HYPER GENERALIZED WEAKLY SYMMETRIC
$(CS)_4$-SPACETIME AND THE RICCI SOLITONS

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Abstract. A hyper generalized weakly symmetric $(CS)_4$-spacetime has been studied. It is found that such a spacetime is a perfect fluid spacetime, space of quasi constant curvature and conformally flat. Also, we point out the sufficient condition for a compact, orientable hyper generalized weakly symmetric $(CS)_4$-spacetime to be conformal to a sphere in 5 dimensional Euclidean space $E_5$.

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1. Introduction

In 2003 Shaikh [14] established the notion of Lorentzian concircular structure manifolds (briefly, $(LCS)_n$-manifolds) with an example. Four dimensional Lorentzian concircular structure manifold is termed as $(CS)_4$-spacetime (See [12]).

Definition 1. A semi-Riemannian manifold $(M^n, g)$, $n = \text{dim} \ M$ is said to be hyper generalized weakly symmetric if its Riemannian curvature tensor $R$ admits the relation


where

$$ (g \wedge S)(Y, U, V, Z) = g(Y, Z)S(U, V) + g(U, V)S(Y, Z) - g(Y, V)S(U, Z) - g(U, Z)S(Y, V) $$

(2)
and $\Pi_i$, $\Psi_i$ and $\wp_i$ are non-zero 1-forms defined as $\Pi_i(X) = g(X, e_i)$, $\Psi_i(X) = g(X, \tau_i)$ and $\wp_i(X) = g(X, \tau_i)$.

The beauty of such manifold is that it has the flavour of,
(i) locally symmetric space [5] (for $\Pi_i = \Psi_i = \wp_i = 0$, where $i = 1, 2$),
(ii) recurrent space [18] (for $\Pi_1 \neq 0$, $\Pi_2 = \Psi_1 = \wp_1 = 0$, where $i = 1, 2$),
(iii) hyper recurrent space [13] (for $\Pi_i \neq 0, \Psi_i = \wp_i = 0$, where $i = 1, 2$),
(iv) pseudo recurrent space [6] (for $\Pi_1 = \Psi_1 = \wp_1 = 0$ and $\Pi_2 = \Psi_2 = \wp_2 = 0$),
(v) semi-pseudo symmetric space [16] (for $\Psi_1 = \wp_1 + \wp_1, \Psi_1 = \wp_1 = 0$, where $i = 1, 2$),
(vi) hyper semi-pseudo symmetric space (for $\Pi_1 = \Psi_1 = \wp_1 = 0$ and $\Psi_2 = \wp_2 = 0$),
(vii) hyper pseudo symmetric space (for $\Pi_1 = \Psi_1 = \wp_1 = 0$ where $i = 1, 2$),
(viii) almost hyper pseudo symmetric space [16] (for $\Pi_1 = \Psi_1 = \wp_1 + \wp_1, \Psi_1 = \wp_1 = 0$ and $\Psi_2 = \wp_2 = 0$),
(ix) weakly symmetric space [15] (for $\Pi_2 = \Psi_2 = \wp_2 = 0$).

**Definition 2.** A four dimensional Lorentzian manifold is said to be a perfect fluid spacetimes if it satisfies

$$S(U, V) = \gamma g(U, V) + \nu \delta(U)\delta(V),$$

for any vector fields $U$ and $V$, where $\gamma$ and $\nu$ are some scalar functions, $\delta$ being a non-zero 1-form corresponding to an unit timelike vector field $\pi$, that is, $g(U, \pi) = \delta(U)$ and $g(\pi, \pi) = -1$.

**Definition 3. ([8])** A Lorentzian manifold is said to infinitesimally spatially isotropic relative to a unit timelike vector field $\varphi$ if the Riemannian curvature tensor $R$ satisfies the condition:

$$R(U, Y)V = \varphi [g(Y, V)U - g(U, V)Y],$$

for all $U, Y, Z$ belongs to $g^{\perp}$ and $R(U, \varphi)\varphi = \pi U$ for all $U \in g^{\perp}$, where $\varphi$ and $\pi$ are real valued functions.

First section deals with some basic definitions and thereafter we mention known results of $(CS)_4$-spacetimes which are used in sequel. In third section, we show that a hyper generalized weakly symmetric $(CS)_4$-spacetime is a perfect fluid spacetimes, a space of quasi-constant curvature, conformally flat and infinitesimally spatially isotropic relative to the unit timelike vector field $\xi$. Then we study Ricci solitons.
and the Poisson equation in that spacetime. Lastly, we obtain the sufficient condition for a compact, orientable hyper generalized weakly symmetric \((\text{CS})_4\)-spacetime to be conformal to a sphere in \(E_5\).

2. \((\text{CS})_4\)-Spacetimes

In a \((\text{CS})_4\)-spacetime, the following relations hold [[14], [3], [4], [2]]:

\[
(\nabla_U \eta)V = \delta \{ g(U, V) + \eta(U)\eta(V) \} \quad (\delta \neq 0),
\]

\[
\eta(\xi) = -1, \quad \phi \circ \xi = 0,
\]

\[
\phi U = U + \eta(U)\xi = \frac{1}{\delta} \nabla_U \xi,
\]

\[
\eta(\phi X) = 0, \quad g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),
\]

\[
\eta(R(X, Y)Z) = (\delta^2 - \epsilon)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)],
\]

\[
R(X, Y)\xi = (\delta^2 - \epsilon)[\eta(Y)X - \eta(X)Y],
\]

\[
(\nabla_X R)(Y, Z)\xi = \delta(\delta^2 - \epsilon)[g(X, Z)Y - g(X, Y)Z]
+ (2\delta \epsilon - \theta)\eta(X)[\eta(Z)Y - \eta(Y)Z] - \delta R(Y, Z)(X),
\]

\[
(\nabla_X R)(Y, Z, V, \xi) = -\delta R(Y, Z, V, X)
- \delta(\delta^2 - \epsilon)[g(X, Z)g(Y, V) - g(X, Y)g(Z, V)]
- (2\delta \epsilon - \theta)\eta(X)[\eta(Z)g(Y, V) - \eta(Y)g(Z, V)],
\]

\[
S(X, \xi) = 3(\delta^2 - \epsilon)\eta(X),
\]

\[
(\nabla_U S)(X, \xi) = 3[\delta(\delta^2 - \epsilon)g(X, U) + (2\delta \epsilon - \theta)\eta(X)\eta(U)] - \delta S(X, U).
\]

for any vector fields \(X, Y, Z, U, V\).
3. Hyper generalized weakly symmetric \((CS)_1\)-spacetimes

We first consider a hyper generalized weakly symmetric \((CS)_1\)-spacetimes with defining condition (1). Using (2) in (1) and then contracting the resultant, we have

\[
(\nabla_X S)(Y, Z) = \Pi_1(X)S(Y, Z) + \Psi_1(Y)S(X, Z) + F_1(Z)S(X, Y) \\
+ \Psi_1(R(X, Y)Z) + F_1(R(X, Z)Y) + \Pi_2(X)\{2S(Y, Z) + rg(Y, Z)\} \\
+ \Psi_2(Y)\{2S(X, Z) + rg(X, Z)\} + F_2(Z)\{2S(Y, X) + rg(Y, X)\} \\
+ \Psi_2(LX)g(Y, Z) + \Psi_2(X)S(Y, Z) - \Psi_2(Y)S(X, Z) \\
- \Psi_2(LY)g(Z, X) + F_2(LX)g(Y, Z) + F_2(X)S(Y, Z) \\
- F_2(LZ)g(Y, X) - F_2(Z)S(X, Y).
\]

(13)

Setting \(Z = \xi\) in (13) and then making use of (8), (11) and (12) we get

\[
3\{\delta(\delta^2 - \epsilon)g(X, Y) + (2\delta\epsilon - \theta)\eta(X)\eta(Y)\} - \delta S(X, Y) \\
= \Pi_1(X)3(\delta^2 - \epsilon)\eta(Y) + \Psi_1(Y)3(\delta^2 - \epsilon)\eta(X) \\
+ (\delta^2 - \epsilon)[\Psi_1(X)\eta(Y) - \Psi_1(Y)\eta(X)] \\
+ (\delta^2 - \epsilon)[\eta(Y)F_1(X) - g(X, Y)F_1(\xi)] + S(X, Y)F_1(\xi) \\
+ \Pi_2(X)\eta(Y)\{6(\delta^2 - \epsilon) + r\} + \Psi_2(Y)\eta(Y)\{6(\delta^2 - \epsilon) + r\} \\
+ F_2(\xi)(2S(Y, X) + rg(Y, X)] + \Psi_2(LX)\eta(Y) \\
+ \Psi_2(X)3(\delta^2 - \epsilon)\eta(Y) - \Psi_2(Y)3(\delta^2 - \epsilon)\eta(X) \\
- \Psi_2(LY)\eta(X) + F_2(LX)\eta(Y) + F_2(X)3(\delta^2 - \epsilon)\eta(Y) \\
- F_2(L\xi)g(Y, X) - F_2(\xi)S(X, Y)
\]

(14)

which yields

\[
3(2\delta\epsilon - \theta) = r[\Psi_2(\xi) - \Pi_2(\xi) - F_2(\xi)] \\
- 3(\delta^2 - \epsilon)[\Pi_1(\xi) + \Psi_1(\xi) + 3\Pi_2(\xi)] \\
+ 3\Psi_2(\xi) + F_1(\xi) + 2F_2(\xi)
\]

(15)

for \(X = Y = \xi\).

Again, setting \(Y = \xi\) and \(X = \xi\) in succession in (14) and then using the relation
(15), we have respectively
\[ \begin{aligned}
3\Pi_1(X)(\delta^2 - \epsilon) + \Psi_1(X)(\delta^2 - \epsilon) + F_1(X)(\delta^2 - \epsilon) \\
+6\Pi_2(X)(\delta^2 - \epsilon) + r\Pi_2(X) + \Psi_2(LX) + F_2(LX) \\
+3\Psi_2(X)(\delta^2 - \epsilon) + 3F_2(X)(\delta^2 - \epsilon) \\
= \eta(X)[3\delta(\delta^2 - \epsilon) - 3(\delta^2 - \epsilon) - \Psi_1(\xi)(\delta^2 - \epsilon) \\
-6\Psi_2(\xi)(\delta^2 - \epsilon) + 2r\Psi_2(\xi) - \Psi_2(L\xi) \\
-r\Pi_2(\xi) - 3\Pi_1(\xi)(\delta^2 - \epsilon) - 9\Pi_2(\xi)(\delta^2 - \epsilon) \\
-3F_2(\xi)(\delta^2 - \epsilon) - F_1(\xi)(\delta^2 - \epsilon) - F_2(L\xi)]
\end{aligned} \] (16)

and
\[ \begin{aligned}
2\Psi_1(Y)(\delta^2 - \epsilon) + 3\Psi_2(Y)(\delta^2 - \epsilon) \\
+6\Psi_2(Y) - \Psi_2(LY) \\
= \eta(Y)[3\delta(\delta^2 - \epsilon) - 3(\delta^2 - \epsilon) - 2\Psi_1(\xi)(\delta^2 - \epsilon) \\
-3\Pi_2(\xi)(\delta^2 - \epsilon) - 6\Psi_2(\xi)(\delta^2 - \epsilon) \\
+r\Psi_2(\xi) + \Psi_2(L\xi)].
\end{aligned} \] (17)

Next, in view of (15), (16) and (17), the relation (14) yields
\[ \begin{aligned}
S(X,Y) &= \frac{1}{3}[r - 3(\delta^2 - \epsilon)]g(X,Y) \\
&\quad + \frac{1}{3}[r - 12(\delta^2 - \epsilon)]\eta(X)\eta(Y).
\end{aligned} \] (18)

This leads to the followings:

**Theorem 1.** Every hyper generalized weakly symmetric \((CS)_4\)-spacetime is a perfect fluid spacetime.

Now with the help of (10), (18) and by the symmetry of the Riemann curvature tensor, one can easily find out
\[ \begin{aligned}
R(Y,V,U,Z) &= \frac{r - 6(\delta^2 - \epsilon)}{6}G(Y,V,U,Z) \\
&\quad + \frac{r - 12(\delta^2 - \epsilon)}{6}H(Y,V,U,Z),
\end{aligned} \] (19)

where \(G = g \wedge g\) and \(H = g \wedge (\eta \otimes \eta)\). Thus we can state:

**Theorem 2.** A hyper generalized weakly symmetric \((CS)_4\)-spacetime is a space of quasi constant curvature.
In an 4-dimensinal semi-Riemannian manifold the Weyl conformal curvature tensor defined as
\[
C(X,Y)Z = R(X,Y)Z - \frac{1}{2}[S(Y,Z)X - S(X,Z)Y \\
+ g(Y,Z)QX - g(X,Z)QY + \frac{r}{3}(g(Y,Z)X - g(X,Z)Y)].
\]

By virtue of (18) and (19), we can calculate that the Weyl conformal curvature tensor vanishes identically. This infer:

**Theorem 3.** Every hyper generalized weakly symmetric \((CS)_4\)-spacetime is conformally flat.

**Theorem 4.** ([17], Theorem 3.3.) In a hyper generalized weakly symmetric \((CS)_4\)-spacetime with constant scalar curvature (mentioned in (15)) the followings are true: i) the characteristic vector field \(\xi\) is irrotational, ii) the integral curves of the characteristic vector field \(\xi\) are geodesic, iii) the characteristic vector field \(\xi\) corresponding to the 1-form \(\eta\) is a unit proper concircular vector field.

Next, we assume that \(\xi^\perp\) is an orthonormal 3-dimensional distribution to \(\xi\) in hyper generalized weakly symmetric \((CS)_4\)-spacetime. Then \(g(U,\xi) = 0\), for all \(U \in \xi^\perp\). Therefore, from (19) we obtain
\[
R(U,Y)V = \frac{r - 6(\delta^2 - \epsilon)}{6}[g(Y,V)U - g(U,V)Y].
\]
From the above equation, we have
\[
R(U,\xi)\xi = \frac{6(\delta^2 - \epsilon) - r}{6}U.
\]
for all \(U \in \xi^\perp\). This leads to the followings;

**Theorem 5.** A hyper generalized weakly symmetric \((CS)_4\)-spacetime is infinitesimally spatially isotropic relative to the unit timelike vector field \(\xi\).

4. Ricci solitons on hyper generalized weakly symmetric \((CS)_4\)-spacetime

Suppose in a \((CS)_4\)-spacetime the pair \((\lambda, \xi)\) defines a Ricci soliton, that is,
\[
2S(X,Y) = -(\mathcal{L}_\xi g)(X,Y) - 2\lambda g(X,Y),
\]
for $\lambda$ a real number. Writting $\mathcal{L}_\xi g$ in terms of the Levi-Civita connection $\nabla$, the above equation yields,

$$2S(X,Y) = -g(\nabla_X \xi, Y) - g(X, \nabla_Y \xi) - 2\lambda g(X,Y),$$

for any $X, Y \in \chi(M)$. As a consequence of (5), the above equation becomes

$$S(X,Y) = -(\lambda + \delta)g(X,Y) - \delta \eta(X)\eta(Y). \tag{20}$$

In view of (18) and (20) we obtain

$$\lambda = -\frac{r}{4} - \frac{3}{4}\delta.$$ 

Therefore;

**Theorem 6.** Ricci soliton in a hyper generalized weakly symmetric (CS)$_4$-spacetime is $(-\frac{r}{4} - \frac{3}{4}\delta, \xi)$.

**Theorem 7.** If $(\lambda = -(\frac{r}{4} + \frac{3}{4}\delta), \xi = \text{grad}(f))$ defines a Ricci soliton in a hyper generalized weakly symmetric (CS)$_4$-spacetime, then the Poisson equation satisfied by $f$ is

$$\Delta(f) = -(4\lambda + r).$$

5. **Sufficient condition for a compact, orientable hyper generalized weakly symmetric (CS)$_4$-spacetime to be conformal to a sphere in 5 dimensinal Euclidean space $E_5$.

**Definition 4.** Suppose, $(M_1,g_1)$ and $(M_2,g_2)$ be any two $n$-dimensional Riemannian manifold. Then $(M_1,g_1)$ is said to be conformal to $(M_2,g_2)$ if, i) there exits a one- one differentiable mapping $\varphi : (M_1,g_1) \rightarrow (M_2,g_2)$, ii) the angle between any two vectors at a point $p$ of $M_1$ is equal to the angle between the corresponding vectors mapped by $\varphi$ in $M_2$.

According to Watanabe [19], if in an $n$-dimensional Riemannian manifold $\hat{M}$, there exists a non parallel vector field $U$ such that the relation

$$\int_{\hat{M}} S(U,U)dx = \frac{1}{2} \int_{\hat{M}} |dU|^2 dx + \frac{n-1}{n} \int_{\hat{M}} (\partial U)^2 dx \tag{21}$$

satisfies, then $\hat{M}$ is conformal to a sphere in $E_{n+1}$, where $dx$ is the volume element of $\hat{M}$ and $dU$ and $\partial U$ are the curl and divergence of $U$ respectively. Here, we
consider a compact orientable hyper generalized weakly symmetric \((CS)_4\)-spactime without boundary.

From (18), we get
\[
S(U, \xi) = 3(\delta^2 - \epsilon)\eta(U).
\]
Hence,
\[
S(\xi, \xi) = 3(\epsilon - \delta^2).
\]
In view of this and letting \(\xi\) for \(U\), the relation (21) becomes
\[
12(\epsilon - \delta^2) \int_M dx = 2 \int_M |d\xi|^2 dx + 3 \int_M (\partial\xi)^2 dx. \tag{22}
\]
Now, assume \(\xi\) is a parallel vector field. Then
\[
\nabla_U \xi = 0.
\]
Hence, from the Ricci identity we have
\[
R(U, X)\xi = 0.
\]
Which gives after contraction
\[
S(V, \xi) = 0.
\]
Since \((\delta^2 - \epsilon) \neq 0\) thus from the above, \(\xi\) cannot be a parallel vector field. Thus in a compact, orientable hyper generalized weakly symmetric \((CS)_4\)-spacetime without boundary the characteristic vector field \(\xi\) is not a parallel vector field. Therefore we can state

**Theorem 8.** If a compact, orientable hyper generalized weakly symmetric \((CS)_4\)-spacetime without boundary admits the relation (22), then it is conformal to a sphere immersed in 5 dimensional Euclidean space \(E_5\).

**Remark 1.** In [10] authors have proved that 4-dimensinal Lorentzian concircular structure (known as \((CS)_4\))-spacetime coincide with Generalized Robertson-Walker (GRW) spacetimes. Consequently, each of the above mentioned results holds also for hyper generalized weakly symmetric GRW-spacetimes.

**References**


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