(F, φ, ω) - GREGUS TYPE CONTRACTION CONDITION APPROACH TO φ-FIXED POINT RESULTS IN METRIC SPACES

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Abstract. In this paper, we introduce (F, φ, ω) - Gregus type contraction, (F, φ, ω) - weak Gregus type contraction condition mappings and establish results of φ - fixed point for such mappings. Our results generalize some results of [1] and [2]. To support our results we illustrate example with numerical experiment for approximating the φ - fixed point.

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1. Introduction

1.1 φ - fixed points and (F, φ) - contraction mappings:

Recently, Jleli et al.[1] introduced an interesting concept of φ - fixed points, φ - Picard mappings and weakly φ - Picard mappings as follows:

Let X be a nonempty set, φ : X → [0, ∞) be a given function and T : X → X be a mapping. We denote the set of all fixed points of T by \( F_T := \{ x \in X : Tx = x \} \) and denote the set of all zeros of the function φ by \( Z_\phi := \{ x \in X : \phi(x) = 0 \} \).

Definition 1. [1] An element \( z \in X \) is said to be a φ - fixed point of the operator T if and only if \( z \in F_T \cap Z_\phi \).

Definition 2. [1] The operator T is said

(1) to be a φ - Picard mapping if and only if

(i) \( F_T \cap Z_\phi = \{ z \} \),
(ii) \( T^n x \to z \) as \( n \to \infty \), for each \( x \in X \).

(2) to be a weakly φ - Picard mapping if and only if
(i) $T$ has at least one $\varphi$ - fixed point,

(ii) the sequence $\{T^n x\}$ converges for each $x \in X$, and the limit is a $\varphi$ - fixed point $T$.

Also, Jleli et.al [1] introduced a new type of control function $F : [0, \infty)^3 \to [0, \infty)$ satisfying the following conditions:

$(F_1)$ $\max\{a, b\} \leq F(a, b, c)$

$(F_2)$ $F(0, 0, 0) = 0$

$(F_3)$ $F$ is continuous

In this paper we are inserting the fourth condition as follows:

$(F_4)$ $F(0, b, c) \leq F(a, b, c)$.

Throughout this paper, the class of all functions satisfying the conditions $(F_1) - (F_4)$ is denoted by $\mathcal{F}$.

**Example 1.** Let $f_1, f_2, f_3 : [0, \infty)^3 \to [0, \infty)$ be defined by

$f_1(a, b, c) = a + b + c$,

$f_2(a, b, c) = \max\{a, b\} + c$,

$f_3(a, b, c) = a + a^2 + b + c$,

for all $a, b, c \in [0, \infty)$. Then $f_1, f_2, f_3 \in \mathcal{F}$.

**Definition 3.** [1] Let $(X, d)$ be a metric space and $\varphi : X \to [0, \infty)$ be a given function and $F \in \mathcal{F}$. We say that the mapping $T : X \to X$ is a $(F, \varphi)$ - contraction with respect to the metric $d$ if and only if for each $x, y \in X$ and for some constant $k \in (0, 1)$ such that

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq kF(d(x, y), \varphi(x), \varphi(y)).$$

(1)

**Theorem 1.** [1] Let $(X, d)$ be a complete metric space and $\varphi : X \to [0, \infty)$ be a given function and $F \in \mathcal{F}$. Suppose that the following condition holds:

(a) $\varphi$ is lower semi-continuous,

(b) $T : X \to X$ is a $(F, \varphi)$ - contraction with respect to the metric $d$.

Then the following assertions hold:

(i) $F_T \subseteq Z_{\varphi}$,

(ii) $T$ is a $\varphi$ - Picard mapping,

(iii) if $x \in X$ and for $n \in \mathbb{N}$, we have

$$d(T^n x, z) \leq \frac{k^n}{1 - k}F(d(Tx, x), \varphi(Tx), \varphi(x)) \text{ where } \{z\} = F_T \cap Z_{\varphi}.$$

Let $\Omega$ be the set of all functions $\omega : [0, \infty) \to [0, \infty)$ satisfying the following conditions:
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\((j_1)\) \(\omega\) is nondecreasing function,
\((j_2)\) \(\omega\) is continuous,
\((j_3)\) \(\lim_{n \to \infty} \omega^n(t) = 0, \quad \forall \ t \in (0, \infty)\)
\((j_4)\) \(\sum_{n=0}^{\infty} \omega^n(t) < \infty, \quad \forall \ t > 0\).

**Definition 4.** [2] Let \((X, d)\) be a metric space, \(\varphi : X \to [0, \infty)\) be a given function, \(F \in \mathcal{F}\) and \(\omega \in \Omega\). The mapping \(T : X \to X\) is said to be a \((F, \varphi, \omega)\) - contraction with respect to the metric \(d\) if and only if
\[
F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq \omega(F(d(x, y), \varphi(x), \varphi(y))) \quad \forall x, y \in X.
\]

**Theorem 2.** [2] Let \((X, d)\) be a metric space, \(\varphi : X \to [0, \infty)\) be a given function, \(F \in \mathcal{F}\) and \(\omega \in \Omega\). Assume that the following conditions are satisfied:
\((H_1)\) \(\varphi\) is lower semi continuous,
\((H_2)\) \(T : X \to X\) is an \((F, \varphi, \omega)\) - contraction with respect to the metric \(d\).

Then the following assertion hold:

\[(i)\] \(F_T \subseteq Z_\varphi,\)
\[(ii)\] \(T\) is a \(\varphi\) - Picard mapping.

**Lemma 3.** [2] If \(\omega \in \Omega\), then \(\omega(t) < t \quad \forall \ t > 0\).

**Remark 1.** [2] From \(j_1\) and Lemma 3, we have \(\omega(0) = 0\).

**The aim of the work:** The main purpose of this paper is to introduce the concept of \((F, \varphi, \omega)\) - Gregus type contraction mapping and \((F, \varphi, \omega)\) - weak Gregus type contraction mapping in metric space setting and establish \(\varphi\) - fixed point results. These results are partially extend and generalize the results of Jleli et al.[1] and Kumrod et al.[2]. Also proved and example to illustrate the results presented herein.

2. **Main Results**

2.1. **\((F, \varphi, \omega)\) - Gregus type contraction condition**

**Definition 5.** Let \((X, d)\) be a metric space and \(\varphi : X \to [0, \infty)\) be a given function, \(F \in \mathcal{F}\) and \(\omega \in \Omega\). We say that the mapping \(T : X \to X\) is an \((F, \varphi, \omega)\) - Gregus
**type contraction condition** with respect to the metric $d$ if and only if for any $x, y \in X$ and some $a \in (0, 1]$

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq \omega \left( a F(d(x, y), \varphi(x), \varphi(y)) \\
+ (1 - a) \max \{ F(d(x, Tx), \varphi(x), \varphi(y)), F(d(y, Tx), \varphi(x), \varphi(y)) \} \right).$$

Now, we give the existence of $\varphi$- fixed point result for $(F, \varphi, \omega)$ - Gregus type contraction mapping.

**Theorem 4.** Let $(X, d)$ be a metric space and $\varphi : X \to [0, \infty)$ be a given function, $F \in \mathcal{F}$ and $\omega \in \Omega$. Suppose that the following conditions are satisfied:

(K1) $\varphi$ is lower semi-continuous,
(K2) $T : X \to X$ is an $(F, \varphi, \omega)$ - Gregus type contraction with respect to the metric $d$.

Then the following conditions hold:

(i) $F_T \subseteq Z_\varphi$,
(ii) $T$ is a $\varphi$ - Picard mapping.

**Proof.** (i) Suppose that $\eta \in F_T$. Taking equation (1) with $x = y = \eta$, we have

$$F(0, \varphi(\eta), \varphi(\eta)) \leq \omega \left( a F(0, \varphi(\eta), \varphi(\eta)) \\
+ (1 - a) \max \{ F(0, \varphi(\eta), \varphi(\eta)), F(0, \varphi(\eta), \varphi(\eta)) \} \right)$$

$$= \omega ( a F(0, \varphi(\eta), \varphi(\eta)) + (1 - a) F(0, \varphi(\eta), \varphi(\eta)) )$$

$$= \omega ( F(0, \varphi(\eta), \varphi(\eta)) ).$$

Using Lemma 3, we obtain that

$$F(0, \varphi(\eta), \varphi(\eta)) = 0. \quad (3)$$

By the property of $(F_1)$, we get

$$\varphi(\eta) \leq F(0, \varphi(\eta), \varphi(\eta)). \quad (4)$$

Using equation (3) and (4), we get $\varphi(\eta) = 0$ and then $\eta \in Z_\varphi$. Hence condition (i) holds.

(ii) Let $x$ be a arbitrary point in $X$, then, we have

$$F(d(T^n x, T^{n+1} x), \varphi(T^n x), \varphi(T^{n+1} x))$$

$$\leq \omega ( a F(d(T^{n-1} x, T^n x), \varphi(T^{n-1} x), \varphi(T^n x)) )$$

$$+ (1 - a) \max \{ F(d(T^n x, T^{n+1} x), \varphi(T^n x), \varphi(T^{n+1} x)), F(d(T^{n+1} x, T^{n+2} x), \varphi(T^{n+1} x), \varphi(T^{n+2} x)) \}.$$
\[+(1-a) \max \{F(d(T^{n-1}x, T^{n}x), \varphi(T^{n-1}x), \varphi(T^{n}x)),
   F(d(T^{n}, T^{n}x), \varphi(T^{n-1}x), \varphi(T^{n}x))\}\]

\[= \omega \left(a(F(d(T^{n-1}x, T^{n}x), \varphi(T^{n-1}x)), \varphi(T^{n}x))
   + (1-a)F(d(T^{n-1}x, T^{n}x), \varphi(T^{n-1}x), \varphi(T^{n}x))\right)\]

Now, from (F_4), we get
\[F(d(T^{n}x, T^{n+1}x), \varphi(T^{n}x), \varphi(T^{n+1}x))\]
\[\leq \omega \left(a(F(d(T^{n-1}x, T^{n}x), \varphi(T^{n-1}x)), \varphi(T^{n}x))
   + (1-a)F(d(T^{n-1}x, T^{n}x), \varphi(T^{n-1}x), \varphi(T^{n}x))\right)\]
\[= \omega(F(d(T^{n-1}x, T^{n}x), \varphi(T^{n-1}x), \varphi(T^{n}x)))\].

By induction for each \(n \in N\) and using the property (F_1), we obtain that
\[\max \{d(T^{n}x, T^{n+1}x), \varphi(T^{n}x)\} \leq F(d(T^{n-1}x, T^{n}x), \varphi(T^{n-1}x), \varphi(T^{n}x))\]
\[\leq \omega^n(F(d(x, Tx), \varphi(x), \varphi(Tx))). \quad (5)\]

From equation (5), we have
\[d(T^{n}x, T^{n+1}x) \leq \omega^n(F(d(x, Tx), \varphi(x), \varphi(Tx))). \quad (6)\]

Now, we prove that \(\{T^{n}x\}\) is a Cauchy sequence. Suppose that \(m,n \in N\) such that \(m > n\), we have
\[d(T^{n}x, T^{m}x) \leq d(T^{n}x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x) + \ldots + d(T^{m-1}x, T^{m}x)\]
\[= \omega^n(F(d(Tx, x), \varphi(Tx), \varphi(x))) + \omega^{n+1}(F(d(Tx, x), \varphi(Tx), \varphi(x)))
   + \ldots + \omega^{m-1}(F(d(Tx, x), \varphi(Tx), \varphi(x)))\]
\[= \omega^n(1 + \omega + \ldots)(F(d(Tx, x), \varphi(Tx), \varphi(x)))\]
\[= \sum_{i=1}^{m-1} \omega^i(F(d(Tx, x), \varphi(Tx), \varphi(x))) - \sum_{k=1}^{n-1} \omega^k(F(d(Tx, x), \varphi(Tx), \varphi(x))).\]

Since \(\omega \in \Omega\), then we get \(\lim_{m,n \to \infty} d(T^{n}x, T^{m}x) = 0\), its leads to the sequence \(\{T^{n}x\}\)
is a Cauchy sequence. Since \((X, d)\) is a complete metric space, then there is some point \(z \in X\) such that
\[\lim_{n \to \infty} d(T^{n}x, z) = 0. \quad (7)\]
Finally, we have to prove that $z$ is $\varphi$ - fixed point of $T$. From (5), we can write,
\[ \varphi(T^nx) \leq \omega^n(F(d(x, Tx), \varphi(x), \varphi(Tx))). \tag{8} \]

On taking limits in (8) and using $j_3$, we get
\[ \lim_{n \to \infty} \varphi(T^nx) = 0. \tag{9} \]

Since $\varphi$ is lower semi continuous and using (7), then we get
\[ \varphi(z) \leq \liminf_{n \to \infty} \varphi(T^nx) = 0. \tag{10} \]

Taking $x = T^{n-1}x$ and $y = z$ in (2), we have
\[
F(d(T^nx, Tz), \varphi(T^nx), \varphi(Tz)) \\
\leq \omega \left( a \ F(d(T^{n-1}x, z), \varphi(T^{n-1}x), \varphi(z)) \right) \\
+ (1 - a) \max \{ F(d(T^{n-1}x, T^nx), \varphi(T^{n-1}x), \varphi(z)), \\
F(d(z, T^nx), \varphi(T^{n-1}x), \varphi(z)) \} 
\]

On taking limits as $n \to \infty$ in above inequality, using (7), (8) and (9), $(F_2)$, $(F_3)$ and using Lemma 3, we get
\[ F(d(z, Tz), 0, \varphi(Tz)) \leq \omega(F(0, 0, 0)) = 0, \]

which imply that
\[ d(z, Tz) = 0. \tag{11} \]

Then from equation (10) and (11) that $z$ is $\varphi$ - fixed point of $T$.

Uniqueness: Assume that $z$ and $z^*$ are two $\varphi$-fixed points of $T$. Applying equation(2) with $x = z$ and $y = z^*$. Then we obtain
\[
F(d(Tz, Tz^*), \varphi(Tz), \varphi(Tz^*)) \\
\leq \omega \left( a \ F(d(z, z^*), \varphi(z), \varphi(z^*)) \right) \\
+ (1 - a) \max \{ F(d(z, Tz), \varphi(z), \varphi(z^*)), F(d(z^*, Tz), \varphi(z), \varphi(z^*)) \} 
\]
\[ F(d(z, z^*), 0, 0)) \\
\leq \omega \left( a \ F(d(z, z^*), 0, 0)) + (1 - a) \max \{ F(0, 0, 0), F(d(z^*, Tz), 0, 0)) \right) \\
= \omega(F(d(z, z^*), 0, 0)) = 0. \]

By Lemma 3 and Remark 1, we obtain that $F(d(z, z^*), 0, 0) = 0$ and hence $d(z, z^*) = 0$. This implies that the $\varphi$ - fixed point of $T$ is unique ($\{z\} = F_T \cap Z_\varphi$). So $T$ is a $\varphi$ - Picard mapping.
Theorem 5. Under the hypothesis of Theorem 4, the following condition also hold T is a weakly $\varphi$ - Picard operator.

Proof. From equation (7) and (9)-(11) of Theorem 4, we get $T$ is weakly $\varphi$ - Picard operator.

Example 2. Let $X = [0, 1]$ and $d : X \times X \rightarrow R$ be defined as $d(x, y) = |x - y|$ for all $x, y \in X$. Assume that $T : X \rightarrow X$ is defined as

$$T(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2}, \\ \frac{1-x}{2} & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

The function $\varphi : X \rightarrow [0, \infty)$ is define $\varphi(x) = \frac{x}{2}$ for all $x \in X$, the function $F : [0, +\infty)^3 \rightarrow [0, +\infty)$ is define by $F(a, b, c) = a + b + c$ and $\omega$ be a identity mapping on $+$. At $a = \frac{3}{8}$.

<table>
<thead>
<tr>
<th>Cases</th>
<th>LHS value of (2)</th>
<th>RHS value of (2)</th>
<th>Positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x, y \in [0, \frac{1}{2}]$</td>
<td>0</td>
<td>$\frac{3(1-y)}{4}$</td>
<td>$a(3y-2x+(1-a)(3y))$</td>
</tr>
<tr>
<td>$x \in [0, \frac{1}{2}], y \in [\frac{1}{2}, 1]$</td>
<td>$\frac{3(1-y)}{4}$</td>
<td>$a(3y-2x+(1-a)(3y))$</td>
<td>$\frac{a(3y-x)+(1-a)(3y+y-1)}{2}$</td>
</tr>
<tr>
<td>$y \in [0, \frac{1}{2}], x \in [\frac{1}{2}, 1]$</td>
<td>$\frac{3(1-x)}{4}$</td>
<td>$\frac{a(3y-x)+(1-a)(3y+y-1)}{2}$</td>
<td>$\frac{2ax+(1-a)(3x-1)}{2}$</td>
</tr>
<tr>
<td>$x, y \in [\frac{1}{2}, 1]$</td>
<td>$\frac{1-x}{2}$</td>
<td>$x = y$</td>
<td>$x = y$</td>
</tr>
<tr>
<td>$x &lt; y$</td>
<td>$\frac{3y-5x+2}{4}$</td>
<td>$x &lt; y$</td>
<td>$x &lt; y$</td>
</tr>
<tr>
<td>$x &gt; y$</td>
<td>$\frac{3x-5y+2}{4}$</td>
<td>$x &gt; y$</td>
<td>$x &gt; y$</td>
</tr>
</tbody>
</table>

It is easy to see that $F \in F, \omega \in \Omega$ and $\varphi$ is lower semi continuous. Finally, the above table shows that the mapping $T$ satisfies the condition (2).

Now, we extend the contractive condition (2) and prove the second main result.

2.2. $(F, \varphi, \omega)$ - weak Gregus type contraction condition

Definition 6. Let $(X, d)$ be a metric space and $\varphi : X \rightarrow [0, \infty)$ be a given function, $F \in F$ and $\omega \in \Omega$. We say that the mapping $T : X \rightarrow X$ is an $(F, \varphi, \omega)$ - weak Gregus type contraction condition with respect to the metric $d$ if and only if

$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq \omega \left( a \left( F(d(x, y), \varphi(x), \varphi(y)) + (1-a) \max \left\{ F(M(x, y), \varphi(x), \varphi(y)), F(N(x, y), \varphi(x), \varphi(y)) \right\} \right), \right.$

(12)

where $M(x, y) = \max \{ d(x, y), d(x, Tx) \}$ and $N(x, y) = \min \{ d(x, Ty), d(y, Tx), d(y, Ty) \}, \forall x, y \in X$ and for some $a \in (0, 1)$.
Now, we give the existence of \( \varphi \)-fixed point result for \((F, \varphi, \omega)\) - weak Gregus type contraction mapping.

**Theorem 6.** Let \((X, d)\) be a metric space and \(\varphi : X \rightarrow [0, \infty)\) be a given function, \(F \in F\) and \(\omega \in \Omega\). Suppose that the following conditions are satisfied:

(K1) \(\varphi\) is lower semi-continuous,

(K2) \(T : X \rightarrow X\) is an \((F, \varphi, \omega)\) - Gregus type contraction with respect to the metric \(d\).

Then the following conditions hold:

(i) \(F_T \subseteq Z_\varphi\),

(ii) \(T\) is a \(\varphi\) - Picard mapping.

**Proof.** (i) Suppose that \(\eta \in F_T\). Taking equation (12) with \(x = y = \eta\), we have

\[
F(0, \varphi(\eta), \varphi(\eta)) \leq \omega \left( a F(0, \varphi(\eta), \varphi(\eta)) \\
+ (1 - a) \max \{ F(0, \varphi(\eta), \varphi(\eta)), F(0, \varphi(\eta), \varphi(\eta)) \} \right)
= \omega(F(0, \varphi(\eta), \varphi(\eta))).
\]

where \(M(x, y) = 0 = N(x, y)\).

Using Lemma 3, we obtain that

\[
F(0, \varphi(\eta), \varphi(\eta)) = 0. \tag{13}
\]

By the property of \((F_1)\), we have

\[
\varphi(\eta) \leq F(0, \varphi(\eta), \varphi(\eta)). \tag{14}
\]

Using equation (13) and (14), we get \(\varphi(\eta) = 0\) and then \(\eta \in Z_\varphi\).

Hence condition (i) holds.

(ii) Let \(x\) be arbitrary point in \(X\), then we have

\[
F(d(T^n x, T^{n+1} x), \varphi(T^n x), \varphi(T^{n+1} x)) \leq \omega \left( a F(d(T^{n-1} x, T^n x), \varphi(T^n x), \varphi(T^{n+1} x)) \\
+ (1 - a) \max \{ F(M(T^{n-1} x, T^n x), \varphi(T^{n-1} x), \varphi(T^n x)), \right.
\]

\[
F(N(T^{n-1} x, T^n x), \varphi(T^{n-1} x), \varphi(T^n x))) \right) \}
\]

where \(M(x, y) = M(T^{n-1} x, T^n x) = d(T^{n-1} x, T^n x)\) and \(N(x, y) = N(T^{n-1} x, T^n x) = min \{d(T^{n-1} x, T^{n+1} x), d(T^n x, T^{n+1} x), d(T^n x, T^{n+1} x) = 0.\)
Now, we prove that
\[ d_m > n \]
From (15), we have
\[ (1 - a) \max \{ F(d(T^{n-1}x, T^n x), \varphi(T^{n-1}x)), F(0, \varphi(T^{n-1}x), \varphi(T^n x)) \} \]
Finally, we have to prove that \[ \varphi(T^n x) \in z \]
By using (17), then above inequality reduced to
\[ \max \{ d(T^{n+1}x, T^n x), \varphi(T^{n+1}x) \} \leq F(d(T^n x, T^{n+1}x), \varphi(T^n x)) \]
By induction for each \( n \in \mathbb{N} \) and using the property \( F_1 \), we obtain that
\[ \max \{ d(T^{n+1}x, T^n x), \varphi(T^{n+1}x) \} \leq \omega(F(d(T^n x, T^{n+1}x), \varphi(T^n x))) \]
\[ \leq \omega^n(F(d(Tx, x), \varphi(Tx), \varphi(x))) \]
From (15), we have
\[ d(T^{n+1}x, T^n x) \leq \omega^n(F(d(Tx, x), \varphi(Tx), \varphi(x))). \] (16)
Now, We prove that \( \{ T^n x \} \) is a Cauchy sequence. Suppose that \( m, n \in \mathbb{N} \) such that \( m > n \), we have
\[ d(T^n x, T^m x) \leq d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{n+2}x) + \ldots + d(T^{m-1}x, T^m x) \]
\[ = \omega^n(F(d(Tx, x), \varphi(Tx), \varphi(x))) + \omega^{n+1}(F(d(Tx, x), \varphi(Tx), \varphi(x))) + \ldots + \omega^{m-1}(F(d(Tx, x), \varphi(Tx), \varphi(x))) \]
\[ = \omega^n(1 + \omega + \ldots)(F(d(Tx, x), \varphi(Tx), \varphi(x))) \]
By using (j3) and (j4), then we get \( \lim_{m,n \to \infty} d(T^n x, T^m x) = 0 \), its leads to the sequence \( \{ T^n x \} \) is a Cauchy sequence. Since \( (X, d) \) is a complete metric space, there is some \( z \in X \) such that
\[ \lim_{n \to \infty} d(T^n x, z) = 0. \] (17)
Finally, we have to prove that \( z \) is \( \varphi \) - fixed point of \( T \). From (5), we can write,
\[ \varphi(T^{n+1}x) \leq \omega^n(F(d(Tx, x), \varphi(Tx), \varphi(x))). \] (18)
On taking limits in (18) and from \( j_2 \), we get
\[
\lim_{n \to \infty} \varphi(T^{n+1}x) = 0. \tag{19}
\]
Since \( \varphi \) is lower semi continuous then the equation (17) - (19), then, we get
\[
\varphi(z) \leq \liminf_{n \to \infty} \varphi(T^{n+1}x) = 0. \tag{20}
\]
On taking \( x = T^{n-1}x \) and \( y = z \) in (12), we get
\[
F(d(T^n x, Tz), \varphi(T^n x), \varphi(Tz)) \leq \omega\left( a \left( F(d(T^n x, z), \varphi(T^{n-1}x), \varphi(z)) \right) + (1 - a) \max \left\{ F(M(T^n x, z), \varphi(T^{n-1}x), \varphi(z)), \right. \\
\left. F(N(T^n x, z), \varphi(T^{n-1}x), \varphi(z)) \right\} \right)
\]
On taking limit as \( n \to \infty \) in above inequality and using equation (19) - (20), the properties \( F_2, F_3 \) and also using Lemma 3. and Remark 1., then, we get
\[
F(d(z, Tz), 0, \varphi(Tz)) \leq \omega(F(0, 0, 0)) = 0,
\]
which implies that
\[
d(z, Tz) = 0. \tag{21}
\]
Then from equation (20) and (21) that \( z \) is \( \varphi \) - fixed point of \( T \) (i.e., \( z \in F_T \cap Z_\varphi \)).
Finally, we have to show that \( T \) - is a \( \varphi \) - Picard mapping. It is sufficient to show that assume that \( z \) and \( z^* \) are two \( \varphi \) - fixed points of \( T \). Applying equation (12) with \( x = z \) and \( y = z^* \). Then we obtain that
\[
F(d(Tz, Tz^*), \varphi(Tz), \varphi(Tz^*)) \\ \leq \omega\left( a \left( F(d(z, z^*), \varphi(z), \varphi(z^*)) \right) + (1 - a) \max \left\{ F(M(z, z^*), \varphi(z), \varphi(z^*)), \right. \\
\left. F(N(z, z^*), \varphi(z), \varphi(z^*)) \right\} \right)
\]
where \( M(x, y) = d(z, z^*) \) and \( N(x, y) = 0 \).
By using \(( F_4 )\) and Lemma 3. and Remark 1., then we get
\[
F(d(Tz, Tz^*), 0, 0)) \leq \omega(F(d(z, z^*), 0, 0)) = 0.
\]
Hence \( d(z, z^*) = 0 \). This implies that the \( \varphi \) - fixed point of \( T \) is unique \(( \{z \} = F_T \cap Z_\varphi \)). So \( T \) is a \( \varphi \) - Picard mapping.

**Theorem 7.** *Under the hypothesis of Theorem 6., the following condition also hold T is a weakly \( \varphi \) - Picard operator.*
Proof. From equation (17) and (19)-(21) of Theorem 6., we get $T$ is weakly $\varphi$ - Picard operator.

3. Example

**Example 3.** Let $X = [0, 1]$ and $d : X \times X \rightarrow \mathbb{R}$ be define by $d(x, y) = |x - y|$ for all $x, y \in X$. Then $(X, d)$ is a complete metric space. Suppose that $T : X \rightarrow X$ is defined by

$$T(x) = \begin{cases} 0 & \text{if } 0 \leq x < \frac{1}{2}, \\ k \log(x + 1) & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

where $k \in [0, 1)$, the function $\varphi : X \rightarrow [0, \infty)$ is define $\varphi(x) = \frac{x}{2}$ for all $x \in X$, the function $F : [0, +\infty)^3 \rightarrow [0, +\infty)$ is define by $F(a, b, c) = a + b + c$ and the function $\omega : [0, +\infty) \rightarrow [0, +\infty)$ is define by

$$\omega(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq 1, \\ 5k \log 6t & \text{if } t > 1. \end{cases}$$

It is easy to see that $F \in F$, $\omega \in \Omega$ and $\varphi$ satisfies lower semi-continuous condition.

Now, we have to show that $T$ satisfied condition equation (12).

**Case 1:** Suppose that $x, y \in [0 \leq x < \frac{1}{2})$, then $T$ holds equation (12) trivially.

**Case 2:** Suppose that $x, y \in \left[\frac{1}{2} \leq x \leq 1\right)$. We assume that $y \leq x$. Then we have

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) = d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)$$

$$= |k \log(x + 1) - k \log(y + 1)| + \frac{k \log(x + 1)}{2} + \frac{k \log(y + 1)}{2}$$

$$\leq k \log(x + 1)$$

$$< 5k \log(6) \leq \text{RHS of (12)}.$$

**Case 3:** Suppose that $x, y \in [\frac{1}{2}, 1)$ and $y \in \left[0, \frac{1}{2}\right)$. Then we have

$$F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) = d(Tx, Ty) + \varphi(Tx) + \varphi(Ty)$$

$$= |k \log(x + 1) - 0| + \frac{k \log(x + 1)}{2} + 0$$

$$= k \log(x + 1) + \frac{k \log(x + 1)}{2}$$

$$= \frac{3}{2} k \log(x + 1)$$

$$< 5k \log(6) \leq \text{RHS of (12)}.$$
All the hypothesis of Theorem 6., are satisfied and 0 is a $\varphi$ - fixed point the operator $T$ and also fixed point of $T$.

We can see from the following table approximating the $\varphi$ - fixed point of $T$ at two different values of $k$.

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<tr>
<th>$k = 0.4$</th>
<th>$x_0 = 0.5$</th>
<th>$x_0 = 0.7$</th>
<th>$x_0 = 0.9$</th>
<th>$k = 0.8$</th>
<th>$x_0 = 0.5$</th>
<th>$x_0 = 0.7$</th>
<th>$x_0 = 0.9$</th>
</tr>
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<td>0.0704</td>
<td>0.0921</td>
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<td>$x_1$</td>
<td>0.1408</td>
<td>0.1843</td>
<td>0.2230</td>
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<tr>
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<td>0.0031</td>
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</tbody>
</table>

Table 1 and Table 2 iterates of Picard iteration for two different values of $k$

And also, the convergence behavior of these iterations in shown in Fig. 1

![aks2-eps-converted-to.pdf](aks2-eps-converted-to.pdf) ![aks1-eps-converted-to.pdf](aks1-eps-converted-to.pdf)

**Fig.1:** left figure for $k = 0.4$ and right figure for $k = 0.8$.

**References**


A. K. Singh, Koti N.V.V.Vara Prasad – \((F, \varphi, \omega)\) - Gregus type contraction …

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