB}^\alpha h(t,\tau,w(\tau))\right)]$$
$$\leq \sup_{t\in I}\frac{1}{E_\alpha\left(\xi(t-a)^\alpha\right)}Q\left[|v(t)-w(t)| + {}_a^tI_{AB}^\alpha P_1|v(\tau)-w(\tau)| + {}_a^bI_{AB}^\alpha P_2|v(\tau)-w(\tau)|\right]$$
$$\leq Q\Big[\sup_{t\in I}\frac{|v(t)-w(t)|}{E_\alpha\left(\xi(t-a)^\alpha\right)} + \sup_{t\in I}\frac{1}{E_\alpha\left(\xi(t-a)^\alpha\right)}P_1{}_a^tI_{AB}^\alpha E_\alpha\left(\xi(t-a)^\alpha\right)\frac{|v(\tau)-w(\tau)|}{E_\alpha\left(\xi(t-a)^\alpha\right)}$$
$$+\sup_{t\in I}\frac{1}{E_\alpha\left(\xi(t-a)^\alpha\right)}P_2{}_a^bI_{AB}^\alpha E_\alpha\left(\xi(t-a)^\alpha\right)\frac{|v(\tau)-w(\tau)|}{E_\alpha\left(\xi(t-a)^\alpha\right)}\Big]$$
$$\leq Q\Big[d_{\xi,\alpha}(v,w) + P_1 d_{\xi,\alpha}(v,w)\sup_{t\in I}\frac{E_\alpha\left(\xi(t-a)^\alpha\right)}{a}{}_a^tI_{AB}^\alpha E_\alpha\left(\xi(t-a)^\alpha\right)$$
$$+P_2 d_{\xi,\alpha}(v,w)\sup_{t\in I}\frac{E_\alpha\left(\xi(t-a)^\alpha\right)}{a}{}_a^bI_{AB}^\alpha E_\alpha\left(\xi(t-a)^\alpha\right)\Big]$$
$$= Qd_{\xi,\alpha}(v,w)\left[1 + (P_1+P_2)\left[d_{\xi,\alpha}(v,w)\sup_{t\in I}\frac{1}{E_\alpha\left(\xi(t-a)^\alpha\right)}\frac{1}{B(\alpha)}\right]\left[(1-\alpha)+\frac{2\alpha}{\xi}\right]\right.$$
$$= Q|x|d_{\xi,\alpha}(v,w)\frac{E_\alpha\left(\xi(t-a)^\alpha\right)-1}{\xi}. \tag{8}$$

It follows from Banach fixed point theorem $\mathfrak{T}$ has a unique fixed point.

3.2. Estimates of Solution

Now we obtain estimates on the solutions of equation (1) with some suitable assumptions

Theorem 3. *Suppose that the functions $\mathfrak{f}, k, h$ are continuous and satisfy the conditions*

$$|\mathfrak{f}(t,u,v,y) - \mathfrak{f}(t,\bar{u},\bar{v},\bar{y}))| \leq G[|u-\bar{u}| + |v-\bar{v}| + |y-\bar{y}|] \tag{9}$$
$$|k(t,\tau,u) - k(t,\tau,v)| \leq h_1(t)|u-v| \tag{10}$$
$$|h(t,\tau,u) - h(t,\tau,v)| \leq h_2(t)|u-v| \tag{11}$$

where $0 \le G < 1$ *is a constant*

$$m_2 = \underset{t\in I}{Sup}\left|\mathfrak{f}\left(t,0,{}_a^tI_{AB}^{\alpha}k(t,s,0),{}_a^bI_{AB}^{\alpha}h(t,s,0)\right)\right| < \infty \tag{12}$$

If $x(t)$ *is any solution of* (1) *and* $H_1(t) = \mathrm{Sup}_{t\in I}\, h_1(t)$, $H_2(t) = \mathrm{Sup}_{t\in I}\, h_2(t)$ *then*

$$\begin{aligned} &|x(t)| \\ \le\ & \frac{m_2}{(1-G)}\Big[\frac{B(\alpha)}{\left(B(\alpha)-(1-\alpha)\left(\frac{G}{1-G}\right)H_1(t)\right)}E_\alpha\left(\frac{\alpha\frac{G}{1-G}H_1(t)(t-a)^\alpha}{\left(B(\alpha)-(1-\alpha)\left(\frac{G}{1-G}\right)H_1(t)\right)}\right) \\ &+\frac{B(\alpha)}{\left(B(\alpha)-(1-\alpha)\left(\frac{G}{1-G}\right)H_2(t)\right)}E_\alpha\left(\frac{\alpha\frac{G}{1-G}H_2(t)(t-a)^\alpha}{\left(B(\alpha)-(1-\alpha)\left(\frac{G}{1-G}\right)H_2(t)\right)}\right)\Big] \end{aligned} \tag{13}$$

Proof. Since the solution $x(t)$ of equation (1) satisfies the equation (5) and the hypothesis we have

$$\begin{aligned} |x(t)| \le & \left|\mathfrak{f}\left(t,0,{}_a^tI_{AB}^{\alpha}k(t,\tau,0),{}_a^bI_{AB}^{\alpha}h(t,\tau,0)\right)\right| \\ &+\Big|\mathfrak{f}\left(t,x(t),{}_a^tI_{AB}^{\alpha}k(t,\tau,x(\tau)),{}_a^bI_{AB}^{\alpha}h(t,\tau,x(\tau))\right) \\ &-\mathfrak{f}\left(t,0,{}_a^tI_{AB}^{\alpha}k(t,\tau,0),{}_a^bI_{AB}^{\alpha}h(t,\tau,0)\right)\Big| \\ &\le m_2 + G\left[|x(t)| + {}_a^tI_{AB}^{\alpha}h_1(\tau)|x(\tau)| + {}_a^bI_{AB}^{\alpha}h_2(\tau)|x(\tau)|\right]. \end{aligned} \tag{14}$$

From (14) and hypotheses $0 \le G < 1$ we have

$$|x(t)| \le \frac{m_2}{(1-G)} + \frac{G}{1-G}H_1(t){}_a^tI_{AB}^{\alpha}|x(\tau)| + \frac{G}{1-G}H_2(t){}_a^bI_{AB}^{\alpha}|x(\tau)|. \tag{15}$$

Now applying the Gronwall inequality Theorem 1 to (15) we get (13).

4. An Example

Example. Consider the nonlinear ABC-fractional Volterra-Fredholm integral equation:

$$x(t) = \frac{1}{10(t+1)}x(t) + \frac{1}{10}{}_0^tI_{AB}^{\alpha}e^{-2t}x(\tau) + \frac{1}{10}{}_0^1I_{AB}^{\alpha}e^{-4t}x(\tau),\ t\in[0,1]. \tag{16}$$

Set

$$\begin{aligned} &f(t,x(\tau),\mathrm{X}x(\tau),\mathrm{Y}x(\tau)) = \frac{1}{10(t+1)}x(t) + \frac{1}{10}{}_0^tI_{AB}^{\alpha}e^{-2t}x(\tau) + \frac{1}{10}{}_0^1I_{AB}^{\alpha}e^{-4t}x(\tau), \\ &\mathrm{X}x(\tau) = {}_0^tI_{AB}^{\alpha}e^{-2t}x(\tau), \\ &\mathrm{Y}x(\tau) = {}_0^1I_{AB}^{\alpha}e^{-4t}x(\tau). \end{aligned}$$

From above it can be easy to see that

$$
\begin{aligned}
&|f(t,u(\tau),\mathrm{X}u(\tau),\mathrm{Y}u(\tau))-f(t,v(\tau),\mathrm{X}v(\tau),\mathrm{Y}v(\tau))|\\
&\leq \frac{1}{10}[|u-v|+|\mathrm{X}u-\mathrm{X}v|+|\mathrm{Y}u-\mathrm{Y}v|],\\
&|\mathrm{X}u-\mathrm{X}v|\leq \frac{1}{10e^{2t}a}{}_0^tI_{AB}^{\alpha}x(\tau)\\
&|\mathrm{Y}u-\mathrm{Y}v|\leq \frac{1}{10e^{4t}a}{}_0^1I_{AB}^{\alpha}x(\tau).
\end{aligned}
$$

Thus from above equation the conditions (2)-(3) holds we have $Q=\frac{1}{10}$ and $P_1=\frac{1}{10e^{2t}}, P_2=\frac{1}{10e^{4t}}$ then for $\xi=2$ we have

$$
Q\left(1+\frac{P_1+P_2}{\xi}\right)\cong 0.1007665<1.
$$

Thus the assumptions of the Theorem 2 are satisfied and thus the integral equation (16) has a unique solution.

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