COEFFICIENT BOUNDS FOR A CERTAIN FAMILIES OF M-FOLD SYMMETRIC BI-UNIVALENT FUNCTIONS ASSOCIATED WITH Q-ANALOGUE OF WANAS OPERATOR

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Abstract. The motivation of the present paper is to define $q$-analogue of Wanas operator in geometric function theory. We also introduce certain families $T_{m,n}^{q}(t, n, \beta, q, \delta)$ and $T_{m,n}^{q}(t, n, \beta, q, \gamma)$ of holomorphic and $m$-fold symmetric bi-univalent functions associated with $q$-analogue of Wanas operator. The upper bounds for the second and third Taylor-Maclaurin coefficients for functions in each of these subfamilies are obtained. Furthermore, Several consequences of our results are pointed out based on the various special choices of the involved parameters.

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1. Introduction and Definitions

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk in the complex plane and let $\mathcal{A} = \{f : \mathbb{U} \rightarrow \mathbb{C} : f$ is holomorphic in $\mathbb{U}, f(0) = 0 = f'(0) - 1\}$ be the family of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

Assume that $\mathcal{S}$ be the subfamily of $\mathcal{A}$ consisting of all functions $f$ univalent in $\mathbb{U}$.

The Koebe on-quarter theorem (see [5]) state that the image of $\mathbb{U}$ under every function $f(z) \in \mathcal{S}$ contains a disk of radius $1/4$. Therefore, all function $f(z) \in \mathcal{S}$ has an inverse $f^{-1}(z)$ which satisfies $f^{-1}(f(z)) = z$ and $f(f^{-1}(w)) = w \ (|w| < r_0(f), \ r_0(f) \geq \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots$$
A function \( f \in \mathcal{A} \) denoted by \( \Sigma \) is said to be bi-univalent in \( \mathbb{U} \) if both \( f^{-1}(z) \) and \( f(z) \) are univalent in \( \mathbb{U} \) (see for details [3, 4, 7, 8, 12, 14, 16, 20, 21, 24, 27, 29, 32]).

For each function \( f \in \mathcal{S} \), the function \( h(z) = (f(z^m))^{1/m} \), \( (z \in \mathbb{U}, \; m \in \mathbb{N}) \) is univalent and maps the unit disk \( \mathbb{U} \) into a region with \( m \)-fold symmetry. A function is said to be \( m \)-fold symmetric (see [11] and [15]) if it has the following normalized form:

\[
f(z) = z + \sum_{k=1}^{\infty} a_{mk+1}z^{mk+1} \quad (z \in \mathbb{U}, m \in \mathbb{N}^+). \quad (3)
\]

We denote by \( \mathcal{S}_m \) the class of \( m \)-fold symmetric univalent function in \( \mathbb{U} \), which are normalized by the series expansion (3). Also, the functions in the class \( \mathcal{S} \) are one-fold symmetric.

Analogous to the concept of \( m \)-fold symmetric univalent function, here we introduced the concept of \( m \)-fold symmetric bi-univalent functions. From (3), Srivastava et al. [25] obtained the series expansion for \( f^{-1} \) as follows:

\[
g(w) = f^{-1}(w) = w - a_{m+1}w^{m+1} + [(m+1)a_{m-1}^2 - a_{2m+1}] w^{2m+1} - \frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1} \bigg] w^{3m+1} + \cdots. \quad (4)
\]

where \( f^{-1} = g \).

We denote by \( \Sigma_m \) the class of \( m \)-fold symmetric bi-univalent function in \( \mathbb{U} \). We can note that for \( m = 1 \), the formular (4) coincides with the formular (2) of the class \( \Sigma \). Some of the examples on \( m \)-fold symmetric bi-univalent functions are given as follows:

\[
\frac{1}{2} \log \left( \frac{1+z^m}{1-z^m} \right)^{1/m}, \quad [\log(1-z^m)]^{1/m} \quad \text{and} \quad \left\{ \frac{z^m}{1-z^m} \right\}^{1/m},
\]

with the corresponding inverse functions

\[
\left( \frac{e^{2w^m} - 1}{e^{2w^m} + 1} \right)^{1/m}, \quad \left( \frac{w^m}{1+w^m} \right)^{1/m} \quad \text{and} \quad \left( \frac{e^{w^m} - 1}{e^{w^m} + 1} \right)^{1/m},
\]

respectively. Recently, different researches related to this field investigated bounds for various subclasses of \( m \)-fold bi-univalent function (see [2, 6, 23, 26, 30]).

Jackson [9, 10] introduced the \( q \)-derivative operator \( D_q \) of a function as follows:

\[
D_qf(z) = \frac{f(qz) - f(z)}{(q - 1)z} \quad (5)
\]
and \( D_q f(z) = f'(0) \). In case \( f(z) = z^{\phi} \) for \( \phi \) is a positive integer, the \( q \)-derivative of \( f(z) \) is given by
\[
D_q z^{\phi} = \frac{z^{\phi} - (qz)^\phi}{(q - 1)z} = [\phi]_q z^{\phi - 1}.
\]
As \( q \to 1^- \) and \( \phi \in \mathbb{N} \), we get
\[
[\phi]_q = \frac{1 - q^\phi}{1 - q} = 1 + q + \cdots + q^{\phi} \to \phi
\]
where \( (z \neq 0, \ q \neq 0) \), for more details on the concepts of \( q \)-derivative (see [1, 13, 17, 22]).

Wanas [28] in 2019 introduced the following operator, which can also be called (Wanas operator) \( W_{\alpha, \sigma}^{\alpha, \sigma} : \mathcal{A} \to \mathcal{A} \) defined by
\[
W_{\alpha, \sigma}^{\alpha, \sigma} f(z) = \sum_{j=2}^{\infty} [\Psi_j(\sigma, \alpha, \beta)]^n a_j z^j,
\]
where
\[
\Psi_j(\sigma, \alpha, \beta) = \sum_{c=1}^{\sigma} \binom{\sigma}{c} (-1)^{c+1} \left( \frac{\alpha^c + j \beta^c}{\alpha^c + \beta^c} \right),
\]
\( c, n \in \mathbb{N}_0, \beta \geq 0, \alpha \in \mathbb{R} \) and \( \alpha + \beta > 0 \).

Special cases of this operator can be found in [31].

Now \( q \to 1^- \), \( [\phi]_q \to \phi \). For \( f(z) \in \mathcal{A} \), we can define \( q \)-difference Wanas operator as given below
\[
W_{1,0,q}^{0,1} f(z) = f(z)
\]
\[
W_{1,1,q}^{0,1} f(z) = z W_q f(z)
\]
\[
W_{1,n,q}^{0,1} f(z) = z W_q (W_q^{n-1} f(z))
\]
\[
W_{\alpha, \sigma}^{\alpha, \sigma} f(z) = z + \sum_{j=2}^{\infty} [\Psi_j(\sigma, \alpha, \beta)]^n a_j z^j,
\]
where
\[
\Psi_j(\sigma, \alpha, \beta) = \sum_{c=1}^{\sigma} \binom{\sigma}{c} (-1)^{c+1} \left( \frac{\alpha^c + j \beta^c}{\alpha^c + \beta^c} \right),
\]
\( c, n \in \mathbb{N}_0, \beta \geq 0, \alpha \in \mathbb{R}, \alpha + \beta > 0, 0 < q < 1, z \in \mathbb{U} \).

**Lemma 1.** Suppose \( l(z) \in \mathcal{P} \), the class of functions which are holomorphic in \( \mathbb{U} \) with \( \Re(l(z)) > 0 \), \( (z \in \mathbb{U}) \) and have the form \( l(z) = 1 + l_1 z + l_2 z^2 + l_3 z^3 + \cdots \), \( (z \in \mathbb{U}) \); then \( |l_n| \leq 2 \) for each \( n \in \mathbb{N} \).
2. COEFFICIENT ESTIMATES FOR THE FUNCTION CLASS $\mathcal{T}_{\Sigma_m}^\sigma(t, n, \beta, q, \delta)$

**Definition 1.** A function $f \in \Sigma_m$ given by (3) is said to be in the class $\mathcal{T}_{\Sigma_m}^\sigma(t, n, \beta, q, \delta)$ if it satisfies the following conditions:

$$\left| \arg\left( \frac{\mathcal{M}_{\beta, t, q}^\sigma f(z)}{\mathcal{M}_{\beta, n, q}^\sigma f(z)} \right) \right| < \frac{\delta\pi}{2}, \quad (10)$$

$$\left| \arg\left( \frac{\mathcal{M}_{\beta, t, q}^\sigma g(w)}{\mathcal{M}_{\beta, n, q}^\sigma g(w)} \right) \right| < \frac{\alpha\pi}{2}, \quad (11)$$

where $0 < \delta \leq 1$, $n, t \in \mathbb{N}_0$, $t \geq n$ and the function $g = f^{-1}$ is given by (4). Also $\mathcal{M}_{\beta, t, q}^\sigma f(z)$ and $\mathcal{M}_{\beta, n, q}^\sigma f(z)$ are $q$-Wanas operators and have the following forms:

$$\mathcal{M}_{\beta, t, q}^\sigma f(z) = z + \sum_{j=1}^{\infty} \left[ \Psi_{jm+1}(\sigma, \alpha, \beta) \right]_q a_{jm+1} z^{jm+1} \quad (12)$$

and

$$\mathcal{M}_{\beta, n, q}^\sigma g(w) = w + \sum_{j=1}^{\infty} \left[ \Psi_{jm+1}(\sigma, \alpha, \beta) \right]_q b_{jm+1} w^{jm+1}. \quad (13)$$

We state and prove the following results.

**Theorem 2.** Let $f(z)$ given by (3) be in the class $\mathcal{T}_{\Sigma_m}^\sigma(t, n, \beta, q, \delta)$ ($0 < \delta \leq 1$, $n, t \in \mathbb{N}_0$). Then

$$|a_{m+1}| \leq \frac{2\delta}{\sqrt{\delta(m+1)\left[ \Psi_{2m+1}(\sigma, \alpha, \beta) \right]_q^t - \left[ \Psi_{2m+1}(\sigma, \alpha, \beta) \right]_q^n} - 2\delta \left( \left[ \Psi_{m+1}(\sigma, \alpha, \beta) \right]_q^{n+t} - \left[ \Psi_{m+1}(\sigma, \alpha, \beta) \right]_q^n \right)^2}$$

and

$$|a_{2m+1}| \leq \frac{2\delta}{\left[ \Psi_{2m+1}(\sigma, \alpha, \beta) \right]_q^t - \left[ \Psi_{2m+1}(\sigma, \alpha, \beta) \right]_q^n} + \frac{2(m+1)\delta^2}{\left( \left[ \Psi_{m+1}(\sigma, \alpha, \beta) \right]_q^t - \left[ \Psi_{m+1}(\sigma, \alpha, \beta) \right]_q^n \right)^2}. \quad (14)$$
Proof. We can write the inequality in (10) and (11) as
\[
\frac{\mathcal{W}_{\beta,t,q}^\alpha f(z)}{\mathcal{W}_{\beta,n,q}^\alpha f(z)} = [s(z)]^\delta
\]
and
\[
\frac{\mathcal{W}_{\beta,t,q}^\alpha g(w)}{\mathcal{W}_{\beta,n,q}^\alpha g(w)} = [t(w)]^\delta
\]
respectively.

Where \(g(w) = f^{-1}\) and \(s(z), t(w)\) in \(P\) have the following series representation:

\[s(z) = 1 + s_m z^m + s_{2m} z^{2m} + s_{3m} z^{3m} + \cdots\]

and

\[t(w) = 1 + t_m w^m + t_{2m} w^{2m} + t_{3m} w^{3m} + \cdots\]

Clearly,

\[\left[ s(z) \right]^\delta = 1 + \delta s_m z^m + \left( \delta s_{2m} + \frac{\delta(\delta - 1)}{2} s_m^2 \right) z^{2m} + \cdots\]

and

\[\left[ t(w) \right]^\delta = 1 + \delta t_m w^m + \left( \delta t_{2m} + \frac{\delta(\delta - 1)}{2} t_m^2 \right) w^{2m} + \cdots\]

Now equating the coefficient in (10) and (11) we get

\(\left( [\Psi_{m+1}(\sigma, \alpha, \beta)]^t_q - [\Psi_{m+1}(\sigma, \alpha, \beta)]^n_q \right) a_{m+1} = \delta s_m,\) (22)

\(\left( [\Psi_{2m+1}(\sigma, \alpha, \beta)]^t_q - [\Psi_{2m+1}(\sigma, \alpha, \beta)]^n_q \right) a_{2m+1}\)

\(\quad - \left( [\Psi_{m+1}(\sigma, \alpha, \beta)]^{n+t}_q - [\Psi_{m+1}(\sigma, \alpha, \beta)]^{2n}_q \right) a_{m+1}^2 = \delta s_{2m} + \frac{\delta(\delta - 1)}{2} s_m^2,\) (23)

\(\quad - \left( [\Psi_{m+1}(\sigma, \alpha, \beta)]^{t}_q - [\Psi_{m+1}(\sigma, \alpha, \beta)]^{n}_q \right) a_{m+1} = \delta t_m,\) (24)

\(\left( [\Psi_{2m+1}(\sigma, \alpha, \beta)]^t_q - [\Psi_{2m+1}(\sigma, \alpha, \beta)]^n_q \right) (m+1) a_{m+1}^2 - a_{2m+1}\)

\(\quad - \left( [\Psi_{m+1}(\sigma, \alpha, \beta)]^{n+t}_q - [\Psi_{m+1}(\sigma, \alpha, \beta)]^{2n}_q \right) a_{m+1}^2 = \delta t_{2m} + \frac{\delta(\delta - 1)}{2} t_m^2.\) (25)

From equation (22) and (24), we find that

\[s_m = -t_m\] (26)
and
\[
2 \left( [\Psi_{m+1}(\sigma, \alpha, \beta)]_q^t - [\Psi_{m+1}(\sigma, \alpha, \beta)]_q^n \right)^2 a_{m+1}^2 = \delta^2(s_m^2 + t_m^2). \tag{27}
\]
Also, from (23), (25) and (27), we have
\[
(m + 1)\delta([\Psi_{2m+1}(\sigma, \alpha, \beta)]_q^t - [\Psi_{2m+1}(\sigma, \alpha, \beta)]_q^n)a_{2m+1}^2 - 2\delta([\Psi_{m+1}(\sigma, \alpha, \beta)]_q^{n+t} - [\Psi_{m+1}(\sigma, \alpha, \beta)]_q^n) \]
\[
- \left(2 \alpha^2\right) a_{m+1}^2 = \delta(s_{2m} + t_{2m}) + \frac{\delta(\delta - 1)}{2} (t_m^2 + s_m^2) = \delta^2(s_{2m} + t_{2m}) \]
\[
+ (\delta - 1) \left( [\Psi_{m+1}(\sigma, \alpha, \beta)]_q^t - [\Psi_{m+1}(\sigma, \alpha, \beta)]_q^n \right)^2 a_{m+1}^2.
\]
Therefore, after simplifying and using Lemma 1 for the coefficient \(s_{2m} \) and \(t_{2m}\), we have (14).

For us to get the bound on \(|a_{2m+1}|\), we subtract (25) from (23) to have
\[
\left(2 \alpha^2\right) a_{2m+1} = \alpha(s_{2m} - t_{2m}) + \frac{\alpha(\alpha - 1)}{2} (t_m^2 - s_m^2). \tag{28}
\]
It follows from (26), (27) and (28)
\[
a_{2m+1} = \frac{\delta(s_{2m} - t_{2m})}{[\Psi_{2m+1}(\sigma, \alpha, \beta)]_q^t - [\Psi_{2m+1}(\sigma, \alpha, \beta)]_q^n} + \frac{(m + 1)\delta^2(t_m^2 - s_m^2)}{4([\Psi_{m+1}(\sigma, \alpha, \beta)]_q^t - [\Psi_{m+1}(\sigma, \alpha, \beta)]_q^n)^2}. \tag{29}
\]
Taking the absolute value of (29) and using Lemma 1 for the coefficient \(s_m, s_{2m}, t_m\) and \(t_{2m}\), we have (15) which completes the proof of Theorem 2.

When \(m = 1\) and \(\sigma = \beta = 1\) which is the one-fold symmetric bi-univalent functions, Theorem 2 gives the following corollary:

**Corollary 3.** Let \(f(z)\) given by (3) be in the class \(T_{2}^{\delta}(t, n, 1, q, \delta)\) \((0 < \delta \leq 1, n, t \in \mathbb{N}_0, \alpha > -1)\). Then
\[
|a_2| \leq \frac{2\delta}{\sqrt{2\delta \left( \left[ \frac{2\alpha^2+3}{\alpha+1} \right]_q^t - \left[ \frac{2\alpha^2+3}{\alpha+1} \right]_q^n \right) - 2\delta([2]_q^{n+t} - [2]_q^n) - (1 - \delta)([2]_q - [2]_q^n)^2}}
\]
and
\[
|a_{2m+1}| \leq \frac{2\delta}{\left[ \frac{2\alpha^2+3}{\alpha+1} \right]_q^t - \left[ \frac{2\alpha^2+3}{\alpha+1} \right]_q^n} + \frac{4\delta^2((2]_q - [2]_q^n)^2}.}
\]

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When \( m = \sigma = 1 \) and \( \alpha = 1 - \beta \) which is the one-fold symmetric bi-univalent functions, Theorem 2 gives the following corollary:

**Corollary 4.** Let \( f(z) \) given by (3) be in the class \( T_{\Sigma}^{1-\beta}(t,n,q,\delta) \) (0 < \( \delta \leq 1 \), \( n, t \in \mathbb{N}_0 \)). Then

\[
|a_2| \leq \frac{2\delta}{\sqrt{2\delta([2 + \beta]^t_q - [2 + \beta]^n_q) - 2\delta([2]^t_q - [2]^n_q)^2}}
\]

and

\[
|a_{2n+1}| \leq \frac{2\delta}{[2 + \beta]^t_q - [2 + \beta]^n_q} + \frac{4\delta^2}{([2]^t_q - [2]^n_q)^2}.
\]

**Remark 1.** In Theorem 2, if we choose

1. \( q = 1, \sigma = \beta = 1 \) and \( \alpha = 0 \) then we have results determined by Seker and Taymur [18], Theorem 2.

2. \( m = q = 1, \sigma = \beta = t = 1 \) and \( \alpha = n = 0 \) then we have results determined by Brannan and Taha [3], Theorem 2.

3. \( m = q = 1, \sigma = \beta = 1 \) and \( \alpha = 0 \) then we have results determined by Seker [19], Theorem 2.

3. **Coefficient estimates for the function class** \( T_{\Sigma_m}^{\sigma,\alpha}(t,n,\beta,q,\gamma) \)

**Definition 2.** A function \( f \in \Sigma_m \) given by (3) is said to be in the class \( T_{\Sigma_m}^{\sigma,\alpha}(t,n,\beta,q,\gamma) \) if it satisfies the following conditions:

\[
\Re \left\{ \frac{M_{\beta,t,q}^{\alpha,\sigma} f(z)}{M_{\beta,n,q}^{\alpha,\sigma} f(z)} \right\} > \gamma; \tag{30}
\]

\[
\Re \left\{ \frac{M_{\beta,t,q}^{\alpha,\sigma} g(w)}{M_{\beta,n,q}^{\alpha,\sigma} g(w)} \right\} > \gamma; \tag{31}
\]

where \( 0 \leq \gamma < 1, n, t \in \mathbb{N}_0, t \geq n \) and the function \( g = f^{-1} \) is given by (4).

We state and prove the following results.
Theorem 5. Let \( f(z) \) given by (3) be in the class \( T_{\Sigma_m}^{\sigma,\alpha}(t,n,\beta,q,\gamma) \) \((0 \leq \gamma < 1, n,t \in \mathbb{N}_0)\). Then

\[
|a_{m+1}| \leq 2 \left( \frac{1 - \gamma}{(m+1) \left( [\Psi_{2m+1}^{(1)}(\sigma,\alpha,\beta)]_q^t - [\Psi_{2m+1}^{(1)}(\sigma,\alpha,\beta)]_q^n \right) + 2 \left( [\Psi_{m+1}^{(1)}(\sigma,\alpha,\beta)]_q^n - [\Psi_{m+1}^{(1)}(\sigma,\alpha,\beta)]_q^{2n} \right)} \right).
\]

and

\[
|a_{2m+1}| \leq \frac{2(1 - \gamma)}{[\Psi_{2m+1}^{(1)}(\sigma,\alpha,\beta)]_q^n - [\Psi_{2m+1}^{(1)}(\sigma,\alpha,\beta)]_q^n} + \frac{(m+1)(1 - \gamma)^2}{( [\Psi_{m+1}^{(1)}(\sigma,\alpha,\beta)]_q^n - [\Psi_{m+1}^{(1)}(\sigma,\alpha,\beta)]_q^{2n})^2}.
\]

Proof. First of all, the argument inequality in (30) and (31) can be written in their equivalent forms as:

\[
\frac{\mathfrak{M}^{\sigma,\alpha,\beta}}{\mathfrak{M}^{\sigma,\alpha,\beta}} f(z) = \gamma + (1 - \gamma) s(z)
\]

and

\[
\frac{\mathfrak{M}^{\sigma,\alpha,\beta}}{\mathfrak{M}^{\sigma,\alpha,\beta}} g(w) = \gamma + (1 - \gamma) t(w).
\]

respectively. Where \( s(z), t(w) \in \mathcal{P} \) and have the forms

\[
s(z) = 1 + s_m z^m + s_{2m} z^{2m} + s_{3m} z^{3m} + \cdots
\]

and

\[
t(w) = 1 + t_m w^m + t_{2m} w^{2m} + t_{3m} w^{3m} + \cdots
\]

Clearly,

\[
\gamma + (1 - \beta \gamma) s(z) = 1 + (1 - \gamma) s_m z^m + (1 - \gamma) s_{2m} z^{2m} + \cdots
\]

and

\[
\gamma + (1 - \gamma) t(w) = 1 + (1 - \gamma) t_m w^m + (1 - \gamma) t_{2m} w^{2m} + \cdots.
\]
Now equating the coefficient in (34) and (35), we get

\[
([\Psi_{m+1}(\sigma, \alpha, \beta)]_q^t - [\Psi_{m+1}(\sigma, \alpha, \beta)]_q^n) a_{m+1} = (1 - \gamma) s_m, \tag{40}
\]

\[
([\Psi_{2m+1}(\sigma, \alpha, \beta)]_q^t - [\Psi_{2m+1}(\sigma, \alpha, \beta)]_q^n) a_{2m+1}
- ([\Psi_{m+1}(\sigma, \alpha, \beta)]_q^{n+t} - [\Psi_{m+1}(\sigma, \alpha, \beta)]_q^{2n}) a_{m+1}^2 = (1 - \gamma) s_{2m}, \tag{41}
\]

\[
- ([\Psi_{m+1}(\sigma, \alpha, \beta)]_q^n - [\Psi_{m+1}(\sigma, \alpha, \beta)]_q^n) a_{m+1} = (1 - \gamma) t_m, \tag{42}
\]

\[
([\Psi_{2m+1}(\sigma, \alpha, \beta)]_q^t - [\Psi_{2m+1}(\sigma, \alpha, \beta)]_q^n) ((m + 1)a_{m+1}^2 - a_{2m+1})
- ([\Psi_{m+1}(\sigma, \alpha, \beta)]_q^{n+t} - [\Psi_{m+1}(\sigma, \alpha, \beta)]_q^{2n}) a_{m+1}^2 = (1 - \gamma) t_{2m}. \tag{43}
\]

From (40) and (42), we get

\[
s_m = -t_m \tag{44}
\]

and

\[
2 ([\Psi_{m+1}(\sigma, \alpha, \beta)]_q^t - [\Psi_{m+1}(\sigma, \alpha, \beta)]_q^n)^2 a_{m+1}^2 = (1 - \gamma)^2 (s_m^2 + t_m^2). \tag{45}
\]

Also, adding (41) and (43), we have

\[
(m + 1)([\Psi_{2m+1}(\sigma, \alpha, \beta)]_q^t - [\Psi_{2m+1}(\sigma, \alpha, \beta)]_q^n) a_{2m+1}^2
- ([\Psi_{m+1}(\sigma, \alpha, \beta)]_q^{n+t} - [\Psi_{m+1}(\sigma, \alpha, \beta)]_q^{2n}) a_{m+1}^2
= (1 - \gamma) (s_{2m} + t_{2m})
\]

Therefore, after simplifying and applying Lemma 1 for the coefficient \(s_{2m}\) and \(t_{2m}\), we obtain (32).

Next, in order to find the bound on \(|a_{2m+1}|\), by subtracting (43) from (41), we have

\[
[\Psi_{2m+1}(\sigma, \alpha, \beta)]_q^t - [\Psi_{2m+1}(\sigma, \alpha, \beta)]_q^n

\left(2a_{2m+1} - (m + 1)a_{m+1}^2\right) = (1 - \gamma)(t_{2m} - s_{2m}). \tag{46}
\]

Applying (45) and Lemma 1 once again for coefficients \(s_m, s_{2m}, t_m\) and \(t_{2m}\), we have (33) which completes the proof of Theorem 5.

When \(m = 1\) and \(\sigma = \beta = 1\) which is the one-fold symmetric bi-univalent functions, Theorem 5 gives the following corollary:
Corollary 6. Let \( f(z) \) given by (3) be in the class \( T^{\alpha}_t(t, n, q, \gamma) \) \((0 \leq \gamma < 1, n, t \in \mathbb{N}_0, \alpha > -1)\). Then

\[
|a_{m+1}| \leq 2 \sqrt{\frac{1 - \gamma}{2 \left( \left\lfloor \frac{2\alpha + 3}{\alpha + 1} \right\rfloor_q - \left\lfloor \frac{2\alpha + 3}{\alpha + 1} \right\rfloor_n^q \right) - 2 \left( \left\lfloor \frac{n+t}{q} \right\rfloor^q - \left\lfloor \frac{2n}{q} \right\rfloor^q \right)}}
\]

and

\[
|a_{2m+1}| \leq \frac{2(1 - \gamma)}{\left\lfloor \frac{2\alpha + 3}{\alpha + 1} \right\rfloor_q - \left\lfloor \frac{2\alpha + 3}{\alpha + 1} \right\rfloor_n^q} + \frac{2(1 - \gamma)^2}{\left( \left\lfloor \frac{t}{q} \right\rfloor - \left\lfloor \frac{2n}{q} \right\rfloor^q \right)^2}.
\]

When \( m = \sigma = 1 \) and \( \alpha = 1 - \beta \) which is the one-fold symmetric bi-univalent functions, Theorem 5 gives the following corollary:

Corollary 7. Let \( f(z) \) given by (3) be in the class \( T^{1-\beta}_t(t, n, q, \gamma) \) \((0 \leq \gamma < 1, n, t \in \mathbb{N}_0)\). Then

\[
|a_{m+1}| \leq 2 \sqrt{\frac{1 - \gamma}{2 \left( \left\lfloor 2 + \beta \right\rfloor_q - \left\lfloor 2 + \beta \right\rfloor_n^q \right) - 2 \left( \left\lfloor \frac{n+t}{q} \right\rfloor^q - \left\lfloor \frac{2n}{q} \right\rfloor^q \right)}}
\]

and

\[
|a_{2m+1}| \leq \frac{2(1 - \gamma)}{\left\lfloor 2 + \beta \right\rfloor_q - \left\lfloor 2 + \beta \right\rfloor_n^q} + \frac{2(1 - \gamma)^2}{\left( \left\lfloor \frac{t}{q} \right\rfloor - \left\lfloor \frac{2n}{q} \right\rfloor^q \right)^2}.
\]

Remark 2. In Theorem 5, if we choose

1. \( q = 1, \sigma = \beta = 1 \) and \( \alpha = 0 \) then we have results determined by Seker and Taymur [18], Theorem 2).
2. \( m = q = 1, \sigma = \beta = t = 1 \) and \( \alpha = n = 0 \) then we have results determined by Brannan and Taha [3], Theorem 2).
3. \( m = q = 1, \sigma = \beta = 1 \) and \( \alpha = 0 \) then we have results determined by Seker [19], Theorem 2).
References


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