HIGHER ORDER LOGARITHMIC KLEIN-GORDON EQUATION: GLOBAL EXISTENCE, DECAY AND NONEXISTENCE

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Abstract. In this work, we study a higher order Klein-Gordon equation with logarithmic nonlinearity. Firstly, we established the global existence of solution by potential well method. In addition, we obtain exponential decay and global nonexistence of solutions.

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1. Introduction

In this paper, we consider the following higher order Klein-Gordon equation with logarithmic source term

\[
\begin{aligned}
    u_{tt} + \mathcal{P} u + u + u_t &= 2u \ln |u|, & x \in \Omega, \ t > 0, \\
    \frac{\partial^i u(x,t)}{\partial \nu^i} &= 0, & i = 0, 1, 2, ..., m - 1, \ x \in \partial \Omega, \\
    u(x,0) &= u_0(x), \ u_t(x,0) = u_1(x), & x \in \Omega
\end{aligned}
\]  

where \( \mathcal{P} = (-\Delta)^m \), \( m \geq 1 \) is a positive integer, \( \Omega \) is a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \), \( \nu \) denotes the unit outward normal vector on \( \partial \Omega \), and \( \frac{\partial^i}{\partial \nu^i} \) denotes the \( i \)-th order normal derivation.

The model equation (1) arises in logarithmic quantum mechanics, nuclear physics, optics, supersymmetry and geophysics [5, 6, 7, 21].

When \( m = 1 \), (1) becomes

\[
u_{tt} - \Delta u + u + u_t = u \ln |u|^2.
\]

In 2020, Ye [36] proved the existence, exponential decay and blow up of solutions of the equation (2). Hu et al. [33] studied the following equation

\[
u_{tt} - \Delta u + u + u_t = u \ln |u|^k.
\]
They studied exponential growth and decay of solutions for the equation (3). In [13], Gorka studied the following Klein-Gordon equation
\[ u_{tt} - u_{xx} + u = \varepsilon u \ln |u|^2. \]

Ye and Li [38] considered the following Klein-Gordon equation
\[ u_{tt} - \Delta u + u = u \ln |u|. \]

They obtained global existence and blow up of solutions. Hiramatsu et al. [16] studied the following Klein-Gordon equation
\[ u_{tt} - \Delta u + u + u_t + |u|^2 u = u \ln u. \] (4)

They proved the dynamics of Q-balls in theoretical physics. Later, Han [15] studied global existence of weak solutions (4). Pişkin and Çalışır [29] investigated the following Petrovsky equation
\[ u_{tt} + \Delta u + \Delta^2 u + = u \ln |u|^2. \]

They proved energy decay and blow up at infinite time of solutions. Recently, some authors studied the hyperbolic or parabolic type equations with logarithmic nonlinearity (see [3, 4, 8, 9, 10, 11, 17, 19, 20, 25, 30, 31, 26, 27, 28, 37, 39]).

The main purpose of this paper is to proved the global existence, the decay and the global nonexistence of solution to the higher order Klein-Gordon equation with logarithmic source term (1).

This paper is organized as follows: In Section 2, we present some notations and lemmas. In Section 3, we prove the global existence and decay of solutions. In Section 4, we prove the global nonexistence of solutions.

2. Preliminaries

In this section, we denote
\[ \|u\| = \|u\|_{L^2(\Omega)}, \quad \|u\|_p = \|u\|_{L^p(\Omega)}, \]
for \(1 < p < \infty\). Also, let \( L^p(\Omega) \) denote the Lebesgue spaces and \( W_0^{m,2}(\Omega) = H_0^m(\Omega) \) the Sobolev spaces (see [1, 32], for details).

Next, we define the potential energy functional and Nehari functional of problem (1)
\[ J(u) = \frac{1}{2} \left\| \mathcal{P}_1 u \right\|^2 + \|u\|^2 - \frac{1}{2} \int_{\Omega} u^2 \ln |u|^2 \, dx, \] (5)
\[ I(u) = \left\| \mathcal{P}_1 u \right\|^2 + \|u\|^2 - \int_{\Omega} u^2 \ln |u|^2 \, dx, \] (6)
and the total energy functional
\[
E(t) = \frac{1}{2} \| u_t \|^2 + \frac{1}{2} \| \mathcal{P}^1 u \|^2 + \| u \|^2 - \frac{1}{2} \int_{\Omega} u^2 \ln |u|^2 \, dx
\]
\[
= \frac{1}{2} \| u_t \|^2 + J(u)
\]
(7)
for \( u \in H^m_0(\Omega), \, t \geq 0 \) and
\[
E(0) = \frac{1}{2} \| u_1 \|^2 + \frac{1}{2} \| \mathcal{P}^1 u_0 \|^2 + \| u_0 \|^2 - \frac{1}{2} \int_{\Omega} u_0^2 \ln |u_0|^2 \, dx
\]
(8)
is the initial total energy.

As in Payne and Sattinger [24], The mountain pass value of \( J(u) \) (also known as potential well depth) is defined as
\[
d = \inf_{\lambda \geq 0} \left\{ \sup_{\lambda u \in H^m_0(\Omega)/\{0\}} J(\lambda u) : u \in H^m_0(\Omega)/\{0\} \right\}.
\]
(9)

Now, we define the so called Nehari manifold (see [23, 24, 34, 35]) as follows
\[
\mathcal{N} = \{ u \in H^m_0(\Omega)/\{0\} : K(u) = 0 \}
\]
\( \mathcal{N} \) separates the two unbounded sets
\[
\mathcal{N}^+ = \{ u \in H^m_0(\Omega)/\{0\} : K(u) > 0 \} \cup \{0\}
\]
\[
\mathcal{N}^- = \{ u \in H^m_0(\Omega)/\{0\} : K(u) < 0 \}.
\]

Then, the stable set \( \mathcal{W} \) and the unstable set \( \mathcal{U} \) as follows
\[
\mathcal{W} = \{ u \in H^m_0(\Omega)/\{0\} : J(u) \leq d \} \cap \mathcal{N}^+
\]
\[
\mathcal{U} = \{ u \in H^m_0(\Omega)/\{0\} : J(u) \leq d \} \cap \mathcal{N}^-.
\]

It is readily seen that the potential well depth \( d \) defined in (9) may also be characterized as
\[
d = \inf_{u \in \mathcal{N}} J(u).
\]
(10)

**Definition 1.** The function \( u(x, t) \) is a weak solution of (1) on \([0, T]\), if
\[
u \in C([0, T], H^m_0(\Omega)), \, u_t \in C([0, T], L^2(\Omega))
\]
and \( u \) satisfies
\[
\int_{\Omega} u_{tt} \varphi dx + \int_{\Omega} \mathcal{P}^{\frac{1}{2}} u \mathcal{P}^{\frac{1}{2}} \varphi dx + \int_{\Omega} u_t \varphi dx + \int_{\Omega} u \varphi dx = \int_{\Omega} u \ln |u|^2 \varphi dx
\]
for each test function \( \varphi \in H^m_0(\Omega) \) and for almost all \( t \in [0, T] \).
The proof of the following lemma can be done as in [17].

**Lemma 1.** Let \( u(x,t) \) be a solution of the problem (1). Then \( E(t) \) is a non-increasing function for \( t > 0 \) and

\[
E'(t) = - \|u_t\|^2 \leq 0.
\]

**Lemma 2.** [1, 32]. Let \( r \) be a number with

\[
\begin{align*}
2 &\leq r < +\infty, \quad \text{if} \quad n \leq 2m, \\
2 &\leq r \leq \frac{2n}{n-2m}, \quad \text{if} \quad n > 2m.
\end{align*}
\]

Then there is constant \( C \) depending on \( \Omega \) and \( r \) such that

\[
\|u\|_r \leq C \left\| \mathcal{P}^{\frac{1}{2}} u \right\|, \quad \forall u \in H^m_0(\Omega).
\]

**Lemma 3.** [12, 14]. If \( u \in H_0^1(\Omega) \), then for each \( a > 0 \), one has the inequality

\[
\int_{\Omega} u^2 \ln |u| dx \leq \|u\|^2 \ln \|u\| + \frac{\alpha^2}{2\pi} \|\nabla u\|^2 - \frac{n}{2} (1 + \ln \alpha) \|u\|^2.
\]

**Lemma 4.** If \( u \in H^m_0(\Omega) \), then for each \( a > 0 \),

\[
\int_{\Omega} u^2 \ln |u| dx \leq \|u\|^2 \ln \|u\| + \frac{c_p \alpha^2}{2\pi} \left\| \mathcal{P}^{\frac{1}{2}} u \right\|^2 - \frac{n}{2} (1 + \ln \alpha) \|u\|^2.
\]

**Proof.** By using the embedding theorem \( \|\nabla u\|^2 \leq c_p \left\| \mathcal{P}^{\frac{1}{2}} u \right\|^2 \), we arrive at

\[
\int_{\Omega} u^2 \ln |u| dx \leq \|u\|^2 \ln \|u\| + \frac{c_p \alpha^2}{2\pi} \left\| \mathcal{P}^{\frac{1}{2}} u \right\|^2 - \frac{n}{2} (1 + \ln \alpha) \|u\|^2,
\]

where \( c_p \) constant.

We conclude this section by stating a local existence result of the problem (1), which can be established by similar way as done in combination of the arguments in [2, 18, 22].

**Theorem 5.** *(Local existence).* Assume that \( u_0 \in H^m_0(\Omega) \), \( u_1 \in L^2(\Omega) \). Then there exists \( T > 0 \) such that the problem (1) has a unique local solution \( u(x,t) \) which satisfies

\[
u \in C \left( \left[ 0, T \right); H^m_0(\Omega) \right), \quad u_t \in C \left( \left[ 0, T \right); L^2(\Omega) \right).
\]

Moreover, at least one of the following statements holds true:

i. \( \|u_t\|^2 + \left\| \mathcal{P}^{\frac{1}{2}} u \right\|^2 + \|u\|^2 \to \infty \) as \( t \to T^- \);

ii. \( T = +\infty \).
3. Global existence and decay of solutions

In this section, we establish the global existence and decay of solutions of (1).

**Lemma 6.** Let \( u \in H_0^m(\Omega) \) and \( \|u\| \neq 0 \). Then

\[
I(\lambda u) = \lambda \frac{d}{d\lambda} J(\lambda u) \begin{cases} > 0, & 0 < \lambda < \lambda^*, \\
= 0, & \lambda = \lambda^*, \\
< 0, & \lambda^* < \lambda < +\infty,
\end{cases}
\]

where

\[
\lambda^* = \exp \left( \frac{\|P^{\frac{1}{2}}u\|^2 + \|u\|^2 - 2 \int_{\Omega} u^2 \ln u \, dx}{2 \|u\|^2} \right).
\]

**Proof.** From (5) it implies

\[
J(\lambda u) = \lambda^2 \left( \|P^{\frac{1}{2}}u\|^2 + \|u\|^2 - \int_{\Omega} u^2 \ln u \, dx \right).
\]

A direct computation on above equality, we have

\[
\frac{d}{d\lambda} J(\lambda u) = \lambda \left( \|P^{\frac{1}{2}}u\|^2 + \|u\|^2 - 2 \ln \lambda \|u\|^2 - 2 \int_{\Omega} u^2 \ln u \, dx \right). \tag{11}
\]

Let \( \frac{d}{d\lambda} J(\lambda u) = 0 \), then we have

\[
\lambda^* = \exp \left( \frac{\|P^{\frac{1}{2}}u\|^2 + \|u\|^2 - 2 \int_{\Omega} u^2 \ln u \, dx}{2 \|u\|^2} \right).
\]

It follows from (6) that

\[
I(\lambda u) = \lambda^2 \left( \|P^{\frac{1}{2}}u\|^2 + \|u\|^2 - 2 \lambda^2 \int_{\Omega} u^2 \ln u \, dx - 2 \lambda^2 \ln \lambda \|u\|^2 \right). \tag{12}
\]

By (11) and (12), the conclusion in Lemma 6 is valid.

**Lemma 7.** Assume that \( u \in H_0^m(\Omega) \). The depth of potential well \( d \) is defined as

\[
d = \frac{1}{2} \left( \frac{\pi}{c_p} \right)^{\frac{3}{2}} e^n. \tag{13}
\]
Proof. By definition of $I(u)$ and using Lemma 4, we get

$$I(u) = \left\| P_{1/2} u \right\|^2 + \| u \|^2 - \int_{\Omega} u^2 \ln |u|^2 dx$$

$$\geq \left( 1 - \frac{cp\alpha^2}{\pi} \right) \left( \left\| P_{1/2} u \right\|^2 + \| u \|^2 \right) + [n(1 + \ln \alpha) - 2 \ln \| u \|] \| u \|^2$$  \hspace{1cm} (14)

for any $\alpha > 0$. Taking $\alpha = \sqrt{\frac{\pi}{cp}}$, we obtain from (14) that

$$I(u) \geq [n(1 + \ln \alpha) - 2 \ln \| u \|] \| u \|^2.$$  \hspace{1cm} (15)

We have from Lemma 6 that

$$\sup_{\lambda \geq 0} J(\lambda u) = J(\lambda^* u) = \frac{1}{2} I(\lambda^* u) + \frac{1}{2} \| \lambda^* u \|^2.$$  \hspace{1cm} (16)

We obtain from (15) and Lemma 6 that

$$0 = I(\lambda^* u) \geq [n(1 + \ln \alpha) - 2 \ln \| \lambda^* u \|] \| \lambda^* u \|^2,$$

then

$$\| \lambda^* u \|^2 \geq \alpha^n e^n$$  \hspace{1cm} (17)

It follows from (16) and (17) that

$$\sup_{\lambda \geq 0} J(\lambda u) \geq \frac{1}{2} \alpha^n e^n$$  \hspace{1cm} (18)

By (9) and (18), we get

$$d = \frac{1}{2} \left( \frac{\pi}{cp} \right)^{\frac{2}{n}} e^n.$$

**Lemma 8.** Let $E(0) < d$. If $u_0 \in N^+$ and $u_1 \in L^2(\Omega)$, then $u(t) \in N^+$ for each $t \in [0, T)$.

Proof. From (7) ve Lemma 1, we obtain

$$E(t) = \frac{1}{2} \| u_t \|^2 + J(u)$$

$$\leq \frac{1}{2} \| u_1 \|^2 + J(u_0)$$

$$= E(0) < d$$
for $\forall t \in [0, T)$, which implies that

$$J(u) < d. \quad (19)$$

Assume that there exists a number $t^* \in [0, T)$ such that $u(t) \in \mathcal{N}^+$ on $[0, t^*)$ and $u(t^*) \notin \mathcal{N}^+$. Then, in virtue of continuity of $u(t)$, we see $u(t^*) \in \partial \mathcal{N}^+$. From the definition of $\mathcal{N}^+$ and the continuity of $I(u)$ with respect to $t$, we have

$$I(u(t^*)) = 0. \quad (20)$$

Suppose that (20) holds, then we get from (18) and (15) that

$$\|u(t^*)\|^2 \geq 2d. \quad (21)$$

By (5), (6), (20) and (21), we have

$$J(u(t^*)) = \frac{1}{2} \|u(t^*)\|^2 + \frac{1}{2} I(u(t^*)) \geq d,$$

which is contradictive with (19). Hence, the case (20) is impossible. Consequently, we conclude that $u(t) \in \mathcal{N}^+$ on $[0, T)$.

**Theorem 9.** *(Global existence).* Assume that $u_0 \in \mathcal{W}$, $u_1 \in L^2(\Omega)$ and $E(0) < d$. Then the local solution furnished in Theorem 5 is a global solution and $T$ may be taken arbitrarily large.

**Proof.** It suffices to show that

$$\|u_t\|^2 + \|P_\frac{1}{2}u\|^2 + \|u\|^2$$

is bounded independently of $t$. Under the hypotheses Theorem 9, we get from Lemma 8 that $u \in \mathcal{W}$ on $[0, T)$. So, the following formula holds on $[0, T)$ by Lemma 4

$$J(u) = \frac{1}{2} \|P_\frac{1}{2}u\|^2 + \|u\|^2 - \frac{1}{2} \int_{\Omega} u^2 \ln |u|^2 \, dx$$

$$\geq \frac{1}{2} \left(1 - \frac{c_0 \alpha^2}{\pi}\right) \|P_\frac{1}{2}u\|^2 + \left(1 - \ln \|u\| + \frac{n}{2} (1 + \ln \alpha)\right) \|u\|^2. \quad (22)$$

By (5), (6) and $u \in \mathcal{W}$, we have

$$J(u) = \frac{1}{2} \|u\|^2 + \frac{1}{2} I(u) \geq \frac{1}{2} \|u\|^2, \quad (23)$$

which implies that

$$\|u\|^2 \leq 2J(u) \leq 2d. \quad (24)$$
It follows from (22) and (24), we obtain
\[ J(u) \geq \frac{1}{2} \left( 1 - \frac{c_p \alpha^2}{\pi} \right) \|P^{1/2} u\|^2 + \left( 1 - \frac{1}{2} \ln 2d + \frac{n}{2} (1 + \ln \alpha) \right) \|u\|^2. \] (25)

By Lemma 7 and \(0 < \alpha < \frac{\pi}{c_p}\), we have
\[ 1 - \frac{c_p \alpha^2}{\pi} \geq 0, \quad 1 - \frac{1}{2} \ln 2d + \frac{n}{2} (1 + \ln \alpha) > 0. \]
Thus, we have from (25) that
\[ J(u) \geq C_1 \left( \|P^{1/2} u\|^2 + \|u\|^2 \right), \] (26)
where
\[ C_1 = \min \left\{ \frac{1}{2} - \frac{c_p \alpha^2}{2\pi}, \ 1 - \frac{1}{2} \ln 2d + \frac{n}{2} (1 + \ln \alpha) \right\}. \]

We have from (26) that
\[ \frac{1}{2} \|u_t\|^2 + C_1 \left( \|P^{1/2} u\|^2 + \|u\|^2 \right) \leq \frac{1}{2} \|u_t\|^2 + J(u) = E(t) \leq E(0) < d, \] (27)
which implies that
\[ \|u_t\|^2 + \|P^{1/2} u\|^2 + \|u\|^2 \leq \frac{d}{C_2} < \infty, \]
where \(C_2 = \min \{C_1, 1\}\). The above inequality and the continuation principle lead to the global existence of solution \(u\) for the problem (1).

**Theorem 10.** (Decay). Suppose that \(E(0) < \frac{1}{2} \left( \frac{\pi}{c_p} \right)^{\frac{3}{2}} e^\beta \leq d\), where \(\beta\) is a positive number which satisfies \(0 < \beta \leq 1\). If \(u_0 \in W, u_1 \in L^2(\Omega)\), then there exist two positive constants \(\kappa\) and \(k\) independent of \(t\) such that the global solution has the following exponential decay property
\[ 0 < E(t) \leq \kappa e^{-kt}, \ \forall t \geq 0. \]

**Proof.** By Lemma 8, we see that \(u(t) \in N^+\) for all \(t \geq 0\). Thus, we have \(0 < E(t) < d\) for all \(t \geq 0\). In order to prove the decay of solution. We define
\[ F(t) = E(t) + \varepsilon \int_{\Omega} u_t u dx, \] (28)
where \(\varepsilon > 0\) will be determined later.
It is easy to prove that there exist two positive constants \( \xi_1 \) and \( \xi_2 \) depending on \( \varepsilon \) such that
\[
\xi_1 E(t) \leq F(t) \leq \xi_2 E(t),
\]  
for \( \forall t \geq 0 \). In fact, we get from (27) and (28) that
\[
F(t) \leq E(t) + \frac{\varepsilon}{2} \left( \|u_t\|^2 + \|u\|^2 \right) \\
\leq \left( 1 + \varepsilon + \frac{\varepsilon}{2}C_1 \right) E(t) \\
= \xi_2 E(t). \quad (30)
\]
On the other hand, by (27) and (28), we obtain the following inequality
\[
F(t) \geq E(t) - \frac{\varepsilon}{2} \|u_t\|^2 - \frac{\varepsilon}{2} \|u\|^2 \\
\geq \frac{1}{2}(1 - \varepsilon) \|u_t\|^2 + J(u) - \frac{\varepsilon}{2}C_1 E(t). \quad (31)
\]
By choosing \( \varepsilon \) small enough such that \( 0 < \varepsilon \leq \min \left\{ 1, \frac{2C_1}{2C_1+1} \right\} \), it follows from (31) that
\[
F(t) \geq \left( 1 - \varepsilon - \frac{\varepsilon}{2}C_1 \right) E(t) \\
= \xi_1 E(t). \quad (32)
\]
From (30) and (32), the inequality (29) is valid.

We now differentiate (28), by using the equation (1) and Lemma 1, to obtain
\[
F'(t) = (\varepsilon - 1) \|u_t\|^2 - \varepsilon \left\| \mathcal{P}^{\frac{1}{2}} u \right\|^2 - \varepsilon \|u\|^2 - \varepsilon \int \Omega u_t u dx + \varepsilon \int \Omega u^2 \ln |u|^2 dx. \quad (33)
\]
For any \( \zeta > 0 \), we have from Young’s inequality that
\[
\left| \int \Omega u_t u dx \right| \leq \frac{1}{4\zeta} \|u_t\|^2 + \zeta \|u\|^2. \quad (34)
\]
Therefore, inserting (34) into (33), we obtain
\[
F'(t) \leq \left( \varepsilon + \frac{\varepsilon}{4\zeta} - 1 \right) \|u_t\|^2 - \varepsilon \left\| \mathcal{P}^{\frac{1}{2}} u \right\|^2 + \varepsilon(\zeta - 1) \|u\|^2 + \varepsilon \int \Omega u^2 \ln |u|^2 dx. \quad (35)
\]
By using (7) and (35), for any positive constant $\eta$, we have

\[
F'(t) \leq -\eta \varepsilon E(t) + \left[ \varepsilon \left( 1 + \frac{\eta}{2} + \frac{1}{4\eta} \right) - 1 \right] \|u_t\|^2 \\
+ \varepsilon \left( \frac{\eta}{2} - 1 \right) \left\| \mathcal{P}^{\frac{1}{2}} u \right\|^2 + \varepsilon (\eta + \zeta - 1) \|u\|^2 \\
+ \varepsilon \left( 1 - \frac{\eta}{2} \right) \int_{\Omega} u^2 \ln |u|^2 dx. 
\]

(36)

Now, choosing $0 < \eta \leq 1$, and by Lemma 3 and (24), we get

\[
F'(t) \leq -\eta \varepsilon E(t) + \left[ \varepsilon \left( 1 + \frac{\eta}{2} + \frac{1}{4\eta} \right) - 1 \right] \|u_t\|^2 \\
- \varepsilon \left( 1 - \frac{\eta}{2} \right) \left( 1 - \frac{\alpha^2}{\pi} \right) \left\| \mathcal{P}^{\frac{1}{2}} u \right\|^2 \\
+ \varepsilon \left\{ \eta + \zeta - 1 + \left( 1 - \frac{\eta}{2} \right) \left[ \ln(2J(t)) - n(1 + \ln \alpha) \right] \right\} \|u\|^2. 
\]

(37)

By $0 < \eta \leq 1$ and $J(t) < E(0) < \frac{1}{2} \left( \frac{\pi}{c_p} \right)^{2} e^{n} \beta \leq d$, we select the constant $\alpha$ to meet

$\sqrt{\frac{\pi}{c_p} \beta^{2}} \leq \alpha \leq \frac{\pi}{c_p}$, and take $\zeta > 0$ small sufficiently such that

\[
\zeta < 1 - \eta + \left( \frac{\eta}{2} - 1 \right) [\ln(2J(t)) - n(1 + \ln \alpha)] \\
< 1 - \eta + \left( \frac{\eta}{2} - 1 \right) \left[ \ln \left( \frac{\pi}{c_p} \right)^{2} e^{n} \beta - n(1 + \ln \alpha) \right] \\
= 1 - \eta + \left( \frac{\eta}{2} - 1 \right) \ln \left( \frac{\pi}{c_p} \right)^{\frac{n}{2} \beta} \alpha. 
\]

Then, we obtain

\[
F'(t) \leq -\eta \varepsilon E(t) + \left[ \varepsilon \left( 1 + \frac{\eta}{2} + \frac{1}{4\eta} \right) - 1 \right] \|u_t\|^2. 
\]

(38)

Now, choosing $\varepsilon$ so small enough that

\[
\varepsilon \left( 1 + \frac{\eta}{2} + \frac{1}{4\eta} \right) - 1 < 0,
\]

then the inequality (38) implies that

\[
F'(t) \leq -\eta \varepsilon E(t), \forall t \geq 0. 
\]

(39)
We conclude from (29) and (39) that
\[
F'(t) \leq -kF(t), \forall t \geq 0, 
\]
where \( k = \eta \epsilon / \xi^2 > 0 \).

Integrating the differential inequality (40) from 0 to \( t \) gives the following exponential decay estimate for function \( F(t) \)
\[
F(t) \leq F(0)e^{-kt}, \forall t \geq 0. 
\]
Consequently, we obtain from (29) once again that
\[
E(t) \leq \kappa e^{-kt}, \forall t \geq 0, 
\]
where \( \kappa = F(0)/\xi_1 \). This completes the proof of Theorem 10.

4. Global nonexistence of solutions

In this section, we establish the global nonexistence of solutions of (1).

**Lemma 11.** Let \( u(t) \) be a solution of (1) which is given by Theorem 5. If \( u_0 \in \mathcal{U} \) and \( E(0) < d \), then \( u(t) \in \mathcal{U} \) and \( E(t) < d \), for all \( t \geq 0 \).

**Proof.** It follows from the conditions in Lemma 11 and Lemma 1 that
\[
E(t) \leq E(0) < d, \forall t \in [0, T). 
\]
Therefore, we have from (7) that
\[
J(u) \leq E(t) < d, \forall t \in [0, T). 
\]
Next, let us assume by contradiction that there exists \( t^* \in [0, T) \) such that \( u(t^*) \notin \mathcal{U} \), then by continuity, we have \( I(u(t^*)) = 0 \). This implies that \( u(t^*) \in \mathcal{N} \). We get from (10) that \( J(u(t^*)) \geq d \), which is contradiction with (42). Consequently, the conclusion in Lemma 11 holds.

**Theorem 12.** (Global nonexistence) Suppose that \( u_0 \in \mathcal{U}, u_1 \in L^2(\Omega) \) satisfies
\[
\int_\Omega u_0(x)u_1(x)dx \neq 0 \text{ and } 
\]
\[
0 < E(0) < \min \left\{ d, \frac{3}{4} \left( \frac{\pi}{c_p} \right)^2 e^n \right\}. 
\]
Then the solution \( u(t) \) in Theorem 5 of the problem (1) blows up in finite \( T_* < +\infty \), this means that
\[
\lim_{t \to T_*} \|u(t)\|^2 = +\infty. 
\]
Proof. By \( u_0 \in U \), \( E(0) < d \) and Lemma 11, we obtain \( u \in U \) for all \( t \in [0, T] \). Thus, we get
\[
I(u) = \|\mathcal{P}^{\frac{1}{2}} u\|^2 + \|u\|^2 - \int_\Omega u^2 \ln |u|^2 \, dx < 0, \quad \forall t \in [0, T]. \tag{43}
\]
We have from (43) and Lemma 4 that
\[
\left(1 - \frac{c_p \alpha^2}{\pi}\right) \|\mathcal{P}^{\frac{1}{2}} u\|^2 + \|u\|^2 + [n(1 + \ln \alpha) - \ln \|u\|^2] \|u\|^2 < 0. \tag{44}
\]
We conclude from \( \alpha = \sqrt{\frac{\pi}{c_p}} \) and (44) that
\[
n(1 + \ln \alpha) - \ln \|u\|^2 < 0,
\]
which implies that
\[
\|u(t)\|^2 > 2d, \forall t \in [0, T]. \tag{45}
\]
Assume by contradiction that the solution \( u(t) \) is global. Then for any \( T > 0 \), we define \( G(t) : [0, T] \rightarrow [0, +\infty] \) by
\[
G(t) = \|u(t)\|^2 + \int_0^t \|u(s)\|^2 \, ds + (T - t) \|u_0\|^2. \tag{46}
\]
Noting that \( G(t) > 0 \) for all \( t \in [0, T] \). By the continuity of the function \( G(t) \), there exists \( \mu > 0 \) (independent of the choice of \( T \)) such that
\[
G(t) \geq \mu > 0, \forall t \in [0, T]. \tag{47}
\]
By differentiating on both sides of (46), we get
\[
G'(t) = 2 \int_\Omega u u_t \, dx + \|u(t)\|^2 - \|u_0\|^2 = 2 \int_\Omega u u_t + 2 \int_0^t \int_\Omega u(s) u_t(s) \, dx \, ds. \tag{48}
\]
Taking the derivative of the function \( G'(t) \) in (48), we obtain
\[
G''(t) = 2 \|u_t\|^2 + 2 \int_\Omega u u_{tt} \, dx + 2 \int_\Omega u u_t \, dx. \tag{49}
\]
We get from (1) and (49) that
\[
G''(t) = 2 \left[ \|u_t(t)\|^2 + \int_\Omega u^2 \ln |u|^2 \, dx - \right. \left. \|\mathcal{P}^{\frac{1}{2}} u(t)\|^2 - \|u(t)\|^2 \right]. \tag{50}
\]
We have from (46), (48) and (50) that
\[
G(t)G''(t) - \frac{3}{2}[G'(t)]^2 = 2G(t) \left[ \|u_t(t)\|^2 + \int_{\Omega} u^2 \ln |u|^2 dx \right] \\
-2G(t) \left[ \|P^{\frac{1}{2}} u(t)\|^2 + \|u(t)\|^2 \right] \\
-6[G(t) - (T - t) \|u_0\|^2] \times \left[ \|u_t(t)\|^2 + \int_0^t \|u_t(s)\|^2 ds \right] \\
+6K(t)
\] (51)

where
\[
K(t) = \left[ \|u(t)\|^2 + \int_0^t \|u(s)\|^2 ds \right] \times \left[ \|u_t(t)\|^2 + \int_0^t \|u_t(s)\|^2 ds \right] \\
- \left[ \int_{\Omega} uu_t dx + \int_0^t \int_{\Omega} u(s)u_t(s)dxds \right]^2.
\] (52)

By using Schwarz inequality, we have
\[
\left( \int_{\Omega} uu_t dx \right)^2 \leq \|u(t)\|^2 \|u_t(t)\|^2,
\] (53)
\[
\left( \int_0^t \int_{\Omega} uu_t dxds \right)^2 \leq \int_0^t \|u(s)\|^2 ds \int_0^t \|u_t(s)\|^2 ds,
\] (54)
and
\[
2 \int_0^t \int_{\Omega} u(s)u_t(s)dxds \int_{\Omega} uu_t dx \leq \|u_t(t)\|^2 \int_0^t \|u(s)\|^2 ds + \|u(t)\|^2 \int_0^t \|u_t(s)\|^2 ds.
\] (55)

These inequalities (52)-(55) entail \(K(t) \geq 0\) for all \(t \in [0, T]\). Therefore, we reach the following differential inequality from (51) that
\[
G(t)G''(t) - \frac{3}{2}[G'(t)]^2 \geq G(t)\chi(t), \forall t \in [0, T],
\] (56)

where
\[
\chi(t) = 2 \left[ \|u_t(t)\|^2 + \int_{\Omega} u^2 \ln |u|^2 dx - \|P^{\frac{1}{2}} u(t)\|^2 - \|u(t)\|^2 \right] \\
-6 \left[ \|u_t(t)\|^2 + \int_0^t \|u_t(s)\|^2 ds \right].
\] (57)

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We have from (7) and Lemma 4 that
\[ \chi(t) \geq -8E(t) + 2 \left( 1 - \frac{c_p \alpha^2}{\pi} \right) \| P^\frac{1}{2} u(t) \|^2 + 6 \| u(t) \|^2 + 2 \left[ n \left( 1 + \ln \alpha \right) - \ln \| u(t) \|^2 \right] \| u(t) \|^2 - 6 \int_0^t \| u_t(s) \|^2 ds. \] (58)

By (13), (45) and \( \alpha = \sqrt{\frac{\pi}{c_p}} \), we have from (58) that
\[ \chi(t) \geq -8E(t) + 2 \left[ n \left( 1 + \ln \alpha \right) - \ln \| u(t) \|^2 \right] \| u(t) \|^2 - 6 \int_0^t \| u_t(s) \|^2 ds. \] (59)

By Lemma 1, we get
\[ \chi(t) \geq -8E(0) + 2 \left[ n \left( 1 + \ln \alpha \right) - \ln \| u(t) \|^2 \right] \| u(t) \|^2 + 2 \int_0^t \| u_t(s) \|^2 ds. \] (60)

Hence, we conclude from (45) and \( E(0) < d \) that
\[ \chi(t) \geq -8E(0) + 12d = 8 \left[ d - E(0) \right] + 4d > 0. \] (61)

Therefore, there exists \( \gamma > 0 \) which is independent of \( T \) such that
\[ \chi(t) \geq \gamma > 0, \forall t \geq 0. \] (62)

It follows from (47), (56) and (62) that
\[ G(t)G''(t) - \frac{3}{2} [G'(t)]^2 \geq \mu \gamma > 0, \forall t \in [0, T]. \] (63)

By the differential inequality (63), we have
\[ G(t) \geq \frac{G(0)}{\left( 1 - \frac{G'(0)}{2G(0)} t \right)^2}. \] (64)

Hence, there exists \( T_* \) such that
\[ 0 < T_* < \frac{2G(0)}{G'(0)} \leq T, \] (65)
and we have
\[ \lim_{t \to T_*^{-}} G(t) = +\infty. \] (66)

From the definition (46) of \( G(t) \), (66) means that
\[ \lim_{t \to T_*^{-}} \| u(t) \|^2 = +\infty. \]

Thus we can not suppose that the solution of (1) is global.
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