IDEALS IN THE BANACH ALGEBRAS OF $\alpha$-LIPSCHITZ VECTOR-VALUED OPERATORS

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ABSTRACT. We study an interesting class of Banach function algebras of vector-valued operators on compact metric spaces, and investigate certain ideals of the Lipschitz algebras. In this paper, we consider a nonempty compact metric space $(X,d)$ and a commutative unital Banach algebra $(B, \| \cdot \|)$ over the scalar field $\mathbb{F} (= \mathbb{R}$ or $\mathbb{C})$. At first, we define the $B$-valued $\alpha$-Lipschitz operator algebras $\text{Lip}_\alpha(X,B)$ and $\text{lip}_\alpha(X,B)$, where $\alpha \in (0,1]$. Then we characterize the norm closed ideals of $\text{lip}_\alpha(X,B)$, and primary ideals of $\text{Lip}_\alpha(X,B)$.

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1. Introduction

Throughout this paper, let $(X,d)$ be a compact metric space which has at least two elements, $(B, \| \cdot \|)$ be a commutative unital Banach algebra over the scalar field $\mathbb{F}(= \mathbb{R}$ or $\mathbb{C})$, $C(X,B)$ be the set of all $B$-valued continuous operators and $C_b(X,B)$ be the set of all bounded $B$-valued continuous operators on $X$, and also $\alpha \in \mathbb{R}$ with $0 < \alpha \leq 1$. When $B = \mathbb{F}$, we write $C(X)$ instead of $C(X,B)$.

The dual space of $B$ is the vector space $B^*$ whose elements are the continuous linear functionals on $B$. The set of all multiplicative functionals on $B$ is called spectrum of $B$; we denote it by $\sigma(B)$. Suppose that throughout this article $\Lambda \in \sigma(B)$ is arbitrary and fixed. Since $\sigma(B)$ is a subset of the closed unit ball of $B^*$, $\| \Lambda \|$ is bounded, where $\| \Lambda \| = \sup \{ | \Lambda x | : x \in B, \| x \| \leq 1 \}$.

When $B = \mathbb{F}$, take $\Lambda$ as the identity function $\Lambda x = x$.

Consider the set $Y$ as follows

$$Y := \{(x,y) : x,y \in X, x \neq y\}.$$ (1)
For an operator $f : X \to B$, and any $(x, y) \in Y$, define
\[
L_\alpha^\circ f(x, y) := \frac{|(\Lambda of)(x) - (\Lambda of)(y)|}{d^\alpha(x, y)},
\]
where $d^\alpha(x, y) = (d(x, y))^\alpha$, and define
\[
p_\alpha(f) := \sup_{x \neq y} L_\alpha^\circ f(x, y),
\]
which is called the \textit{Lipschitz constant} of $f$. Also, for $0 < \alpha \leq 1$ define
\[
\text{Lip}_\alpha(X, B) := \{ f \in C_b(X, B) : p_\alpha(f) < +\infty \},
\]
and for $0 < \alpha < 1$, define
\[
\text{lip}_\alpha(X, B) := \{ f \in \text{Lip}_\alpha(X, B) : \lim_{d(x, y) \to 0} L_\alpha^\circ f(x, y) = 0 \}.
\]
The elements of $\text{Lip}_\alpha(X, B)$ and $\text{lip}_\alpha(X, B)$ are called \textit{big} and \textit{little} $\alpha$-Lipschitz $B$-valued operators, respectively.

Now, for each $\lambda \in F$ and $f, g \in C(X, B)$ define
\[
(f + g)(x) := f(x) + f(x), \quad (\lambda f)(x) := \lambda f(x), \quad \forall x \in X,
\]
\[
\| f \|_{\infty} := \sup_{x \in X} \| f(x) \|,
\]
and for any $f \in \text{Lip}_\alpha(X, B)$ define
\[
\| f \|_\alpha := p_\alpha(f) + \| f \|_{\infty}.
\]
It is easy to see that $(C(X, B), \| . \|_{\infty})$ becomes a Banach algebra over $F$.

Cao, Zhang and Xu in [9] proved that $(\text{Lip}_\alpha(X, B), \| . \|_\alpha)$ is a Banach space over $F$ and $(\text{lip}_\alpha(X, B), \| . \|_\alpha)$ is a closed linear subspace of $(\text{Lip}_\alpha(X, B), \| . \|_\alpha)$, when $B$ is a Banach space.

We studied some of the properties of these algebras in [16, 17, 18, 19]. Also some properties of these algebras were studied by certain mathematicians including Abtahi [2], Ranjbary and Rejali [13].

Note that for $\alpha = 1$ and $B = F$, the space $\text{Lip}_1(X, F)$ consisting of all Lipschitz functions from $X$ into $F(= \mathbb{R}$ or $\mathbb{C})$ has a series of interesting and important properties, which has been studied by many mathematicians. Including the characterization of the ideals of these algebras in [1, 3 - 8, 11, 12, 14, 15] were researched and studied. In [10, 20] some properties of Lipschitz scalar-valued functions are mentioned.

Finally, in this paper we study the algebras of $\alpha$-Lipschitz $B$-valued operators, and we will characterize the norm closed ideals of $\text{lip}_\alpha(X, B)$, and primary ideals of $\text{Lip}_\alpha(X, B)$. 

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2. Norm closed ideals

In this section, we characterize the norm closed ideals of little $\alpha$-Lipschitz operator algebras $\text{lip}_\alpha(X,B)$. So suppose that $\alpha \in \mathbb{R}$ with $0 < \alpha < 1$.

In the complex plan $\mathbb{C}$, let $D(0,r)$ be the closed disk with center at the origin and radius $r > 0$. Define the map $\Pi_r : \mathbb{C} \to D(0,r)$ by

$$\Pi_r(z) = \begin{cases} \frac{z}{r} & ; \quad |z| \leq r \\ \frac{|z|}{r} & ; \quad |z| > r. \end{cases} \quad (3)$$

Lemma 1. Let $f \in \text{lip}_\alpha(X,B)$, and define $\Lambda_0 f := \Pi_1 \Lambda f$; $n \in \mathbb{N}$. Then $\Lambda_0 f_n \in \text{lip}_\alpha(X,B)$ for any $n \in \mathbb{N}$.

Proof. Since $f \in \text{lip}_\alpha(X,B)$, for any $(x,y) \in Y$ ($Y$ is defined in (1)) we have

$$\lim_{d(x,y) \to 0} \left| \frac{(\Lambda_0 f)(x) - (\Lambda_0 f)(y)}{d^\alpha(x,y)} \right| = 0.$$  

Then for each $n \geq 1$ and $(x,y) \in Y$, we have

$$\lim_{d(x,y) \to 0} \left| \frac{(\Lambda_0 f_n)(x) - (\Lambda_0 f_n)(y)}{d^\alpha(x,y)} \right| = \lim_{d(x,y) \to 0} \left| \frac{\Pi_1 \frac{1}{n} (\Lambda_0 f)(x) - \Pi_1 \frac{1}{n} (\Lambda_0 f)(y)}{d^\alpha(x,y)} \right|. \quad (4)$$

Now we have three case:

Case 1. Suppose $|(|(\Lambda_0 f)(x)| \leq \frac{1}{n}$ and $|(|(\Lambda_0 f)(y)| \leq \frac{1}{n}$. Then

$$\lim_{d(x,y) \to 0} \left| \frac{(\Lambda_0 f)(x) - (\Lambda_0 f)(y)}{d^\alpha(x,y)} \right| = 0. \quad (4)$$

Case 2. Suppose $|(|(\Lambda_0 f)(x)| > \frac{1}{n}$ and $|(|(\Lambda_0 f)(y)| > \frac{1}{n}$. Then

$$\lim_{d(x,y) \to 0} \left| \frac{\frac{1}{n}(\Lambda_0 f)(x) - \frac{1}{n}(\Lambda_0 f)(y)}{(\Lambda_0 f)(x) - (\Lambda_0 f)(y)} \right| = \frac{1}{n} \left( \frac{d^\alpha(X,B)}{(\Lambda_0 f)(x)} \right), \quad (5)$$

if $|(|(\Lambda_0 f)(x)| = |(|(\Lambda_0 f)(y)|$, then

$$\lim_{d(x,y) \to 0} \left| \frac{(\Lambda_0 f)(x) - (\Lambda_0 f)(y)}{d^\alpha(x,y)} \right| = 0,$$

and so (4) = 0.
If \( |(\Lambda_0 f)(x)| \neq |(\Lambda_0 f)(y)| \), then we can assumed that \( |(\Lambda_0 f)(x)| > |(\Lambda_0 f)(y)| \).

Therefore

\[
(5) \leq \frac{1}{n} \frac{1}{|(\Lambda_0 f)(y)|} \times \lim_{d(x,y) \to 0} \frac{|(\Lambda_0 f)(x) - (\Lambda_0 f)(y)|}{d^\alpha(x,y)} = 0,
\]

and so (4) = 0.

**Case 3.** Suppose \( |(\Lambda_0 f)(x)| > \frac{1}{n}, \ |(\Lambda_0 f)(y)| \leq \frac{1}{n} \). Then

\[
(4) = \lim_{d(x,y) \to 0} \frac{\frac{1}{n} |(\Lambda_0 f)(x)|}{|(\Lambda_0 f)(x)| - (\Lambda_0 f)(y)} \leq \lim_{d(x,y) \to 0} \frac{|(\Lambda_0 f)(x) - (\Lambda_0 f)(y)|}{d^\alpha(x,y)} = 0,
\]

and so (4) = 0.

Consequently, in any case we have

\[
\lim_{d(x,y) \to 0} \frac{|(\Lambda_0 f_n)(x) - (\Lambda_0 f_n)(y)|}{d^\alpha(x,y)} = 0 ; \ n \in \mathbb{N}.
\]

This means for any \( n \in \mathbb{N} \), \( \Lambda_0 f_n \in \text{lip}_\alpha(X,B) \). \( \triangle \)

Let \( H \) be a non-empty closed subset of \( X \). Put

\[
i(H) := \{ f \in \text{lip}_\alpha(X,B) : (\Lambda_0 f)|_H = 0 \},
\]

where \( (\Lambda_0 f)|_H \) is the restriction of \( \Lambda_0 f \) to \( H \). It is easy to see that, \( i(H) \) is an ideal of \( \text{lip}_\alpha(X,B) \).

**Lemma 2.** Suppose \( H \) is a closed subset of \( X \), and \( f \in i(H) \). Then there is a sequence \( \{f_n\} \subset \text{lip}_\alpha(X,B) \) such that each \( f_n \) is equal to \( f \) on a neighborhood of \( H \), and \( \lim_{n \to +\infty} p_\alpha(\Lambda_0 f_n) = 0 \).

**Proof.** For any \( n \in \mathbb{N} \), define \( \Lambda_0 f_n := \Pi_1 \frac{1}{n} (\Lambda_0 f) \), where the map \( \Pi_1 \) is defined in (3).

Then for each \( n \in \mathbb{N} \), \( \Lambda_0 f_n \in \text{lip}_\alpha(X,B) \) by Lemma 1. Since \( f \in i(H) \), \( (\Lambda_0 f)|_H = 0 \).

So for any \( n \in \mathbb{N} \) and \( x \in H \), \( |(\Lambda_0 f_n)(x)| < \frac{1}{n} \). Therefor on a neighborhood of \( H \), we have

\[
\Lambda(f_n(x)) = (\Lambda_0 f_n)(x) = \Pi_1 \frac{1}{n} ((\Lambda_0 f)(x)) = (\Lambda_0 f)(x) = \Lambda(f(x)).
\]

Since \( \Lambda \in \sigma(B) \) is arbitrary, \( f_n(x) = f(x) \) on a neighborhood of \( H \), where \( n \in \mathbb{N} \).
Now, since for any \( n \in \mathbb{N} \) we have \( \Lambda of_n \in \text{lip}_\alpha(X, B) \), for each \( \epsilon > 0 \) there exists \( \delta > 0 \) such that for any \( (x, y) \in Y \) (\( Y \) is defined in (1)) with \( d(x, y) < \delta \) we have

\[
\frac{|(\Lambda of_n)(x) - (\Lambda of_n)(y)|}{d^\alpha(x, y)} < \epsilon.
\]

Especially for \( \epsilon = \frac{1}{n} \) (to large enough \( n \)) we have

\[
\frac{|(\Lambda of_n)(x) - (\Lambda of_n)(y)|}{d^\alpha(x, y)} < \frac{1}{n}.
\]

So, for to large enough \( n \), \( p_\alpha(\Lambda of_n) < \frac{1}{n} \). Therefore \( \lim_{n \to +\infty} p_\alpha(\Lambda of_n) = 0 \). \( \triangle \)

For each subset \( E \subset \text{lip}_\alpha(X, B) \), let its hull be the set

\[
hull(E) := \{ x \in X : (\Lambda of)(x) = 0, \ \forall f \in E \}.
\]

A subset \( E \) of \( \text{lip}_\alpha(X, B) \) is a norm closed ideal, if it is an ideal and it is closed in the topology induced by the norm on \( \text{lip}_\alpha(X, B) \).

**Lemma 3.** Let \( E \) be a norm closed ideal of \( \text{lip}_\alpha(X, B) \), and suppose \( f \in \text{lip}_\alpha(X, B) \) such that \( \Lambda of \) vanishes in a neighborhood of \( hull(E) \). Then \( f \in E \).

**Proof.** Let \( H := hull(E) \), \( \epsilon > 0 \), and \( (\Lambda of)(x) = 0 \) for any \( x \in X \) such that \( d(x, H) < \epsilon \), where \( d(x, H) := \inf \{ d(x, y) : y \in H \} \). Suppose that \( G := \{ x \in X : d(x, H) \geq \frac{\epsilon}{2} \} \). It is obvious that \( G \) is a compact subset of \( X \), and for any \( x \in G \) there is a function \( f_x \in E \) that \( \Lambda of_{f_x} \) is nonzero on an open neighborhood of \( x \). As these neighborhoods cover \( G \), by compactness. So we can find a finite set of points \( x_1, x_2, \ldots, x_n \in G \) such that \( \Lambda og \) is nowhere zero on \( G \), where \( g := f_{x_1} + f_{x_2} + \ldots + f_{x_n} \). Then \( g \in E \) and \( g(x) \) is invertible for any \( x \in G \). Define the function \( h \in \text{lip}_\alpha(X, B) \) such that \( (\Lambda oh)(x) = 0 \) for \( x \notin G \), and \( h(x) := (g(x))^{-1} f(x) \) for \( x \in G \). Then \( f = gh \) on \( G \). By ideal properties, we have \( f \in E \). \( \triangle \)

Now we prove one of the main results of the article.

**Theorem 4.** Let \( E \) be a norm closed ideal of \( \text{lip}_\alpha(X, B) \). Then \( E = i(H) \), where \( H = hull(E) \).

**Proof.** It is obvious that \( E \subseteq i(H) \). We prove that \( i(H) \subseteq E \). For this purpose, let \( f \in i(H) \) be arbitrary, so we will show that \( f \in E \).

It is clear that \( hull(E) \) is a closed subset of \( X \). So by Lemma 2, there is a sequence \( \{ f_n \} \subset \text{lip}_\alpha(X, B) \) such that \( f_n = f \) on a neighborhood of \( H \) (\( n \geq 1 \)), and
\[
\lim_{n \to +\infty} p_\alpha(\Lambda o f_n) = 0. \text{ So } \Lambda o (f - f_n) = 0 \text{ on a neighborhood of } H \ (n \geq 1). \text{ Then } f - f_n \in E \ (n \geq 1) \text{ by Lemma 3. Since } \lim_{n \to +\infty} p_\alpha(\Lambda o f_n) = 0 \text{ on a neighborhood of } H,
\]

\[
\lim_{n \to +\infty} \left| \frac{(\Lambda o f_n)(x) - (\Lambda o f_n)(y)}{d^\alpha(x, y)} \right| = 0 ; \ (x \neq y),
\]

\[
\implies \lim_{n \to +\infty} |(\Lambda o f_n)(x) - (\Lambda o f_n)(y)| = 0 ; \ (x \neq y),
\]

\[
\implies \lim_{n \to +\infty} (\Lambda o f_n)(x) = \lim_{n \to +\infty} (\Lambda o f_n)(y) ; \ (x \neq y),
\]
on neighborhood of \( H \). This relation shows that \( f_n \) is a constant function on a neighborhood of \( H \) for each \( n \geq 1 \). So, by definition of \( H = \text{hull}(E) \) and \( f \in i(H) \), we have \( \lim_{n \to +\infty} (\Lambda o f_n)(x) = 0 \) in a neighborhood of \( H \). Then \( \sup |(\Lambda o f_n)(x)| \to 0 \) on a neighborhood of \( H \). Thus \( \| \Lambda o f_n \|_\alpha \to 0 \) on a neighborhood of \( H \). On the other hand we have \( \lim_{n \to +\infty} p_\alpha(\Lambda o f_n) = 0 \), so

\[
\| \Lambda o f_n \|_\alpha = \| \Lambda o f_n \|_\infty + p_\alpha(\Lambda o f_n) \to 0
\]
on a neighborhood of \( H \).

Now define \( g_n := f - f_n \ (n \geq 1) \). Then \( \{g_n\} \subset E \), and so we have

\[
\| \Lambda o (f - g_n) \|_\alpha = \| \Lambda o f_n \|_\alpha \to 0
\]
on a neighborhood of \( H \). Since \( \Lambda \) is arbitrary, \( \| f - g_n \|_\alpha \to 0 \) on a neighborhood of \( H \). Since \( \{g_n\} \subset E \) and \( E \) is a norm closed ideal, \( f \in E \). This completes the proof. \( \Delta \)

\section*{3. Primary ideals}

In this section, we characterize the primary ideals of big \( \alpha \)-Lipschitz operator algebras \( \text{Lip}_\alpha(X, B) \). So suppose that \( \alpha \in \mathbb{R} \) with \( 0 < \alpha \leq 1 \).

Let \( H \) be a non-empty closed subset of \( X \). Put

\[
I(H) := \{ f \in \text{Lip}_\alpha(X, B) : (\Lambda o f)|_H = 0 \}.
\]

Define the mapping \( \lambda \) as follows:

\[
\lambda : \text{Lip}_\alpha(X, B) \to C(Y)
\]

\[
f \mapsto \lambda f
\]
where $Y$ is defined in (1), and $\lambda f : Y \mapsto \mathbb{F}$ with the criterion
\[
(\lambda f)(x, y) := \frac{(\Lambda of)(x) - (\Lambda of)(y)}{d^\alpha(x, y)}.
\]
Then $L_\alpha^\gamma(x, y) = \left| (\lambda f)(x, y) \right|$ for all $(x, y) \in Y$, which $L_\alpha^\gamma(x, y)$ is defined in (2). Also put
\[
J(H) := \{ f \in I(H) : \left| (\lambda f)(x, y) \right| \to 0 \quad \text{as} \quad d(x, H), d(y, H) \to 0 \}.
\]
Clearly for each ideal $E$ in $Lip_\alpha(X, B)$ with $\text{hull}(E) = H$, we have:

**Remark 1.** (i) $J(H)$ is the minimum ideal, and $\overline{J(H)}$ is the minimum closed ideal of $Lip_\alpha(X, B)$, where the norm closure $\overline{J(H)}$ of $J(H)$ is the intersection of all closed sets that contain $\overline{J(H)}$.
(ii) $I(H)$ is the maximum ideal of $Lip_\alpha(X, B)$, and (iii) $J(H) \subset E \subset I(H)$.

Below we prove a theorem, which we need to prove the main result of the article.

**Theorem 5.** Let $H$ be a non-empty closed subset of $X$. Then $J(H) = \overline{I(H)^2}$, that by $\overline{I(H)^2}$ we mean the norm closure of the set of linear combinations of products $fg$ where $f, g \in I(H)$.

**Proof.** Since $J(H)$ and $\overline{I(H)^2}$ are ideals in $Lip_\alpha(X, B)$, Remark 1 implies that $J(H) \subseteq \overline{I(H)^2}$.

Now to prove the other side of the relationship, let $f, g \in I(H)$ be arbitrary such that for each $\epsilon > 0$ and any $(x, y) \in Y$
\[
\left| (\Lambda of)(x) \right| < \frac{\epsilon}{2 \ L_\alpha^\gamma(x, y)} \quad \text{and} \quad \left| (\Lambda og)(y) \right| < \frac{\epsilon}{2 \ L_\alpha^\gamma(x, y)}
\]
when $d(x, H), d(y, H) \to 0$. Then for any $(x, y) \in Y$ as $d(x, H), d(y, H) \to 0$ we have
\[ |(\lambda(fg))(x,y)| = \frac{|(\Lambda o (fg))(x) - (\Lambda o (fg))(y)|}{d^\alpha(x,y)} \]
\[ = \frac{|(\Lambda o f)(x) (\Lambda o g)(x) - (\Lambda o f)(y) (\Lambda o g)(y)|}{d^\alpha(x,y)} \]
\[ \leq \frac{1}{d^\alpha(x,y)} \left( |(\Lambda o f)(x)| \cdot |(\Lambda o g)(x) - (\Lambda o g)(y)| \
+ |(\Lambda o g)(y)| \cdot |(\Lambda o f)(x) - (\Lambda o f)(y)| \right) \]
\[ \leq |(\Lambda o f)(x)| \cdot L^\alpha_g(x,y) + |(\Lambda o g)(y)| \cdot L^\alpha_f(x,y) \]
\[ < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \]

This implies that \( fg \in J(H) \). It follows that \( \overline{I(H)^2} \subseteq J(H) \), and the proof is complete. △

Let \( E \) be an ideal in \( Lip_\alpha(X,B) \). \( E \) is called primary if its hull contains exactly one point.

Now we prove the second main result of the article. The primary ideals of \( Lip_\alpha(X,B) \) are characterized as follows.

**Theorem 6.** Let \( a \in X \), and take \( H = \{a\} \). Suppose that \( E \) be a norm closed subspace of \( Lip_\alpha(X,B) \) such that \( J(H) \subseteq E \subseteq I(H) \). Then \( E \) is a primary ideal of \( Lip_\alpha(X,B) \). Conversely, every primary ideal of \( Lip_\alpha(X,B) \) is of this form.

**Proof.** Let \( f \in E \) and \( g \in Lip_\alpha(X,B) \) be arbitrary. Then \( g - (\Lambda o g)(a) \in I(H) \).

Hence, since \( J(H) = \overline{I(H)^2} \) by Theorem 2,

\[ (g - (\Lambda o g)(a))f \in I(H)E \subseteq I(H)^2 \subseteq J(H) \subseteq E. \]

Thus \( (g - (\Lambda o g)(a))f \in E \). Since \( (\Lambda o g)(a) \) is a constant and \( f \in E \), we have \( (\Lambda o g)(a)f \in E \). So \( gf \in E \). As the same way, \( fg \in E \). This shows that \( E \) is an ideal. Since

\[ \text{hull}(E) = \{ x \in X : (\Lambda o f)(x) = 0, \, \forall f \in E \} = \{a\}, \]

\( E \) is clearly primary.

The converse of theorem is true by Remark 1. △

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REFERENCES


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