CANAL SURFACES WITH MODIFIED ORTHOGONAL FRAME IN MINKOWSKI 3-SPACE

N. Yüksel, N. Oğraş

ABSTRACT. In this paper, we study canal surfaces with modified orthogonal frame in Minkowski 3-space. Some characterizations of the canal surface will be given using curvature-modified orthogonal frame in Minkowski 3-space.

2010 Mathematics Subject Classification: 53A05, 53B25, 53C50.

Keywords: Modified orthogonal frame, canal surface, minkowski3-space.

1. Introduction

The theory of surfaces has an important role in differential geometry. Canal surfaces, on the other hands, are special class of surface theory and are quite often used in studies. Canal surfaces may be formed by either by sweeping a particular circular cross-section of the sphere along the same path. It is parameterized with the help of the spheres forming itself. A canal surface $M$ can be parameterized as follows:

$$x(s, \theta) = c(s) + r(s)((-r'(s))t + \sqrt{1 - r'(s)^2}(\cos \theta n + \sin \theta b))$$

(1)

where $c(s)$ is a unit speed curve parameterized by arc-length $s$. $\{t, n, b\}$ is the frame of $c(s)$. $\{t, n, b\}$ is the unit tangent, principal normal, and binormal vector fields, respectively. Here the curve $c(s)$ called center curve and $r(s)$ is called the radial function of $M$. If the radius function $r(s) = r$, then surface of the canal given by 1;

$$L(s, \theta) = c(s) + r(\cos \theta n + \sin \theta b)$$

(2)

surface is also called tube(pipe )surface[5].

Canal and tube surfaces also have applications in various science. It is used in computer aided design (CAGD) especially for surface modeling, shape reconstruction, planning of robot movements, construction of blending surfaces. It is also convenient for viewing long and thin objects, such as pipe, ropes, poles or living intestines.
The canal surfaces first addressed by Monge have been considered from different angles by many mathematicians. Xu Z., Feng R. and Sun J. studied the analytical and geometric properties of the canal surface. Karacan M.K., Es H., Yaylı Y., Yoon D.W., Tuncer Y., Yüksel N., Bükcü B. considered the canal surfaces and tubular surfaces in Euclidean 3-space, Minkowski space, Galilean and Pseudo space[12, 13, 15, 16]. Doğan F. and Yaylı Y., investigated canal surfaces and tube surfaces given by the different frames of their curve and generalized some properties of them, for instance the equation of tubular surfaces given by Bishop frame of its spine curve and the equation of tube surface given by Darboux frame of its spine curve[2, 3].

The Lorentz-Minkowski space is the basic space model of quantum that plays an important role in general relativity. In recent years, with development of the theory of relativity physicists and geometers extended the topics in classical differential geometry of Riemannian manifolds. It is clearly demonstrated by the fact that many works in Euclidean space have found their counterparts in Minkowski space[7]. At present, the properties of canal surfaces have been researched in $E^3[6]$. Similar to the generating process of canal surfaces in Euclidean space, a canal surface in Minkowski 3-space $E^3_1$ can be obtained as the envelop of a family of pseudo spheres $S^2_1$, pseudohyperbolic spheres $H^2_0$ or lightlike cones $Q^2$ whose centers lie on a space curve (resp. spacelike curve, timelike curve or null curve). The classification of canal surfaces was obtained by Ucum A. and Ilarslan K. in[10]. Fu X., Jung S., Qian J. and Su M. study classify the canal surfaces foliated by pseudo spheres $S^2_1$ along a space curve in $E^3_1[4]$. In this study, we examine and illustrate the canal surfaces that were previously made various characterizations and studied according to modified orthogonal frame by sphere $S^2_1(s)$ in Minkowski 3-space. All the surfaces under consideration are assumed to be smooth, regular and topologically connected unless otherwise stated.

2. Preliminaries

In this section, we review some basic facts for curves and canal surfaces in Minkowski 3-space. The Minkowski 3-space $E^3_1$ is the Euclidean 3-space $E^3$ equipped with standard flat metric given by $<,>= −dx_1^2 + dx_2^2 + dx_3^2$ where $\mathbf{x} = (x_1, x_2, x_3)$ is a rectangular coordinate system of $E^3$. Recall that a vector $v \in E^3_1 − \{0\}$ can be spacelike if $g(v, v) > 0$, timelike if $g(v, v) < 0$ and null (lightlike) if $g(v, v) = 0$ and $v \neq 0$. The norm of a vector $v$ is given $\|v\| = \sqrt{g} < v, v >$ and two vectors $v$ and $w$ are said to be orthogonal if $g(v, w) = 0$. An arbitrary curve $\alpha(s)$ in $E^3_1$ can be locally be spacelike, timelike or null (lightlike), if all of its velocity vectors $\alpha'(t) = t$ are respectively spacelike, timelike or null (lightlike)[8]. The Lorentzian vector product $U$ and $V$ defined by
The formulas of any curve in Minkowski 3-space are given\cite{11}. Now let \( \alpha \) be any curve in Minkowski 3-space and suppose that we can reparameterize \( \alpha \) by the arc length \( s \). We define modified orthogonal frame \( \{T, N, B\} \) as follows

\[
T = \frac{d\alpha}{ds}, \quad N = \frac{dT}{ds}, \quad B = T \wedge N \quad (3)
\]

The relations between those and the Frenet frame \( \{t, n, b\} \) at non-zero points of curvature \( \kappa \)

\[
T = t \quad N = \kappa n \quad B = \kappa b \quad (4)
\]

From equation (4) we obtained next theorem.

**Theorem 1.** Let \( \alpha(s) \) be a unit speed curve classical Frenet frame is \( \{t, n, b\} \) and modified orthogonal frame is \( \{T, N, B\} \) in Minkowski 3-space\cite{1}.

**Case 1:** If \( \alpha \) is a spacelike curve with a spacelike principal normal \( n \), then the modified orthogonal frame is

\[
\begin{bmatrix}
    T'(s) \\
    N'(s) \\
    B'(s)
\end{bmatrix} =
\begin{bmatrix}
    0 & 1 & 0 \\
    -\kappa^2 & \kappa & \tau \\
    0 & \kappa' & \kappa
\end{bmatrix}
\begin{bmatrix}
    T(s) \\
    N(s) \\
    B(s)
\end{bmatrix} \quad (5)
\]

\[
< T, T > = 1, \quad < N, N > = \kappa^2, \quad < B, B > = -\kappa^2
\]

\[
< T, N > = < T, B > = < N, B > = 0
\]

\[
T \wedge N = B, \quad N \wedge B = -\kappa^2 T, \quad B \wedge T = -N
\]

**Case 2:** If \( \alpha \) is a spacelike curve with a spacelike binormal \( b \), then the modified orthogonal frame is

\[
\begin{bmatrix}
    T'(s) \\
    N'(s) \\
    B'(s)
\end{bmatrix} =
\begin{bmatrix}
    0 & 1 & 0 \\
    \kappa^2 & \frac{\kappa'}{\kappa} & \tau \\
    0 & \tau & \frac{\kappa'}{\kappa}
\end{bmatrix}
\begin{bmatrix}
    T(s) \\
    N(s) \\
    B(s)
\end{bmatrix} \quad (6)
\]

\[
< T, T > = 1, \quad < N, N > = \kappa^2, \quad < B, B > = -\kappa^2
\]

\[
< T, N > = < T, B > = < N, B > = 0
\]

\[
T \wedge N = B, \quad N \wedge B = -\kappa^2 T, \quad B \wedge T = -N
\]
\[<T, T> = 1, <N, N> = -\kappa^2, <B, B> = \kappa^2\]
\[<T, N> = <T, B> = <N, B> = 0\]
\[T \wedge N = -B, N \wedge B = -\kappa^2 T, B \wedge T = N\]

**Case 3:** If \(\alpha\) is a timelike curve, then the modified orthogonal frame is

\[
\begin{bmatrix}
T'(s) \\
N'(s) \\
B'(s)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
\frac{\kappa'}{\kappa} & \frac{\tau}{\kappa} & 0 \\
0 & -\tau & \frac{\kappa'}{\kappa}
\end{bmatrix}
\begin{bmatrix}
T(s) \\
N(s) \\
B(s)
\end{bmatrix}
\]

(7)

\[<T, T> = -1, <N, N> = <B, B> = \kappa^2\]
\[<T, N> = <T, B> = <N, B> = 0\]
\[T \wedge N = -B, N \wedge B = \kappa^2 T, B \wedge T = -N\]

**Case 4:** If \(\alpha\) is a pseudo null then the modified orthogonal frame is

\[
\begin{bmatrix}
T'(s) \\
N'(s) \\
B'(s)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
0 & \frac{\kappa'}{\kappa} + \tau & 0 \\
-\kappa^2 & 0 & \frac{\kappa'}{\kappa} - \tau
\end{bmatrix}
\begin{bmatrix}
T(s) \\
N(s) \\
B(s)
\end{bmatrix}
\]

(8)

\[<T, T> = 1, <N, B> = \kappa^2, <N, N> = <B, B> = <T, N> = <T, B> = 0\]

**Case 5:** If \(\alpha\) is a null then the modified orthogonal frame is

\[
\begin{bmatrix}
T'(s) \\
N'(s) \\
B'(s)
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 \\
\kappa \tau & \frac{\kappa'}{\kappa} & -\kappa \\
0 & -\tau & \frac{\kappa'}{\kappa}
\end{bmatrix}
\begin{bmatrix}
T(s) \\
N(s) \\
B(s)
\end{bmatrix}
\]

(9)

\[<T, B> = \kappa, <N, N> = \kappa^2, <T, T> = <N, B> = <T, N> = <B, B> = 0\]
Let $p$ be a fixed point in $E_3^1$ and $r > 0$ be a constant. The pseudo-Riemannian sphere is defined by

$$S_1^2(p, r) = \{ x \in E_3^1 : \langle x - p, x - p \rangle = r^2 \}$$

The surface $M$ in $E_3^1$ is called a canal surface which is formed as the envelope of a family pseudo spheres $S_1^2$ whose centers lie on a space curve $c(s)$ framed by $\{T, N, B\}$. Then $M$ can be parameterized by

$$x(s, \theta) = c(s) + m(s, \theta)T(s) + p(s, \theta)N(s) + q(s, \theta)B(s)$$

(10)

where $m, p$ and $q$ differentiable functions of $s$ and $\theta$. Additionally if $M$ is foliated by pseudo spheres $S_1^2$ is said to be of type $M_+$ and canal surface of type $M_+$ can be divided into three types. If $c(s)$ is spacelike (resp. timelike or null) it is said to be of type $M_1^1$ (resp. $M_2^1$ or $M_3^1$). Also $M_1^1$ can be divided into $M_1^{11}, M_1^{12}$ and $M_1^{13}$. When $c(s)$ is the first kind spacelike curve, the second spacelike curve and null type spacelike curve, respectively [4].

3. Canal Surfaces with Modified Orthogonal Frame in Minkowski 3-Space

In this section we study some properties of different types of canal surfaces formed by the movement of pseudo spheres $S_1^2$ along a space curve in Minkowski 3-space according to the modified orthogonal frame.

3.1. Canal Surfaces of Type $M_1^{11}$ and $M_1^{12}$

We assume $M$ be canal surface formed by the movement of the pseudo spheres $S_1^2$ along a first kind spacelike curve $c(s)$, i.e. $M_1^{11}$. Now we parameterized $M_1^{11}$ canal surface using the modified orthogonal frame of the $c(s)$ center curve;

$$x(s, \theta) = c(s) + m(s, \theta)T(s) + p(s, \theta)N(s) + q(s, \theta)B(s)$$
\[ \| x(s, \theta) - c(s) \| = r(s) \]
\[ < x(s, \theta) - c(s), x_s > = 0 \]
\[ m^2 + \kappa^2 p^2 - \kappa^2 q^2 = r^2 \]
\[ x_s = (1 + m_s - pk^2)T(s) + (m + p_s + p\frac{\kappa'}{\kappa} + \tau q)N(s) + (\tau p + q_s + q\frac{\kappa'}{\kappa})B(s) \]
\[ m = -rr' \]
\[ p = \pm \frac{r(s)}{\kappa(s)} \sqrt{1 - r'(s)^2} \cosh \theta \]
\[ q = \pm \frac{r(s)}{\kappa(s)} \sqrt{1 - r'(s)^2} \sinh \theta \]

Then \( M_{11}^1 \) can be parameterized by
\[ M_{11}^1 = X(s, \theta) = c(s) - r(s)r'(s)T \pm \frac{r(s)}{\kappa(s)} \sqrt{1 - r'(s)^2} (\cosh \theta N(s) + \sinh \theta B(s)) \] (11)

where \( c(s) \) parameterized by arc length \( s \) and \( \kappa \neq 0 \). If the radius function \( r(s) = r \) for \( \kappa \neq 0 \), then the parameterization of the tube surface can be as following;
\[ L_{11}^1(s, \theta) = c(s) + r(s) \sin \left( \frac{\mu}{\kappa} \cosh \theta N(s) + \sinh \theta B(s) \right) \] (12)

From (11) we may assume \( -r'(s) = \cos \mu(s) \) for some smooth function \( \mu = \mu(s) \). Then the canal surface \( M_{11}^1 \) can be written as;
\[ X(s, \theta) = c(s) + r(s)(\cos \mu T + \frac{1}{\kappa} \sin \mu \cosh \theta N + \frac{1}{\kappa} \sin \mu \sinh \theta B) \] (13)

\( \mu [0, \pi] \). Initially we have,
\[ X_s = X_s^1 T + X_s^2 N + X_s^3 B \quad X_\theta = X_\theta^1 T + X_\theta^2 N + X_\theta^3 B \] (14)

\[ X_s^1 = \sin^2 \mu - rr'' - r\kappa \sin \mu \cosh \theta \]
\[ X_s^2 = \frac{1}{\kappa} (r' \sin \mu \cosh \theta - \kappa r r' - \mu'rr' \cosh \theta + \tau r \sin \mu \sinh \theta) \]
\[ X_s^3 = \frac{1}{\kappa} (r' \sin \mu \sinh \theta + \tau r \sin \mu \cosh \theta - \mu'rr' \sinh \theta) \] (15)
\[ X_\theta^1 = 0 \]
\[ X_\theta^2 = \frac{r}{\kappa} \sin \mu \sinh \theta \]
\[ X_\theta^3 = \frac{r}{\kappa} \sin \mu \cosh \theta \]

Then the component functions of the first fundamental form are given by:

\[ E = \langle X_s, X_s \rangle = (X_s^1)^2 + \kappa^2 (X_s^2)^2 - \kappa^2 (X_s^3)^2 \]

\[ F = \langle X_s, X_\theta \rangle = \kappa^2 X_s^2 X_\theta^2 - \kappa^2 X_s^3 X_\theta^3 \]
\[ G = \langle X_\theta, X_\theta \rangle = \kappa^2 (X_\theta^2)^2 - \kappa^2 (X_\theta^3)^2 \]

\[ = -r^2 (\sin^2 \mu + r^2 (\kappa^2 \sin^2 \mu \cosh^2 \theta + r'' \kappa^2 - \tau^2 \sin^2 \mu + \mu^2 + 2 \mu' \kappa \cosh \theta - 2 \mu' \tau \sin \mu \sinh \theta) - 2 (rr'' + \kappa r \sin \mu \cosh \theta) \]

And \( EG - F^2 = -r^2 (rr'' + \kappa r \sin \mu \cosh \theta - \sin^2 \mu)^2 \) From (15) and (16) we have,

\[ X_s \wedge X_\theta = \begin{vmatrix} \dot{X}_s^1 & \dot{X}_s^2 & \dot{X}_s^3 \\ \dot{X}_\theta^1 & \dot{X}_\theta^2 & \dot{X}_\theta^3 \end{vmatrix} \]

\[ = \kappa^2 (X_\theta^2 X_\theta^3 - X_\theta^3 X_\theta^2)T + (X_\theta^3 X_\theta^1)N + (X_\theta^1 X_\theta^2)B \]
\[ = -rr' (-rr'' - \kappa r \sin \mu \cosh \theta + \sin^2 \mu)T \]
\[ + \frac{r}{\kappa} (-rr'' - \kappa r \sin \mu \cosh \theta + \sin^2 \mu)(\sin \mu \cosh \theta)N \]
\[ + \frac{r}{\kappa} (-rr'' - \kappa r \sin \mu \cosh \theta + \sin^2 \mu)(\sin \mu \sinh \theta)B \]

(18)
The unit normal vector field $U$ to $M_{11}^1$ is given by

$$U = \frac{X_s \wedge X_\theta}{\|X_s \wedge X_\theta\|}$$

$$U(s) = \cos \mu T + \frac{1}{\kappa} \sin \mu \cosh \theta N + \frac{1}{\kappa} \sin \mu \sinh \theta B$$

(19)

$$U_s = (-r'' - \kappa \sin \mu \cosh \theta)T + \frac{1}{\kappa} (-r' - \mu' r' \cosh \theta + \tau \sin \mu \sinh \theta)N$$

$$+ \frac{1}{\kappa} (-r' \mu' \sinh \theta + \tau \sin \mu \cosh \theta)B$$

$$U_\theta = \frac{1}{\kappa} (\sin \mu \sinh \theta)N + \frac{1}{\kappa} (\sin \mu \cosh \theta)B$$

(20)

Then the component functions of the second fundamental form are given by

$$L = - <X_s, U_s>$$

$$= -r(\kappa^2 \sin^2 \mu \cosh^2 \theta + r'^2 \kappa^2 - \tau^2 \sin^2 \mu + \mu'^2 +$$

$$2\mu' \kappa \cosh \theta - 2r' \tau \kappa \sin \mu \sinh \theta) + (r'' + \kappa \sin \mu \cosh \theta)$$

$$M = - <X_\theta, U_s>$$

$$= \tau r \sin^2 \mu + \kappa rr' \sin \mu \sinh \theta$$

$$N = - <X_\theta, U_\theta>$$

$$= r \sin^2 \mu$$

(21)

And as well component functions of the third fundamental form are given by

$$e = <U_s, U_s>$$

$$= \kappa^2 \sin^2 \mu \cosh^2 \theta + r'^2 \kappa^2 - \tau^2 \sin^2 \mu + \mu'^2$$

$$+ 2\mu' \kappa \cosh \theta - 2r' \tau \kappa \sin \mu \sinh \theta$$

(22)
Lemma 2. The first, second and third fundamental forms of canal surface $M_{11}^1$ satisfy:

\[
\begin{align*}
L &= E + P_1 \\
M &= F \\
N &= G \\
e &= L - Q_1 \\
f &= M \\
g &= N \\
EG - F^2 &= -r^2 P_1^2 \\
LN - M^2 &= -r P_1 Q_1 \\
eg - f^2 &= -Q_1^2
\end{align*}
\]  

(24)

\[
P_1 = rr'' + \kappa r \sin \mu \cosh \theta - \sin^2 \mu = r Q_1 - \sin^2 \mu
\]

(25)

Remark 1. Due to regularity, we see that $P_1 \neq 0$ everywhere by (24);

From Lemma 2 the Gaussian curvature $K$ the mean curvature $H$ of $M_{11}^1$ are given by respectively.

\[
K = \frac{LN - M^2}{EG - F^2} = \frac{Q_1}{r P_1}
\]

(26)

\[
H = \frac{EN + GL - 2 FM}{2(EG - F^2)} = \frac{\sin^2 \mu + 2 P_1}{-2r P_1}
\]

(27)

Now we denote $M_{12}^1$ canal surface according to modified orthogonal frame. By the definitaion of $M_{12}^1$ and from (10) we get,

\[
m^2 - \kappa^2 p^2 + \kappa^2 q^2 = r^2
\]

\[
m = -rr'
\]

\[
p = \pm \frac{r(s)}{\kappa(s)} \sqrt{1 - r'(s)^2 \sinh \theta}
\]

\[
q = \pm \frac{r(s)}{\kappa(s)} \sqrt{1 - r'(s)^2 \cosh \theta}
\]

\[
M_{12}^1 = X(s, \theta) = c(s) - r(s)r'(s)T \pm \frac{r(s)}{\kappa(s)} \sqrt{1 - r'(s)^2} (\sinh \theta N(s) + \cosh \theta B(s)
\]

(28)

73
where \( c(s) \) parameterized by arc length \( s \) and \( \kappa \neq 0 \). If the radius function \( r(s) = r \) for \( \kappa \neq 0 \), then the parameterization of the tube surface can be as following:

\[
L^{12}_{+}(s, \theta) = c(s) + \frac{r}{\kappa(s)}(\sinh \theta N(s) + \cosh \theta B(s))
\]  

(29)

From (28) we may assume \(-r'(s) = \cos \mu(s)\) for some smooth function \( \mu = \mu(s) \). Then the canal surface \( M^{12}_{+} \) can be written as:

\[
X(s, \theta) = c(s) + r(s)(\cos \mu T + \frac{1}{\kappa} \sin \mu \sinh \theta N + \frac{1}{\kappa} \sin \mu \cosh \theta B)
\]  

(30)

\( \mu \in [0, \pi] \). From (14) we have,

\[
\begin{align*}
X^1_s &= \sin^2 \mu - \tau r'' + r \kappa \sin \mu \sinh \theta \\
X^2_s &= \frac{1}{\kappa}(r' \sin \mu \sinh \theta - \kappa r r' - \mu' r r' \sinh \theta + \tau r \sin \mu \cosh \theta) \\
X^3_s &= \frac{1}{\kappa}(r' \sin \mu \cosh \theta + \tau r \sin \mu \sinh \theta - \mu' r r' \cosh \theta)
\end{align*}
\]  

(31)

\[
\begin{align*}
X^1_\theta &= 0 \\
X^2_\theta &= \frac{r}{\kappa} \sin \mu \cosh \theta \\
X^3_\theta &= \frac{r}{\kappa} \sin \mu \sinh \theta
\end{align*}
\]  

(32)

Then the component functions of the first fundamental form are given by:

\[
E = \sin^2 \mu + r^2(\kappa^2 \sin^2 \mu \sinh^2 \theta - \tau^2 \kappa^2 - \tau^2 \sin^2 \mu + \mu^2 - 2\mu' \kappa \sinh \theta + 2r' \tau \kappa \sin \mu \cosh \theta) - 2(\tau r'' - \kappa r \sin \mu \sinh \theta)
\]

\[
F = -\tau r^3 \sin^2 \mu + \kappa r' r^2 \sin \mu \cosh \theta
\]

\[
G = -r^2 \sin^2 \mu
\]  

(33)

\[
EG - F^2 = -r^2(\tau r'' - \kappa r \sin \mu \sinh \theta - \sin^2 \mu)^2
\]

\[
U(s) = \cos \mu T + \frac{1}{\kappa} \sin \mu \sinh \theta N + \frac{1}{\kappa} \sin \mu \cosh \theta B
\]  

(34)
Then the component functions of the second fundamental form are given by

\[
U_s = (-r'' + \kappa \sin \mu \sinh \theta)T + \frac{1}{\kappa}(-r' \kappa - \mu' r' \sinh \theta + \tau \sin \mu \cosh \theta)N + \frac{1}{\kappa}(-r' \mu' \cosh \theta + \tau \sin \mu \sinh \theta)B
\]

\[
U_\theta = \frac{1}{\kappa}(\sin \mu \cosh \theta)N + \frac{1}{\kappa}(\sin \mu \sinh \theta)B
\]  

(35)

And as well component functions of the third fundamental form are given by

\[
L = -\tau(\kappa^2 \sin^2 \mu \sinh^2 \theta - r'^2 \kappa^2 - \tau^2 \sin^2 \mu + \mu'^2 - 2\mu' \kappa \sinh \theta + 2r' \tau \kappa \sin \mu \sinh \theta) + (r'' - \kappa \sin \mu \sinh \theta)
\]

\[
M = \tau r \sin^2 \mu - \kappa r \mu' \sin \mu \cosh \theta
\]

\[
N = r \sin^2 \mu
\]  

(36)

Lemma 3. The first, second and third fundamental forms of canal surface \(M_{12}^1\) satisfy:

\[
L = \frac{E + P_2}{-r} \quad M = \frac{F}{-r} \quad N = \frac{G}{-r}
\]

\[
e = \frac{L - Q_2}{-r} \quad f = \frac{M}{-r} \quad g = \frac{N}{-r}
\]

\[
EG - F^2 = -r^2 P_2^2 \quad LN - M^2 = -r P_2 Q_2 \quad eg - f^2 = -Q_2^2
\]  

(38)

\[
P_2 = r r'' - \kappa r \sin \mu \sinh \theta - \sin^2 \mu = r Q_2 - \sin^2 \mu
\]

\[
Q_2 = r'' - \kappa \sin \mu \sinh \theta
\]  

(39)

75
Remark 2. Due to regularity, we see that $P_2 \neq 0$ everywhere by (38);

From Lemma 3 the Gaussian curvature $K$ and the mean curvature $H$ of $M_+^{12}$ are given by respectively.

$$K = \frac{LN - M^2}{EG - F^2} = \frac{Q_2}{rP_2} \quad (40)$$

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)} = \frac{\sin^2 \mu + 2P_2}{-2rP_2} \quad (41)$$

Theorem 4. The Gaussian curvature $K$ and mean curvature $H$ of canal surface $M_+^{11}$ and $M_+^{12}$ satisfy:

$$H = -\frac{1}{2} (Kr + \frac{1}{r}) \quad (42)$$

Proof. For $M_+^{11}$ from (26) and (27), for $M_+^{12}$ from (40) and (41) we get conclusion.

Theorem 5. $M_+^{11}$ canal surface is not flat surface.

Proof. We assume that $M_+^{11}$ is flat. In case Gaussian curvature $K \equiv 0$ By (26) we have $Q_1 \equiv 0$. From (25) we get

$$r''(s) + \kappa(s) \sin \mu(s) \cosh \theta = 0$$

It follows that $r'' = 0$ and $\kappa(s) \sin \mu(s) = 0$. Since $M_+^{11}$ is regular, $\sin \mu \neq 0$. In case $r''(s)$ and $\kappa(s) = 0$ if $\kappa(s) = 0$, then $M_+^{11}$ canal surface is non regular. Therefore $M_+^{11}$ is not flat.

Theorem 6. $M_+^{11}$ canal surface is not minimal surface.

Proof. We assume that $M_+^{11}$ is minimal. So that mean curvature $H \equiv 0$, then (27) implies

$$2P_1 + \sin^2 \mu = 0$$

from (25) we get

$$2rr'' - \sin^2 \mu + 2kr \sin \mu \cosh \theta = 0$$

Therefore $2rr'' - \sin^2 \mu = 0$ and $\kappa r \sin \mu \cosh \theta = 0$. Since $r \neq 0, \sin \mu \neq 0$ then $\kappa = 0$. Hence $M_+^{11}$ canal surface is non regular. So $M_+^{11}$ is not minimal.

For $M_+^{12}$ by doing similar calculations according to $M_+^{11}$ same proofs and results are obtained.

Theorem 7. $M_+^{12}$ canal surface is not flat surface.

Theorem 8. $M_+^{12}$ canal surface is not minimal surface.
Corollary 9. Since $\kappa = 0$ is not, $M^{11}_+$ and $M^{12}_+$ is not revolution surface.

Definition 1. For a pair $(X,Y)$, $X \neq Y$ of the curvatures $K, H$ of a canal surface $M$, if $M$ satisfies

$$\Phi = (X, Y) = 0$$

then it is said to be a $(X,Y)$-Weingarten canal surface, where $\Phi$ is the Jacobi function defined by $\Phi = XY - YX [6]$

Definition 2. For a pair $(X,Y)$, $X \neq Y$ of the curvatures $K,H$ of a canal surface $M$, if $M$ satisfies

$$aX + bY = c$$

then it is said to be a $(X, Y)$-linear Weingarten canal surface, where $(a, b, c) \in \mathbb{R}$ and $(a, b, c) \neq (0, 0, 0) [9]$

Lemma 10. Partial derivates of the $K$ Gaussian curvature and $H$ mean curvature of the surface of the canal surface $M^{11}_+$ are as follows;

$$K_s = \frac{1}{r^2 P_1^2} (-2rr'\kappa^2 \sin^2 \mu \cosh^2 \theta + (r'\kappa - r\kappa') \sin^3 \mu \cosh \theta$$

$$- 5rr'' \sin \mu \cosh \theta + r' r'' \sin^2 \mu - rr'' \sin^2 \mu - 4rr' r''^2) \quad (43)$$

$$K_\theta = -\frac{\kappa \sin^3 \mu \sinh \theta}{r P_1^2} \quad (44)$$

$$H_s = \frac{1}{2r^2 P_1^2} (2r^2 r' \kappa^2 \sin^2 \mu \cosh^2 \theta - (2rr' \kappa - r^2 \kappa') \sin^3 \mu \cosh \theta + 5rr' r'' \kappa \sin \mu \cosh \theta$$

$$- 2rr' r'' \sin^2 \mu + r^2 r'' \sin^2 \mu + 4r^2 r' r''^2 + r' \sin^4 \mu \quad (45)$$

$$H_\theta = \frac{\kappa \sin^3 \mu \sinh \theta}{2 P_1^2} \quad (46)$$

Theorem 11. $M^{11}_+$ canal surface is a $(K,H)$ -Weingarten canal surface if and only if it is a tube surface.

Proof. $\Rightarrow$: $(K,H)$ -Weingarten canal surface $M^{11}_+$ satisfies Jacobi equation

$$H_s K_\theta = H_\theta K_s \quad (47)$$

77
from (43), (44), (45) and (46)

\[(Kr' - \frac{r'}{r^2}K_\theta) = 0 \quad (48)\]

by (48) we consider \(K_\theta = 0\) of \(M_{++}^{11}\) from (44) \(\sin \mu \neq 0\) (or else \(P_1 = 0, M_{++}^{11}\) is not regular) we have \(\kappa = 0\) and \(M_{++}^{11}\) is non regular. Thus \(K_\theta \neq 0\). Then we have

\[r'(K - \frac{1}{r^2}) = 0\]

from (48) \(K \neq \frac{1}{r^2}\) otherwise \(M_{++}^{11}\) is not regular.

\[K = \frac{Q_1}{rP_1} = \frac{1}{r^2}\]

from (25) \(\sin \mu \neq 0\). Hence \(r' = 0\) on \(M_{++}^{11}, r = constant\) and \(M_{++}^{11}\) is tube.

\[\iff M_{++}^{11}\) a tube (i.e \(r = constant\), then from (25)-(27) we have,

\[P_1 = \kappa r \cosh \theta - 1\]
\[Q_1 = \kappa \cosh \theta\]
\[K = \frac{\kappa \cosh \theta}{r(r\kappa \cosh \theta - 1)}\]
\[H = \frac{-2r\kappa \cosh \theta + 1}{2r(r\kappa \cosh \theta - 1)}\]

Therefore their partial derivative are given by

\[K_s = -\frac{\kappa' \cosh \theta}{rP_1^2} \quad (49)\]
\[K_\theta = -\frac{\kappa \sinh \theta}{rP_1^2} \quad (50)\]
\[H_s = \frac{\kappa' \cosh \theta}{2P_1^2} \quad (51)\]
\[H_\theta = \frac{\kappa \sinh \theta}{2P_1^2} \quad (52)\]

By (49)-(52) the Jacobi equation (47) is satisfied everywhere.

78
Theorem 12. Let $M_{+}^{11}$ be a linear Weingarten canal surface. Then a tube with radius $r = -\frac{b}{a}$.

Proof. A $(K, H)$-linear Weingarten canal surfaces satisfies

$$aK + bH = 1$$

where $a, b \in \mathbb{R}$ and $(a, b) \neq 0$ from (3.32) we obtained

$$K(2ar - br^2) = b + 2r$$

By (26) we get

$$\frac{(2ar - br^2)(r'' + \kappa \sin \mu \cosh \theta)}{r^2(r'' + \kappa r \sin \mu \cosh \theta - \sin^2 \mu)} = b + 2r$$

$$2\kappa(r^2 + br - a) \sin \mu \cosh \theta + 2(r^2 + br - a)r'' - (b + 2r)(1 - r'^2) = 0$$

Therefore we get

$$\kappa(r^2 + br - a) \sin \mu = 0,$$

$$2(r^2 + br - a)r'' - (b + 2r)(1 - r'^2) = 0$$

Case 1: If $r^2 + br - a \neq 0$ then $\kappa = 0$. Thus $M_{+}^{11}$ is non regular.

Case 2: If $\kappa \neq 0$ then $r^2 + br - a = 0$. Hence $r = -\frac{b}{2}$ is a nonzero constant, $M_{+}^{11}$ is a tube and $a, b$ satisfy $b^2 + 4a = 0$

Lemma 13. Partial derivatives of the $K$ Gaussian curvature and $H$ mean curvature of the surface of the canal surface $M_{+}^{12}$ are as follows;

$$K_s = \frac{1}{r^2 P_2^2} (-2rr' \kappa^2 \sin^2 \mu \sinh^2 \theta + (-r' \kappa + r\kappa') \sin^3 \mu \sin \theta$$

$$+ 5rr' r'' \kappa \sin \mu \sin \theta + r'' \kappa \sin \mu - rr'' \sin^2 \mu = 4rr' r''^2) (53)$$

$$K_\theta = -\frac{\kappa \sin^3 \mu \cosh \theta}{r P_2^2} (54)$$

$$H_s = \frac{1}{2r^2 P_2^2} (2r^2 r' \kappa^2 \sin^2 \mu \sinh^2 \theta - (2rr' \kappa + r^2 \kappa') \sin^3 \mu \sin \theta - 5r^2 r'' \kappa \sin \mu \sin \theta$$

$$- 2rr' r'' \sin^2 \mu + r^2 r'' \sin^2 \mu + 4r^2 r' r''^2 + r' \sin^4 \mu) (55)$$

$$H_\theta = \frac{\kappa \sin^3 \mu \cosh \theta}{2 P_2^2} (56)$$

79
Theorem 14. $M_+^{12}$ canal surface is a $(K,H)$ -Weingarten canal surface if and only if it is a tube surface.

Theorem 15. Let $M_+^{12}$ be a linear Weingarten canal surface. Then a tube with radius $r = \frac{b}{a}$.

3.2. Canal Surfaces of Type $M_+^{2}$

In this part we denote $M_+^{2}$ canal surface according to modified orthogonal frame. By the definitaion of $M_+^{2}$ and from (10) we get,

$$-m^2 + \kappa^2 p^2 + \kappa^2 q^2 = r^2$$

$$m = rr'$$

$$p = \pm \frac{r(s)}{\kappa(s)} \sqrt{1 + r'(s)^2} \cos \theta$$

$$q = \pm \frac{r(s)}{\kappa(s)} \sqrt{1 + r'(s)^2} \sin \theta$$

$$M_+^{2} = X(s, \theta) = c(s) + r(s)r'(s)T \pm \frac{r(s)}{\kappa(s)} \sqrt{1 + r'(s)^2} (\cos \theta N(s) + \sin \theta B(s))$$

(57)

where $c(s)$ parameterized by arc length $s$ and $\kappa \neq 0$. If the radius function $r(s) = r$ for $\kappa \neq 0$, then the parameterization of the tube surface can be as following;

$$L_+^{2}(s, \theta) = c(s) + \frac{r}{\kappa(s)} (\cos \theta N(s) + \sin \theta B(s))$$

(58)

From (57) we may assume $r'(s) = \tan \mu(s)$ for some smooth function $\mu = \mu(s)$. Then the canal surface $M_+^{2}$ can be written as;

$$X(s, \theta) = c(s) + r(s)(\tan \mu T + \frac{1}{\kappa} \sec \mu \cos \theta N + \frac{1}{\kappa} \sec \mu \sin \theta B)$$

(59)

$\mu \epsilon [-\frac{\pi}{2}, \frac{\pi}{2}]$. From (14) we have,

$$X_s^1 = \sec^2 \mu + rr'' + r \kappa \sec \mu \cos \theta$$

$$X_s^2 = \frac{1}{\kappa}(r' \sec \mu \cos \theta + \kappa r r' + \mu' \kappa r' \sec \mu \cos \theta - \tau r \sec \mu \sin \theta)$$

$$X_s^3 = \frac{1}{\kappa}(r' \sec \mu \sin \theta + \tau r \sec \mu \cos \theta + \mu' \kappa r' \sec \mu \sin \theta)$$

(60)
\[
\begin{align*}
X_\theta^1 &= 0 \\
X_\theta^2 &= -\frac{r}{\kappa} \sec \mu \sin \theta \\
X_\theta^3 &= \frac{r}{\kappa} \sec \mu \cos \theta
\end{align*}
\]

(61)

Then the component functions of the first fundamental form are given by:

\[
E = \sec^2 \mu + r^2(-\kappa^2 \sec^2 \mu \cos^2 \theta + r'^2 \kappa^2 + \tau^2 \sec^2 \mu - \mu^2 \sec^2 \theta - 2\mu' \kappa \sec \mu \cos \theta - 2r' \tau \kappa \sec \mu \sin \theta) - 2(r'' + \kappa r \sec \mu \cos \theta)
\]

\[
F = \tau r^2 \sec^2 \mu - \kappa r' r^2 \sec \mu \sin \theta
\]

\[
G = r^2 \sec^2 \mu
\]

(62)

\[
EG - F^2 = -r^2(r'' + \kappa r \sec \mu \cos \theta + \sec^2 \mu)^2
\]

\[
U(s) = \tan \mu T + \frac{1}{\kappa} \sec \mu \cos \theta N + \frac{1}{\kappa} \sec \mu \sin \theta B
\]

(63)

\[
U_s = (r'' + \kappa \sec \mu \cos \theta) T + \frac{1}{\kappa}(r' \kappa + \mu' \sec \mu \cos \theta - \tau \sec \mu \sin \theta) N
\]

\[
+ \frac{1}{\kappa}(-r' \mu' \cos \theta + \tau \sin \mu \sin \theta) B
\]

\[
U_\theta = -\frac{1}{\kappa} \sec \mu \sin \theta N + \frac{1}{\kappa} \sec \mu \cos \theta B
\]

(64)

Then the component functions of the second fundamental form are given by

\[
L = r(\kappa^2 \sec^2 \mu \cos^2 \theta - r'^2 \kappa^2 - \tau^2 \sec^2 \mu + \mu^2 \sec^2 \mu + 2\mu' \kappa \sec \mu \cos \theta + 2r' \tau \kappa \sec \mu \sin \theta) + (r'' + \kappa \sec \mu \cos \theta)
\]

\[
M = -\tau r \sec^2 \mu + \kappa r r' \sec \mu \sin \theta
\]

\[
N = -r^2 \sec^2 \mu
\]

(65)
\[ e = -\kappa^2 \sec^2 \mu \cos^2 \theta + r'^2 \kappa^2 + \tau^2 \sec^2 \mu - \mu'^2 \sec^2 \mu - 2 \mu' \kappa \sec \mu \cos \theta - 2 \tau \kappa \sec \mu \sin \theta \]

\[ f = \tau \sec^2 \mu - \kappa r' \sec \mu \sin \theta \]

\[ g = \sec^2 \mu \]

(66)

**Lemma 16.** The first, second and third fundamental forms of canal surface \( M^2 \) satisfy:

\[
\begin{align*}
L &= \frac{E + P_3}{-r} \\
M &= \frac{F}{-r} \\
N &= \frac{G}{-r} \\
e &= L - Q_3 \\
f &= M \\
g &= N \\
EG - F^2 &= -r^2 P_3^2 \\
LN - M^2 &= -r P_3 Q_3 \\
\text{eg} - f^2 &= -Q_3^2 \\
\end{align*}
\]

(67)

\[ P_3 = rr'' + \kappa r \sec \mu \cos \theta + \sec^2 \mu = r Q_3 + \sec^2 \mu \]

\[ Q_3 = r'' + \kappa \sec \mu \cos \theta \]

(68)

**Remark 3.** Due to regularity, we see that \( P_3 \neq 0 \) everywhere by (67);

From Lemma 16 the Gaussian curvature \( K \) the mean curvature \( H \) of \( M^2 \) are given by respectively.

\[ K = \frac{LN - M^2}{EG - F^2} = \frac{Q_3}{r P_3} \]

(69)

\[ H = \frac{EN + GL - 2FM}{2(EG - F^2)} = \frac{\sec^2 \mu - 2 P_3}{2r P_3} \]

(70)

**Theorem 17.** The Gaussian curvature \( K \) and mean curvature \( H \) of canal surface \( M^2 \) satisfy:

\[ H = \frac{-1}{2} (Kr + \frac{1}{r}) \]

(71)

For \( M^2 \) by doing similar calculations according to \( M^{11} \) same proofs and results are obtained.

**Theorem 18.** \( M^2 \) canal surface is not flat surface.

**Theorem 19.** \( M^2 \) canal surface is not minimal surface.

**Corollary 20.** Since \( \kappa = 0 \) is not, \( M^2 \) is not revolution surface.
Lemma 21. Partial derivatives of the $K$ Gaussian curvature and $H$ mean curvature of the surface of the canal surface $M_+^2$ are as follows;

\[
K_\varphi = \frac{1}{r^2 P^3_3} (-2rr'' \kappa^2 \sec^2 \mu \cos^2 \theta + (-r' \kappa + r \kappa') \sec^3 \mu \cos \theta \\
- 5rr'' \kappa \sec \mu \cos \theta - r' r'' \sec^2 \mu + rr'' \sec^2 \mu - 4rr' r''^2) \tag{72}
\]

\[
K_\theta = -\frac{\kappa \sec^3 \mu \sin \theta}{r P^3_3} \tag{73}
\]

\[
H_\varphi = \frac{1}{2r^2 P^3_3} (2r^2 r' \kappa^2 \sec^2 \mu \cos \theta - (-2rr' \kappa + r^2 \kappa') \sec^3 \mu \cos \theta + 5r^2 r' \kappa \sec \mu \cos \theta \\
- 2rr' r'' \sec^2 \mu - r^2 r'' \sec^2 \mu + 4r^2 r' r''^2 + r' \sec^4 \mu) \tag{74}
\]

\[
H_\theta = \frac{\kappa \sec^3 \mu \sin \theta}{2 P^3_3} \tag{75}
\]

Theorem 22. $M_+^2$ canal surface is a $(K,H)$-Weingarten canal surface if and only if it is a tube surface.

Theorem 23. Let $M_+^2$ be a linear Weingarten canal surface. Then a tube with radius $r = -\frac{b}{a}$

Corollary 24. When the canal surface center curve is first, second type of spacelike and timelike, canal surface is not flat, minimal, revolution and developable surface according to modified orthogonal frame. Also surface is not circular cylinders, circular cones and catenoids.

3.3. Canal Surfaces of Type $M_{+}^{13}$ and $M_{+}^{3}$

In this part we only give $M_{+}^{13}$ and $M_{+}^{3}$ canal surfaces parameterizations according to modified orthogonal frame. From (10) we get,

\[
m^2 + 2\kappa^2 pq = r^2 \\
m = -rr' \\
2pq = \frac{r^2(s)}{\kappa^2 (s)} (1 - r'(s)^2)
\]
\[
M_{+}^{13} = X(s, \theta) = c(s) - r(s)r'(s)T + p(s, \theta)N(s) + q(s, \theta)B(s)
\]
where
\[
2p(s, \theta)q(s, \theta) = \frac{r^2(s)}{\kappa^2(s)}(1 - r'^2(s)) \tag{76}
\]
and \( \kappa \neq 0 \)

For \( M_{+}^{3} \) from (10) we get,
\[
\kappa^2 p^2 + 2\kappa m q = r^2
\]
\[
\kappa q = -rr'
\]

\[
M_{+}^{2} = X(s, \theta) = c(s) + m(s, \theta)T + p(s, \theta)N(s) - \frac{r(s)r'(s)}{\kappa(s)}B(s)
\]
where
\[
\kappa^2(s)p^2(s, \theta) + 2\kappa(s)m(s, \theta)q(s, \theta) = r^2(s) \tag{77}
\]

**Example 1.** Let us consider \( c(s) \) center curve spacelike curve with spacelike binormal \( \alpha \) as;
\[
\alpha(s) = (\cosh \frac{\sqrt{5}}{3}s, \sinh \frac{\sqrt{5}}{3}s, \frac{2s}{3})
\]
Then its Modified frame is;
\[
T = (\frac{\sqrt{5}}{3} \sinh \frac{\sqrt{5}}{3}s, \frac{\sqrt{5}}{3} \cosh \frac{\sqrt{5}}{3}s, \frac{2}{3})
\]
\[
N = (\frac{5}{9} \cosh \frac{\sqrt{5}}{3}s, \frac{5}{9} \sinh \frac{\sqrt{5}}{3}s, 0)
\]
\[
B = (\frac{10}{27} \sinh \frac{\sqrt{5}}{3}s, \frac{10}{27} \cosh \frac{\sqrt{5}}{3}s, -\frac{5\sqrt{5}}{27})
\]
when the radius function \( r(s) = \cos s \) the canal surface as;
\[
X(s, \theta) = (\cosh \frac{\sqrt{5}}{3}s + \frac{\sqrt{5}}{3} \cos s \sin s \sinh \frac{\sqrt{5}}{3}s + \cos^2 s \sinh \theta \cosh \frac{\sqrt{5}}{3}s + \frac{2}{3} \cos^2 s \cosh \theta \sinh \frac{\sqrt{5}}{3}s,
\]
\[
\sin \frac{\sqrt{5}}{3}s + \frac{\sqrt{5}}{3} \cos s \sin s \cosh \frac{\sqrt{5}}{3}s + \cos^2 s \sinh \theta \cosh \frac{\sqrt{5}}{3}s + \frac{2}{3} \cos^2 s \cosh \theta \cosh \frac{\sqrt{5}}{3}s,
\]
\[
\frac{2s}{3} + \frac{2}{3} \cos s \sin s - \frac{\sqrt{5}}{3} \cos^2 s \cosh \theta)
\]
We draw graphics in Figure.1
Figure 1 Canal surface $M_{12}^1$ with $r(s) = \cos s$.

$L_{12}^{12}(s, \theta) = (\cosh \frac{\sqrt{5}}{3}s + \sinh \theta \cosh \frac{\sqrt{5}}{3}s + \frac{2}{3} \cosh \theta \sinh \frac{\sqrt{5}}{3}s, \\
\sinh \frac{\sqrt{5}}{3}s + \sinh \theta \sinh \frac{\sqrt{5}}{3}s + \frac{2}{3} \cosh \theta \cosh \frac{\sqrt{5}}{3}s, \frac{2s}{3} - \frac{\sqrt{5}}{3} \cosh \theta)\\

We draw graphics in Figure 2

Figure 2 Tube surface $L_{12}^{12}$ with $r(s) = 1$.

REFERENCES


Nural Yüksel  
Department of Mathematics, Faculty of Science,  
University of Erciyes,  
Kayseri, Türkiye  
email: yukseln@erciyes.edu.tr

Nurdan Oğraş  
Department of Mathematics, Faculty of Science,  
University of Erciyes,  
Kayseri, Türkiye  
email: nurdanogras@gmail.com