Acta Universitatis Apulensis No. 66/2021
ISSN: 1582-5329 pp. 15-31
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2020.65.02

PROJECTIVELY RELATED PURELY HERMITIAN COMPLEX FINSLER METRICS

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Abstract. In this paper we determined the projectively related purely Hermitian complex Finsler metrics. We prove the complex versions of the Rapsak's theorem and characterize the weakly Kähler and generalized Berwald projectively related purely Hermitian complex Finsler metrics.

2010 *Mathematics Subject Classification*: 53B40, 53C56.

Keywords: "projectively related complex Finsler metric", "purely Hermitian metric", "generalized Berwald metric", "complex Berwald metric".

1. Introduction

The problem of projective changes between two real Finsler metrics is quite old in geometry and it has been studied by many geometers, ([6],[7],[12],[18]). Its origin is formulated in Hilbert's Fourth problem determine the metrics on an open subset in R^n, whose geodesics are straight lines. Two Finsler metrics on a common underlying manifold are called projectively related if they have the same geodesics as point sets.

The study of projective real Finsler spaces was initiated by Berwald, ([9], [10]) and his studies mainly concern the two dimensional Finsler spaces. Further substantial contributions on this topic are from ([22], [18]) and especially, from ([25], [17]). The problem of projective Finsler spaces is strongly connected to projectively related sprays, as Shen pointed out in [25]. The topic of projective real Finsler spaces continoues to be interest for special classes of metrics ([7]-[11]).

In complex geometry, Aikou studied in [2] the projective flatness of complex Finsler metrics by the projective flatness of Finsler connections.

Part of the general themes from projective real Finsler geometry can be broached in complex Finsler geometry. However, there are meaningful differences comparing to real reasonings, mainly on account of the fact that the chern-Finsler complex nonlinear connection (the main tool in this geometry), generally does not derive from a spray. Another problem is that in complex Finsler geometry, the notion of complex geodesic curve comports two different nuances, one is in Abate Patrizio's [1] and the second is due to Royden, [23]. But, these notions don't differ too much. Since a complex geodesics curve in Royden's sense assures that the weakely Kähler condition is satisfied along the curve.

Our aim in the present paper is to study the projectively related purely Hermitian complex Finsler metric F and $\tilde{F}$ on the complex manifold M, using some ideas from the real case. And also in order to obtain a general characterization of the projectively related complex Finsler metrics.

In the second section we studied some preliminary properties of the n- dimentional complex Finsler spaces. In the third section we introduce the notoin of projectively related purely Hermitian complex Finsler metric and then we find some necessary and sufficient condition of projectiveness. The last part of this paper we prove the weakly Kähler and generalized Berwald projectively related purely Hermitian complex Finsler metric.

2. Preliminaries

For the begining we will make a survey of complex Finsler geometry and we will set the basic notions and terminology.

Let $z = (z^k)_{k=1,n}$ are the complex coordinates of the local chart of an n-dimensional complex manifold M. The complexified tangent bundle $T_C M$ expressed as the sum of $T^{'}M$ and its conjugate $T^{''}M$. Since the tangent bundle $T^{'}M$ is itself a complex manifold, its local coordinates of $u \in T^{'}M$ in its chart is defined by $u = (z^k, \eta^k)_{k=\overline{1,n}}$. The coordinates can be transformed into $(z^{'k}, \eta^{'k})_{k=\overline{1,n}}$ by the conditions $z^{'k} = z^{'k}(z)$ and $\eta^{'k} = \frac{\partial z^{'k}}{\partial z^l}\eta^l$.

A complex Finsler space is a pair (M, F), where $F : T^{'}M \to \mathbb{R}$ is a continuous function satisfying conditions;

(i) $L := F^2$ is smooth on $\widetilde{T^{'}M} := T^{'}M \backslash \{0\}$;

(ii) $F(z, \eta) \geq 0$, the equality holds if and only if $\eta = 0$;

(iii) $F(z,\ \lambda\eta)$=$|\lambda|F(z, \eta)$; for $\lambda \in \mathbb{C}$;

(iv) the Hermitian matrix $(g_{i\bar{j}}(z, \eta))$ is positive definite, where $g_{i\bar{j}} = \frac{\partial^2 L}{\partial \eta^i \partial \bar{\eta}^j}$, is the fundamental metric tensor equivalently, it means that the indicatrix is strongly pseudo-convex.

We say that a function f on $T^{'}M$ is (p, q)- homogeneous with respect to η iff $f(z, \lambda\eta) = \lambda^p \bar{\lambda}^q f(z, \eta)$ for any $\lambda \in \mathbb{C}$, For instance, $L := F^2$ is a $(1, 1)$- homogeneous function.

In complex Finsler geometry, projective flatness of complex Finsler metrics are studied based on projective flatness of various Finsler connections. Let $T^{'}M$ be the complexified tangent bundle then the sections of $T^{'}M$ are vertical bundle denoted by $VT^{'}M \subset T^{'}(T^{'}M)$ which is locally spanned by $\{\frac{\partial}{\partial \eta^k}\}$ and its conjugate denoted by $VT^{''}M$. Here the concept of complex nonlinear connections (c.n.c) is an important method used to linearize the geometry of the complex manifold $T^{'}M$. That is c.n.c generates additional subbundle $HT^{'}M$ in $T^{'}(T^{'}M)$. It implies that $T^{'}(T^{'}M) = HT^{'}M \oplus VT^{'}M$. $\{\frac{\delta}{\delta z^k} = \frac{\partial}{\partial z^k} - N^j_k \frac{\partial}{\partial \eta^j}\}$ spans the horizontal distribution $H_u T^{'}M$ where $N^j_k(z, \eta)$ are the coefficients of the $(c.n.c)$.

Next we study derivative law D on the complex tangent bundle of $T_C(T^{'}M)$. A $(1, 0)$-type Hermitian connection D satiesfies $D_{JX}Y = JD_XY$, for all X and J then D is called a Chern-Finsler connection. Here J denotes complex Finsler structure the coefficients defined as follows:

$$N^i_j := g^{\bar{m}i} \frac{\partial g_{l\bar{m}}}{\partial z^j} \eta^l = L^i_{lj} \eta^l \, ; \;\; L^i_{jk} := g^{\bar{l}i} \delta_k g_{j\bar{l}} \, ; \;\; C^i_{jk} = g^{\bar{l}i} \dot{\partial}_k g_{j\bar{l}} \, , \tag{1}$$

where here and further on δ_k is related to the Chern-Finsler (c.n.c) and $D_{\delta_k}\delta_j = L^i_{jk}\delta_i$, $D_{\dot{\partial}_k}\dot{\partial}_j = C^i_{jk}\dot{\partial}_i$.

Let us recall that terminology the complex Finsler space (M, F) is strongly Kähler iff $T^i_{jk} = 0$, Kähler iff $T^i_{jk}\eta^j = 0$ and weakly Kähler iff $g_{i\bar{l}} T^i_{jk} \eta^j \bar{\eta}^l = 0$, where $T^i_{jk} := L^i_{jk} - L^i_{kj}$. In [14] it is proved that strongly Kähler and Kähler notions actually coincide. We notice that in the particular case of complex Finsler metrics which come from Hermitian metrics on M, so-called purely Hermitian metrics. Hence $g_{i\bar{j}} = g_{i\bar{j}}(z)$, all these kinds of Kähler coincide.

The Chern-Finsler $(c.n.c)$ generally, does not derive from a spray, but it always determines a complex spray with the local coefficients $G^i = \frac{1}{2} N^i_j \eta^j$. Instead, G^i induce a $(c.n.c)$ by $\overset{c}{N^i_j} := \dot{\partial}_j G^i$ called canonical in [19], where it is proved that it coincides with Chern-Finsler $(c.n.c)$ if and only if the complex Finsler metric is Kähler. Note that $2G^i = N^i_j \eta^j = \overset{c}{N^i_j} \eta^j$, and so $\eta^k \overset{c}{\delta_k} = \eta^k \delta_k$, where $\overset{c}{\delta_k}$ is related to canonical (c.n.c), i.e. $\overset{c}{\delta_k} := \frac{\partial}{\partial z^k} - \overset{c}{N^j_k} \dot{\partial}_j$. Additinally, in the Kähler case, we have $\overset{c}{\delta_k} = \delta_k$.

In [4] authors proven that the complex Finsler space (M, F) is generalized Berwald iff $\dot{\partial}_{\bar{h}} G^i = 0$ and (M, F) is a complex Berwald space iff it is Kähler and generalized Berwald.

3. Projectively related purely Hermitian complex Finsler metrics

In [1], author define a complex geodesic curve is given by $D_{T^h+\overline{T^h}}T^h = \theta^*(T^h, \overline{T^h})$, where $\theta^* = g^{\bar{m}k} g_{i\bar{p}}(L^{\bar{p}}_{\bar{j}\bar{m}} - L^{\bar{p}}_{\bar{m}\bar{j}}) dz^i \wedge d\bar{z}^j \otimes \delta_k$, for which it is proved that $\theta^{*k} = 2g^{\bar{j}k}\ \overset{c}{\delta_{\bar{j}}} L$ and θ^* is vanishing if and only if the space is weakly Kähler. Thus, the equations of a complex geodesic $z = z(s)$ of (M, L), with s is a real parameter can be rewritten as:

$$\frac{d^2 z^i}{ds^2} + 2G^k\left(z(s), \frac{dz}{ds}\right) = \theta^{*i}\left(z(s), \frac{dz}{ds}\right); \quad i = \overline{1, n}, \tag{2}$$

where by $z^i(s)$, $i = \overline{1, n}$, we denote the coordinates along of curve $z = z(s)$.

We note that the functions θ^{*i} are (1, 1)- homogeneous with respect to η, that is $(\dot{\partial}_k \theta^{*i})\eta^k = \theta^{*i}$ and $(\dot{\partial}_{\bar{k}} \theta^{*i})\bar{\eta}^k = \theta^{*i}$.
Let $\tilde{L}$ be another complex Finsler metric on the underlying manifold M.

Definition 1. *The complex Finsler metrics L and $\tilde{L}$ on the manifold M, are called projectively related if they have the same complex geodesics as point sets.*

This means that for any complex geodesic $z = z(s)$ of (M, L) there is a transformation of its parameter s, $\tilde{s} = \tilde{s}(s)$, with $\frac{d\tilde{s}}{ds} > 0$, such that $z = z(s)$ is a geodesic of $(M, \tilde{L})$ and, conversely.
We suppose that $z = z(s)$ is a complex geodesic of (M, L) thus, it satiesfies (2). Taking an arbitrary transformation of the parameter $t = t(s)$, with $\frac{dt}{ds} > 0$, the equation(2) cannot in general be preserved indeed, for the new parameter t we have

$$\frac{dz^i}{ds} = \frac{dz^i}{dt}\frac{dt}{ds}; \quad \frac{d^2 z^i}{ds^2} = \frac{d^2 z^i}{dt^2}\left(\frac{dt}{ds}\right)^2 + \frac{dz^i}{dt}\frac{d^2 t}{ds^2}; \quad \theta^{*k}\left(z, \frac{dz}{ds}\right) = \left(\frac{dt}{ds}\right)^2 \theta^{*k}\left(z, \frac{dz}{dt}\right). \tag{3}$$

Then

$$\begin{aligned}\left[\frac{d^2 z^i}{dt^2} + 2G^i\left(z, \frac{dz}{dt}\right) - \theta^{*i}\left(z, \frac{dz}{dt}\right)\right]\left(\frac{dt}{ds}\right)^2 &= \frac{d^2 z^i}{ds^2} - \frac{dz^i}{dt}\frac{d^2 t}{ds^2} \\ &\quad + 2G^i\left(z, \frac{dz}{ds}\right) - \theta^{*i}\left(z, \frac{dz}{ds}\right) \\ &= -\frac{dz^i}{dt}\frac{d^2 t}{ds^2}.\end{aligned} \tag{4}$$

Therefore, the equation (2) in parameter t are

$$\frac{d^2 z^i}{dt^2} + 2G^i\left(z, \frac{dz}{dt}\right) - \theta^{*i}\left(z(t), \frac{dz}{dt}\right) = \theta^{*i}\left(z(t), \frac{dz}{dt}\right) - \frac{dz^i}{dt^2}\frac{d^2 t}{ds^2}\frac{1}{(\frac{dt}{ds})^2}; \quad i = \overline{1, n}, \tag{5}$$

which is equivalent to

$$\frac{\frac{d^2z^i}{dt^2}+2G^i\Big(z,\frac{dz}{dt}\Big)-\theta^{*i}\Big(z,\frac{dz}{dt}\Big)-\theta^{*i}\Big(z(t),\frac{dz}{dt}\Big)}{\frac{dz^i}{dt}}=-\frac{d^2t}{ds^2}\frac{1}{(\frac{dt}{ds})^2}\quad i=\overline{1,n}. \tag{6}$$

We can rewrite (6), taking for i two different values, as

$$\begin{aligned}\frac{\frac{d^2z^j}{dt^2}+2G^j\Big(z,\frac{dz}{dt}\Big)-\theta^{*j}\Big(z,\frac{dz}{dt}\Big)}{\frac{dz^j}{dt}} &= \frac{\frac{d^2z^k}{dt^2}+2G^k\Big(z,\frac{dz}{dt}\Big)-\theta^{*k}\Big(z(t),\frac{dz}{dt}\Big)}{\frac{dz^k}{dt}}\\ &= -\frac{d^2t}{ds^2}\frac{1}{(\frac{dt}{ds})^2}, \end{aligned}\tag{7}$$

for any $j,k=\overline{1,n}$.
Corresponding to the complex Finsler metric $\tilde{L}$ on the same manifold M, we have the spray coefficients $\tilde{G}^i$ and the functions $\tilde{\theta}^{*i}$. If L and $\tilde{L}$ are projectively related, then $z=z(\tilde{s})$ is a complex geodesics of $(M,\tilde{L})$, where $\tilde{s}$ is the parameter with respect to $\tilde{L}$. Now, we assume that the same parameter t is transformed by $t=t(\tilde{s})$ and as above we obtain

$$\frac{\frac{d^2z^i}{dt^2}+2\tilde{G}^i\Big(z,\frac{dz}{dt}\Big)-\theta^{*i}\Big(z(t),\frac{dz}{dt}\Big)}{\frac{dz^i}{dt}}=-\frac{d^2t}{ds^2}\frac{1}{(\frac{dt}{ds})^2}\quad i=\overline{1,\,n}. \tag{8}$$

The difference between (6) and (8) gives

$$\begin{aligned}2\tilde{G}^i\Big(z,\frac{dz}{dt}\Big)-\tilde{\theta}^{*i}\Big(z,\frac{dz}{dt}\Big) &= 2G^i\Big(z,\frac{dz}{dt}\Big)-\theta^{*i}\Big(z,\frac{dz}{dt}\Big)\\ &\quad+\Big[\frac{d^2t}{ds^2}\frac{1}{(\frac{dt}{ds})^2}-\frac{d^2t}{d\tilde{s}^2}\frac{1}{(\frac{dt}{d\tilde{s}})^2}\Big]\frac{dz^i}{dt}, \end{aligned}\tag{9}$$

On the geodesics curves, it can be rewritten more generally as

$$2\tilde{G}^i\Big(z,\frac{dz}{dt}\Big)-\tilde{\theta^{*i}}\Big(z,\frac{dz}{dt}\Big)=2G^i\Big(z,\frac{dz}{dt}\Big)-\theta^{*i}\Big(z,\frac{dz}{dt}\Big)+2P\Big(z,\frac{dz}{dt}\Big)\frac{dz^i}{dt}. \tag{10}$$

for any $i=\overline{1,\,n}$, where P is a smooth function on $T^{'}M$ with complex values.

Denoting by $B^i:=\frac{1}{2}(\tilde{\theta}^{*i}-\theta^{*i})$, the homogeneity properties of the functions $\tilde{\theta}^{*i}$ and θ^{*i} give $(\dot{\partial}_kB^i)\eta^k=B^i$ and $(\dot{\partial}_{\bar{k}}B^i)\bar{\eta}^k=B^i$ Moreover the relations (10) become

$$\tilde{G}^i=G^i+B^i+P\eta^i. \tag{11}$$

Now, we use their homogeneity properties, going from η to $\lambda\eta$. Thus, differentiating in (11) with respect to η and $\bar{\eta}$ and then setting $\lambda = 1$, we obtain

$$B^i = [(\dot{\partial}_k P) - P]\eta^i \;\; and \;\; B^i = -(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k \eta^i, \tag{12}$$

and so,

$$(\dot{\partial}_k P)\eta^k + (\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = P, \;\; for\ any \;\; i = \overline{1,\, n}. \tag{13}$$

Proposition 1. *Between the spray coefficients $\tilde{G}^i$ and G^i of the metrics L and $\tilde{L}$ on the manifold M there are the relations $\tilde{G}^i = G^i + B^i + P\eta^i$, for any $i = \overline{1, n}$ where P is a smooth function on $T^{'}M$ with complex values, if and only if $\tilde{G}^i = G^i + (\dot{\partial}_k P)\eta^k \eta^i$, and $B^i(z, \eta) = -(\dot{\partial}_k P)\bar{\eta}^k \eta^i$, for any $i = \overline{1,\, n}$ and $(\dot{\partial}_k P)\eta^k + (\dot{\partial}_{\bar{k}})\bar{\eta}^k = P$.*

From above considerations we obtain.

Proposition 2. *If the complex Finsler metrics L and $\tilde{L}$ on the manifold M are projectively related, then there is a smooth function P on $T^{'}M$ with complex values, satisfying $(\dot{\partial}_k P)\eta^k + (\dot{\partial}_{\bar{k}})\bar{\eta}^k = P$, such that*

$$\tilde{G}^i(z, \eta) = G^i(z, \eta) + (\dot{\partial}_k P)\eta^k \eta^i \;\; and \;\; B^i(z, \eta) = -(\dot{\partial}_{\bar{k}} P)\eta^k \eta^i; \;\; i = \overline{1,\, n}, \tag{14}$$

Lemma 1. *We denotes $S = (\dot{\partial}_k P)\eta^k$ and $Q = -(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k$. The $(2,\,0)$-homogeneity with respect to η of the functions $\tilde{G}^i$ and G^i implies the $(1, 0)$-homogeneity of S, and the $(1, 1)$-homogeneity of B^i give that Q is $(0, 1)$-homogeneous.*

Conversely, under assumption that $z = z(s)$ is a complex geodesics of (M, L), we show that the complex Finsler metric $\tilde{L}$ with the spray coefficients $\tilde{G}^i$ given by

$$\tilde{G}^i = G^i + B^i + P\eta^i.$$

Where P is a smooth function on $T^{'}M$ with complex values, is projectively related to L, that is there is a parametrization $\tilde{s} = \tilde{s}(s)$, with $\frac{d\tilde{s}}{ds} > 0$, such that $z = z(\tilde{s}(s))$ is a geodesics of $(M,\, \tilde{L})$.

If there is a parametrization $\tilde{s} = \tilde{s}(s)$ then we have $\frac{d^2 z^i}{d\tilde{s}^2} = -2G^i(z, \frac{dz}{ds}) + \theta^{*i}(z, \frac{dz}{ds}) - \frac{d^2\tilde{s}}{d\tilde{s}^2}\frac{1}{(\frac{d\tilde{s}}{ds})^2}\frac{dz^i}{d\tilde{s}}$, for any $i = \overline{1, n}$. now, using (14), it results

$$\frac{d^2 z^i}{d\tilde{s}^2} = -2\tilde{G}^i\left(z, \frac{dz}{dt}\right) + \tilde{\theta}^{*i}\left(z, \frac{dz}{d\tilde{s}}\right) + \left(2P\left(z, \frac{dz}{d\tilde{s}}\right) - \frac{d^2\tilde{s}}{ds^2}\frac{1}{(\frac{d\tilde{s}}{ds})^2}\right)\frac{dz^i}{d\tilde{s}}; \;\; i = \overline{1,\, n}, \tag{15}$$

So, $z = z(\tilde{s}(s))$ is a geodesic of $(M,\, \tilde{L})$ if and only if

$$\left(2P\left(z, \frac{dz}{d\tilde{s}}\right) - \frac{d^2\tilde{s}}{ds^2}\frac{1}{\frac{(d\tilde{s})}{ds}}\right)\frac{dz^i}{d\tilde{s}} = 0; \;\; i = \overline{1,\, n}, \tag{16}$$

Supposing the complex geodesic curve is not a line it resluts

$$2P\left(z, \frac{dz}{ds}\right)\frac{d\tilde{s}}{ds} = \frac{d^2\tilde{s}}{ds^2}. \tag{17}$$

Denoting by $u(s) = \frac{d\tilde{s}}{ds}$, we have $\frac{d^2\tilde{s}}{ds^2} = \frac{du}{ds}$ and so, $2P(z, \frac{dz}{ds})u = \frac{du}{ds}$. We obtain $u = ae^{\int 2P(z,\frac{dz}{ds})ds}$. From here, it results that there is

$$\tilde{s}(s) = a\int e^{\int 2P(z,\frac{dz}{ds})ds}\, ds + b, \tag{18}$$

where a, b are arbitrary constants. Corroborating all above results we have proven.

Lemma 2. *Let L and $\tilde{L}$ be complex Finsler metrics on the manifold M. Then L and $\tilde{L}$ are projectively related if and only if there is a smooth function P on $T^{'}M$ with complex values, such that*

$$\tilde{G}^i = G^i + B^i + P\eta^i; \quad i = \overline{1,\, n}. \tag{19}$$

As a consequence of proposition (1) we have the following.

Proposition 3. *Let L and $\tilde{L}$ be complex Finsler metrics on the manifold M. Then L and $\tilde{L}$ are projectively related if and only if there is a smooth function P on $T^{'}M$ with complex values, such that $\tilde{G}^i = G^i + (\dot{\partial}_k P)\eta^k\eta^i$, $B^i(z,\eta) = -(\dot{\partial}_{\bar{k}}P)\bar{\eta}^k\eta^i$ for any $i = \overline{1,\, n}$ and $(\dot{\partial}_k P)\eta^k + (\dot{\partial}_{\bar{k}}P)\bar{\eta}^k = P$.*

The relations (19) between the spray coefficients $\tilde{G}^i$ and G^i of the projectively related complex Finsler metrics L and $\tilde{L}$ will be called projective change.

Theorem 3. *Let L and $\tilde{L}$ be two complex Finsler metrics on the manifold M, which are projectively related. Then, L is weakly Kähler if and only if $\tilde{L}$ is also weakly Kähler. In this case, the projective change is $\tilde{G}^i = G^i + P\eta^i$, where P is a $(1,0)$-homogeneous function.*

Proof. We assume that $\tilde{G}^i = G^i + (\dot{\partial}_k P)\eta^k\eta^i$, $B^i = \frac{1}{2}(\tilde{\theta}^{*i} - \theta^{*i}) = -(\dot{\partial}_{\bar{k}}P)\bar{\eta}^k\eta^i$ and $(\dot{\partial}_k P)\eta^k\eta^i + (\dot{\partial}_{\bar{k}}P)\bar{\eta}^k\eta^i = P$.
If L is weakly Kähler then $\theta^{*i} = 0$ and so $\tilde{\theta}^{*i} = -2(\dot{\partial}_{\bar{k}}P)\bar{\eta}^k\eta^i$ which contracted by $g_{i\bar{r}}\bar{\eta}^r = \dot{\partial}_l\tilde{L}$, gives $\tilde{\theta}^{*i}\tilde{g}_{i\bar{r}}\bar{\eta}^r = -2(\dot{\partial}_{\bar{k}}P)\bar{\eta}^k\tilde{L}$. But $\tilde{\theta}^{*i}\tilde{g}_{i\bar{r}}\bar{\eta}^r = 0$. Thus, $(\dot{\partial}_{\bar{k}}P)\bar{\eta}^k = 0$, which implies $\tilde{\theta}^{*i} = 0$, i.e. $\tilde{L}$ is weakly Kähler and $P = (\dot{\partial}_k P)\eta^k$, so we obtain $\tilde{G}^i = G^i + P\eta^i$. The converse implication results immediately in the same way.

Proposition 4. *Let (M, L) be a complex Finsler space and $\tilde{L} = \alpha^2 + \epsilon|\beta|^2$ (if $\epsilon = \pm 1$) be a purely Hermitian complex Finsler metric on M. The spray coefficients $\tilde{G}^i$ and G^i of the purely Hermitian metrics L and $\tilde{L}$ satisfy*

$$\begin{aligned}\tilde{G}^i &= G^i + \frac{1}{2}(a^{\bar{r}i} + \frac{1}{2}b^i b^{\bar{r}})\left[\frac{\partial}{\partial\bar{\eta}^r}\left(\frac{\partial a_{l\bar{k}}}{\partial z^k}\eta^l\bar{\eta}^k + \epsilon(\bar{\beta}\frac{\partial b_l}{\partial z^k}\eta^l + \beta\frac{\partial b_{\bar{l}}}{\partial z^k}\bar{\eta}^l)\right)\right] \\ &\quad + 2\left[\frac{\partial}{\partial\bar{\eta}^r}(\frac{1}{2}N_k^l)(l_l + \epsilon\bar{\beta}b_l)\right],\ for\ \ i = \overline{1,\, n}, \end{aligned} \tag{20}$$

where

$$\begin{aligned}\overset{CF}{N_k^l} &= 2\overset{a}{N_k^l} + \frac{3+\omega}{2+\omega}\left[\left(\frac{\partial b_r}{\partial z^k}\bar{\eta}^r + \frac{\partial b_s}{\partial z^k}\eta^s\right)\frac{\epsilon}{2}b^i + (b_r\bar{\eta}^r + b_s\eta^s)\frac{\partial b^i}{\partial z^k}\right] \\ &\quad + \frac{1}{2+\omega}\left(\frac{\partial a_{r\bar{m}}}{\partial z^k}\bar{\eta}^r + \frac{\partial a_{s\bar{m}}}{\partial z^k}\eta^s\right)b^i b^{\bar{m}}.\end{aligned}$$

Proof. Having $\overset{c}{\delta_k}\tilde{L} = \frac{\partial a_{lk}}{\partial z^k}\eta^l\eta^k + \epsilon\left(\bar{\beta}\frac{\partial b_l}{\partial z^k} + \beta\frac{\partial b_k}{\partial z^k}\right) - \overset{c}{N_k^l}\left(l_l + \epsilon\bar{\beta}b_l\right)$ by a direct computation we obtain

$$\begin{aligned}\dot{\partial}_{\bar{r}}(\overset{c}{\delta_k}\tilde{L}) &= \frac{\partial^2 a_{l\bar{k}}}{\partial z^k\partial\bar{\eta}^r}\eta^l\bar{\eta}^k + \epsilon\left(2\frac{\partial b_l}{\partial z^k}\eta^l\frac{\partial b_{\bar{l}}}{\partial z^k}\bar{\eta}^l + \bar{\beta}\frac{\partial^2 b_l}{\partial z^k\partial\bar{\eta}^r}\bar{\eta}^l + \beta\frac{\partial^2 b_l}{\partial z^k\partial\bar{\eta}^r}\bar{\eta}^l\right) \\ &\quad -(\dot{\partial}_{\bar{r}}\overset{CF}{N_k^l})(\dot{\partial}_l\tilde{L}) - \overset{CF}{N_k^l}(a_{l\bar{r}} + b_l b_{\bar{r}}),\end{aligned} \tag{21}$$

where $\dot{\partial}_l\tilde{L} = l_l + \epsilon\,\bar{\beta}b_l$,
which contracted with $\tilde{g}^{\bar{r}i}\eta^k$, and taking in to account $\eta^k\overset{c}{\delta_k} = \eta^k\delta_k$, implies that

$$\tilde{g}^{\bar{r}i}\dot{\partial}_{\bar{r}}(\overset{c}{\delta_k}\tilde{L})\eta^k = 2\tilde{G}^i - 2(a^{\bar{r}i} + \frac{1}{2}b^i b^{\bar{r}})(\dot{\partial}_{\bar{r}}G^l)(l_l + \epsilon\bar{\beta}b_l) - 2G^i, \tag{22}$$

where

$$\begin{aligned}G^i &= \overset{a}{G_j^i} + \frac{1}{2}(\frac{3+\omega}{2+\omega})\left[\left(\frac{\partial b_r}{\partial z^k}\bar{\eta}^r + \frac{\partial b_s}{\partial z^k}\eta^s\right)\frac{\epsilon}{2}b^i + (b_r\bar{\eta}^r + b_s\eta^s)\frac{\partial b^i}{\partial z^k}\right]\eta^j \\ &\quad + \left[\frac{1}{2+\omega}\left(\frac{\partial a_{r\bar{m}}}{\partial z^k}\bar{\eta}^r + \frac{\partial a_{s\bar{m}}}{\partial z^k}\eta^s\right)\frac{1}{2}b^i b^{\bar{m}}\right]\eta^j, \\ G^l &= \frac{1}{2}N_k^l\eta^l\end{aligned}$$

and so (20) is proved. We discus some complex versions of the Rapcsak's theorem.

Proposition 5. *Let* $L = \alpha^2 + \epsilon|\beta|^2$ *and* $\tilde{L} = F^2 + \epsilon|\beta|^2$ *be a purely Hermitian complex Finsler metrics on the manifold* M*. Then,* L *and* $\tilde{L}$ *are projectively related if and only if*

$$\frac{1}{2}\left[\dot{\partial}_{\bar{r}}\left(\frac{\partial a_{lk}}{\partial z^k}\eta^l\eta^k + \epsilon\left(\bar{\beta}\frac{\partial b_l}{\partial z^k} + \beta\frac{\partial b_k}{\partial z^k}\right) - \overset{c}{N}{}_k^l(l_l + \epsilon\bar{\beta}b_l)\right)\eta^k + 2(\dot{\partial}_{\bar{r}}G^l)(l_l + \epsilon\bar{\beta}b_l)\right] = P(l_{\bar{r}} + \epsilon\beta b_{\bar{r}}) + B^i\tilde{g}_{l\bar{r}}, \tag{23}$$

with $P = \frac{1}{2\tilde{L}}[(\delta_k\tilde{L})\eta^k + \theta^{*i}(\dot{\partial}_i\tilde{L})]$,

$$\begin{aligned} B^i(z,\eta) &= -\Bigg\{\frac{\partial}{\partial\bar{\eta}^k}\Bigg[\frac{1}{2(\alpha^2+\epsilon|\beta|^2)}\Bigg(\frac{\partial a_{lk}}{\partial z^k}\eta^l\eta^k + \epsilon(\bar{\beta}\frac{\partial b_l}{\partial z^k}\eta^l + \beta\frac{\partial b_{\bar{l}}}{\partial z^k}\bar{\eta}^l) - 2\overset{a}{N}{}_k^l \\ &+\frac{3+\omega}{2+\omega}\left[\left(\frac{\partial b_r}{\partial z^k}\bar{\eta}^r + \frac{\partial b_s}{\partial z^k}\eta^s\right)\frac{\epsilon}{2}b^i + (b_r\bar{\eta}^r + b_s\eta^s)\frac{\partial b^i}{\partial z^k}\right] \\ &+\frac{1}{2+\omega}\left(\frac{\partial a_{r\bar{m}}}{\partial z^k}\bar{\eta}^r + \frac{\partial a_{s\bar{m}}}{\partial z^k}\eta^s\right)b^i b^{\bar{m}}(l_l + \bar{\beta}b_l)\Bigg)\eta^k\Bigg]\Bigg\} - 2B^i(a_{i\bar{m}} + b_i b_{\bar{m}})\bar{\eta}^m. \end{aligned}$$

Proof. We assume that the purely Hermitian complex Finsler metric L and $\tilde{L}$ are projectively related. Then, by lemma (2) and (20) we have

$$\begin{aligned} B^i + P\eta^i &= \frac{1}{2}(a^{\bar{r}i} + \frac{1}{2}b^i b^{\bar{r}})\Bigg\{\frac{\partial}{\partial\bar{\eta}^r}\Bigg(\frac{\partial a_{lk}}{\partial z^k}\eta^l\eta^k + \epsilon(\bar{\beta}\frac{\partial b_l}{\partial z^k}\eta^l + \beta\frac{\partial b_{\bar{l}}}{\partial z^k}\bar{\eta}^l) - 2\overset{a}{N}{}_k^l \\ &+\frac{3+\omega}{2+\omega}\left[\left(\frac{\partial b_r}{\partial z^k}\bar{\eta}^r + \frac{\partial b_s}{\partial z^k}\eta^s\right)\frac{\epsilon}{2}b^i + (b_r\bar{\eta}^r + b_s\eta^s)\frac{\partial b^i}{\partial z^k}\right] \\ &+\frac{1}{2+\omega}\left(\frac{\partial a_{r\bar{m}}}{\partial z^k}\bar{\eta}^r + \frac{\partial a_{s\bar{m}}}{\partial z^k}\eta^s\right)b^i b^{\bar{m}}(l_l + \bar{\beta}b_l)\Bigg\}\eta^k + 2(\frac{\partial}{\partial\bar{\eta}^r}G^l)(l_l + \epsilon\bar{\beta}b_l). \end{aligned} \tag{24}$$

First, if these relations are contracted by $\tilde{g}_{i\bar{m}}\bar{\eta}^m$, we get

$$-\frac{1}{2}\theta^{*i}(\dot{\partial}_i\tilde{L}) + P\tilde{L} = \frac{1}{2}\dot{\partial}_{\bar{m}}(\delta_k\tilde{L})\eta^k\bar{\eta}^m + (\dot{\partial}_{\bar{m}}G^l)\bar{\eta}^m(l_l + \epsilon\bar{\beta}b_l). \tag{25}$$

Because $B^i\tilde{g}_{i\bar{m}}\bar{\eta}^m = -\frac{1}{2}\theta^{*i}(l_l + \epsilon\bar{\beta}b_l)$. But the (2, 0)-homogeneity of the functions G^l leads to $(\dot{\partial}_{\bar{m}}G^l)\bar{\eta}^m = 0$ and $\dot{\partial}_{\bar{m}}(\delta_k\tilde{L})\eta^k\bar{\eta}^m = (\delta_k\tilde{L})\eta^k$.

And thus

$$P = \frac{1}{2(\alpha^2+\epsilon|\beta|^2)}\Bigg\{\Bigg(\frac{\partial a_{lk}}{\partial z^k}\eta^l\eta^k+\epsilon(\bar\beta\frac{\partial b_l}{\partial z^k}\eta^l+\beta\frac{\partial b_{\bar l}}{\partial z^k}\bar\eta^l)-2\overset{a}{N}{}^l_k$$
$$+\frac{3+\omega}{2+\omega}\Bigg[\left(\frac{\partial b_r}{\partial z^k}\bar\eta^r+\frac{\partial b_s}{\partial z^k}\eta^s\right)\frac{\epsilon}{2}b^i+(b_r\bar\eta^r+b_s\eta^s)\frac{\partial b^i}{\partial z^k}\Bigg]$$
$$+\frac{1}{2+\omega}\left(\frac{\partial a_{r\bar m}}{\partial z^k}\bar\eta^r+\frac{\partial a_{s\bar m}}{\partial z^k}\eta^s\right)b^ib^{\bar m}(l_l+\bar\beta b_l)\Bigg)\eta^k\Bigg\}-2B^i(z,\eta)\tilde g_{i\bar m}\bar\eta^m,$$

next we have contracting into (24) only by $\tilde g_{i\bar m}$, we obtain (23). Conversely, substituting the formulas (23) into (20), we obtain (19) with $P=\frac{1}{2\tilde L}[(\delta_k\tilde L)\eta^k+\theta^{*i}(\dot\partial_i\tilde L)]$, that is the purely Hermitian complex Finsler metric L and $\tilde L$ are projectively related.

Theorem 4. *The purely Hermitian complex Finsler metrics L and $\tilde L$ on the manifold M are projectively related if and only if*

$$\frac{\partial}{\partial\bar\eta^r}\Bigg\{\frac{\partial a_{lk}}{\partial z^k}\eta^l\eta^k+\epsilon(\bar\beta\frac{\partial b_l}{\partial z^k}\eta^l+\beta\frac{\partial b_{\bar l}}{\partial z^k}\bar\eta^l)-2\overset{a}{N}{}^l_k$$
$$+\frac{3+\omega}{2+\omega}\Bigg[\left(\frac{\partial b_r}{\partial z^k}\bar\eta^r+\frac{\partial b_s}{\partial z^k}\eta^s\right)\frac{\epsilon}{2}b^i+(b_r\bar\eta^r+b_s\eta^s)\frac{\partial b^i}{\partial z^k}\Bigg]$$
$$+\frac{1}{2+\omega}\left(\frac{\partial a_{r\bar m}}{\partial z^k}\bar\eta^r+\frac{\partial a_{s\bar m}}{\partial z^k}\eta^s\right)b^ib^{\bar m}(l_l+\bar\beta b_l)\Bigg\}\eta^k+2(\frac{\partial}{\partial\bar\eta^r}G^l)(l_l+\epsilon\bar\beta b_l)$$
$$=\left(\frac{1}{\alpha^2+|\beta|^2}\right)\frac{\partial}{\partial\bar\eta^r}\Bigg\{\frac{\partial a_{lk}}{\partial z^k}\eta^l\eta^k+\epsilon(\bar\beta\frac{\partial b_l}{\partial z^k}\eta^l+\beta\frac{\partial b_{\bar l}}{\partial z^k}\bar\eta^l)-2\overset{a}{N}{}^l_k$$
$$+\frac{3+\omega}{2+\omega}\Bigg[\left(\frac{\partial b_r}{\partial z^k}\bar\eta^r+\frac{\partial b_s}{\partial z^k}\eta^s\right)\frac{\epsilon}{2}b^i+(b_r\bar\eta^r+b_s\eta^s)\frac{\partial b^i}{\partial z^k}\Bigg]$$
$$+\frac{1}{2+\omega}\left(\frac{\partial a_{r\bar m}}{\partial z^k}\bar\eta^r+\frac{\partial a_{s\bar m}}{\partial z^k}\eta^s\right)b^ib^{\bar m}(l_l+\bar\beta b_l)\Bigg\}\eta^k(l_{\bar r}+\epsilon\beta b_{\bar r}), \tag{26}$$

where $B^r=-\frac{1}{2\tilde L}\theta^{*i}(l_l+\epsilon\bar\beta b_l)\eta^r$; $r=\overline{1,\,n}$,
and $P=\frac{1}{2\tilde L}\left[\delta_k\tilde L\eta^k+\theta^{*i}(l_l+\epsilon\beta b_{\bar r})\right]$,
moreover the projective change is $\tilde G^i=G^i+\frac{1}{2\tilde L}(\delta_k\tilde L)\eta^k\eta^i$.

Proof. By (3), if L and $\tilde L$ be a purely Hermitian complex Finsler space are projectively related, then there is a smooth function P on $T'M$ with complex values, such that $\tilde{G}^i = G^i + (\frac{\partial}{\partial \eta^k}(P))\eta^k\eta^i$ and $B^i = -(\frac{\partial}{\partial \eta^k}(P))\bar{\eta}^k\eta^i$, for any $i = \overline{1,\, n}$ and $(\frac{\partial}{\partial \eta^k}(P))\eta^k + (\frac{\partial}{\partial \eta^k}(P))\bar{\eta}^k = P$. Using (20) it results

$$S\eta^i = \frac{1}{2}\tilde{g}^{\bar{r}i}\Big((\frac{\partial}{\partial \bar{\eta}^r}(\delta_K \tilde{L})\eta^k) + 2(\dot{\partial}_{\bar{r}} G^l)(l_l + \epsilon\bar{\beta}b_l)\Big); \;\; i = \overline{1,\, n}, \tag{27}$$

which contracted firstly by $\tilde{g}_{i\bar{m}}$ and secondly by $\tilde{g}_{i\bar{m}}\bar{\eta}^m$ give $\frac{\partial}{\partial \bar{\eta}^r}(\delta_k \tilde{L})\eta^k + 2(\frac{\partial}{\partial \bar{\eta}^r}G^l)(l_l + \epsilon\bar{\beta}b_l) = 2S(l_{\bar{r}} + \epsilon\beta b_{\bar{r}})$ and $(\dot{\partial}_k P)\eta^k = \frac{1}{2\tilde{L}}(\delta_k \tilde{L})\eta^k$ respectively, where $\delta_k \tilde{L}$ is (1, 1)-homogeneous. Now, contracting $B^i = -(\partial_{\bar{k}} P)\bar{\eta}^k\eta^i$ with $\tilde{g}_{i\bar{m}}\bar{\eta}^m$, it leads to $(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = \frac{1}{2\tilde{L}}\theta^{*i}(l_l + \epsilon\bar{\beta}b_l)$, because $B^i\tilde{g}_{i\bar{m}}\bar{\eta}^m = -\frac{1}{2}\theta^{*i}(l_l + \epsilon\bar{\beta}b_l)$. We obtained $P = \frac{1}{2\tilde{L}}\Big[\delta_k \tilde{L}\eta^k + \theta^{*i}(l_l + \epsilon\beta b_{\bar{r}})\Big]$.

Conversely, substituting condition of (26) into (20) we get $\tilde{G}^i = G^i + S\eta^i$, where $S := \frac{1}{2\tilde{L}}(\delta_k \tilde{L})\eta^k$. Now, having $P = \frac{1}{2\tilde{L}}\Big[\delta_k \tilde{L}\eta^k + \theta^{*i}(l_l + \epsilon\beta b_{\bar{r}})\Big]$, we obtain

$$\begin{aligned} S \;\; = \;\; & \frac{1}{2(\alpha^2 + \epsilon|\beta|^2)}\Bigg\{\frac{\partial a_{lk}}{\partial z^k}\eta^l\eta^k + \epsilon(\bar{\beta}\frac{\partial b_l}{\partial z^k}\eta^l + \beta\frac{\partial b_{\bar{l}}}{\partial z^k}\bar{\eta}^l) - 2\overset{a}{N}{}^l_k \\ & + \frac{3+\omega}{2+\omega}\Bigg[\Bigg(\frac{\partial b_r}{\partial z^k}\bar{\eta}^r + \frac{\partial b_s}{\partial z^k}\eta^s\Bigg)\frac{\epsilon}{2}b^i + (b_r\bar{\eta}^r + b_s\eta^s)\frac{\partial b^i}{\partial z^k}\Bigg] \\ & + \frac{1}{2+\omega}\Bigg(\frac{\partial a_{r\bar{m}}}{\partial z^k}\bar{\eta}^r + \frac{\partial a_{s\bar{m}}}{\partial z^k}\eta^s\Bigg)b^i b^{\bar{m}}(l_l + \bar{\beta}b_l)\Bigg\}\eta^k, \end{aligned} \tag{28}$$

and $(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = \frac{1}{2\tilde{L}}\theta^{*i}(l_l + \epsilon\bar{\beta}b_l)$ where $\theta^{*i} = \dot{\partial}_k(\theta^{*i})\eta^k$.

Thus, these lead to $\tilde{G}^i = G^i + (\dot{\partial}_k P)\eta^k\eta^i$, $B^i = -(\dot{\partial}_{\bar{k}} P)\bar{\eta}^k\eta^i$ and $(\dot{\partial}_k P)\eta^k + (\dot{\partial}_{\bar{k}} P)\bar{\eta}^k = P$.

Substituting $\tilde{L} = \tilde{F}^2$ into (26) we have proven another equivalent complex versions of Rapsak's theorem.

Theorem 5. *Let L be a weakly Kähler purely Hermitian complex Finsler metric on the manifold M and $\tilde{L}$ be another purely Hermitian complex Finsler metric on M.*

Then, L and $\tilde{L}$ are projectively related if and only if $\tilde{L}$ is weakly Kähler and

$$\frac{\partial}{\partial\bar{\eta}^r}\Bigg\{\frac{\partial a_{lk}}{\partial z^k}\eta^l\eta^k+\epsilon(\bar{\beta}\frac{\partial b_l}{\partial z^k}\eta^l+\beta\frac{\partial b_{\bar{l}}}{\partial z^k}\bar{\eta}^l)-2\overset{a}{N}{}^l_k+\frac{3+\omega}{2+\omega}\Bigg[\Bigg(\frac{\partial b_r}{\partial z^k}\bar{\eta}^r+\frac{\partial b_s}{\partial z^k}\eta^s\Bigg)\frac{\epsilon}{2}b^i$$

$$(b_r\bar{\eta}^r+b_s\eta^s)\frac{\partial b^i}{\partial z^k}\Bigg]+\frac{1}{2+\omega}\Bigg(\frac{\partial a_{r\bar{m}}}{\partial z^k}\bar{\eta}^r+\frac{\partial a_{s\bar{m}}}{\partial z^k}\eta^s\Bigg)b^ib^{\bar{m}}(l_l+\bar{\beta}b_l)\Bigg\}\eta^k$$

$$+2(\frac{\partial}{\partial\bar{\eta}^r}G^l)(l_l+\epsilon\bar{\beta}b_l)=\frac{1}{\alpha^2+\epsilon|\beta|^2}(\delta_k\tilde{L})\eta^k(l_{\bar{r}}+\epsilon\beta b_{\bar{r}}),\tag{29}$$

where $P=\frac{1}{2\tilde{L}}(\delta_k\tilde{L})\eta^k$. Moreover, the projetive change is $\tilde{G}^i=G^i+P\eta^i$ and P is (1, 0)*-homogeneous.*

Proof. Having in mind the theorem (3) and (4) the direct implication is obvious. For the converse, we have $B^i=\theta^{*i}=\tilde{\theta}^{*i}=0$, because L and $\tilde{L}$ are weakly Kähler, which together with (29) are sufficients conditions for the projectiveness of the purely Hermitian complex Finsler metrics L and $\tilde{L}$. Now, plugging (29) into (20) it results $\tilde{G}^i=G^i+P\eta^i$ and The (1, 0)-homogeneity of P.

4. Projectively related generalized Berwald spaces with purely Hermitian complex Finsler metric

We consider $z\in M$, $\eta\in T^{'}_zM=\eta^i\frac{\partial}{\partial z^i}$, $\tilde{a}:=a_{i\bar{j}}(z)dz^i\otimes d\bar{z}^j$ a purely Hermitian metric and $b=b_i(z)dz^i$ a differential (1, 0)-form. By these objects we have defined (for more details see [5]) the complex (α,β)-metric F on $T^{'}M$

$$\tilde{F}(z,\eta):=F(\alpha(z,\eta),\ |\beta(z,\eta)|),\tag{30}$$

where $\alpha(z,\eta):=\sqrt{a_{i\bar{j}}(z)\eta^i\bar{\eta}^j}$ and $\beta(z,\eta)=b_i(z)\eta^i$.
Let us recall that the coefficients of the $C-F$ connection corresponding to the purely Hermitian metric α are.

$$\overset{a}{N}{}^k_j:=a^{\bar{m}k}\frac{\partial a_{l\bar{m}}}{\partial z^j}\eta^l\ ;\ \overset{a}{L}{}^i_{jk}:=a^{\bar{l}i}(\overset{a}{\delta}_k a_{j\bar{l}})\ ;\ \overset{a}{C}{}^i_{jk}=0,\tag{31}$$

and we consider the settings

$$b^l:=a^{\bar{r}l}b_{\bar{j}}\ ;\ ||b||^2:=a^{\bar{r}l}b_lb_{\bar{r}}\ ;\ b^{\bar{l}}:=\bar{b}^l\ ;\ l_l=a_{l\bar{r}}\bar{\eta}^r\quad and\quad l_{\bar{r}}=a_{k\bar{r}}\eta^k.\tag{32}$$

Differentiating $\alpha(z,\eta)$ and $|\beta(z,\eta)|$ partially with respect to η^l and $\bar{\eta}^r$

$$\frac{\partial\alpha}{\partial\eta^l}=\frac{l_l}{2\alpha},\ \frac{\partial|\beta|}{\partial\eta^l}=\frac{\bar{\beta}b_l}{2|\beta|},\ \frac{\partial\alpha}{\partial\bar{\eta}^r}=\frac{l_{\bar{r}}}{2\alpha},\ \frac{\partial|\beta|}{\partial\bar{\eta}^r}=\frac{\beta b_{\bar{r}}}{2|\beta|},$$
$$\frac{\partial^2\alpha}{\partial\eta^l\partial\bar{\eta}^r}=\frac{a_{l\bar{r}}}{2\alpha}-\frac{l_l l_{\bar{r}}}{4\alpha^3},\ \textit{and}\ \frac{\partial^2|\beta|}{\partial\eta^l\partial\bar{\eta}^r}=\frac{b_l b_{\bar{r}}}{4|\beta|}, \tag{33}$$

we introduce generalized purely Hermitian metric function $\tilde{F}$ on the complex manifold M by

$$\tilde{F}=\sqrt{\alpha^2+\epsilon|\beta|^2}\ \ \textit{where}\ \ \epsilon=\pm1, \tag{34}$$

$$\eta_l=\frac{l_l}{\tilde{F}}+\epsilon\frac{\bar{\beta}b_l}{2\tilde{F}}. \tag{35}$$

Theorem 6. *Let F be a generalized Berwald metric on the manifold M and $\tilde{F}$ another purely Hermitian complex Finsler metric on M. Then F and $\tilde{F}$ are projective if and only if*

$$\begin{aligned}\dot{\partial}_{\bar{r}}(\delta_k\tilde{F})\eta^k &= \frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k(\frac{1}{2\tilde{F}}(l_{\bar{r}}+\epsilon\beta b_{\bar{r}}))\\ B^r &= -\frac{1}{\sqrt{\alpha^2+\epsilon|\beta|^2}}\theta^{*l}(l_l+\epsilon\bar{\beta}b_l)\eta^r,\end{aligned} \tag{36}$$

$$\begin{aligned}P &= \frac{1}{\sqrt{\alpha^2+\epsilon|\beta|^2}}\Bigg\{\Bigg[\frac{1}{2\tilde{F}}\Big(\frac{\partial a_{lk}}{\partial z^k}\eta^l\eta^k+\epsilon(\bar{\beta}\frac{\partial b_l}{\partial z^k}\eta^l+\beta\frac{\partial b_{\bar{l}}}{\partial z^k}\bar{\eta}^l)\Big)\\ &-\overset{a}{N}{}^i_j+\frac{1}{\rho_0}a^{\bar{m}i}(\frac{\partial\bar{\eta}_m}{\partial z^j}-\rho_0\frac{\partial a_{\bar{m}}}{\partial z^j}\eta^l)-\frac{1}{\rho_0}\Bigg[R\eta^i\bar{\eta}^m-\frac{\rho_0''}{\rho\gamma}(b^i-R\bar{\beta}\eta^i)(b^{\bar{m}}-R\beta\bar{\eta}^m)\\ &-\frac{\mu_{-2}''}{\rho_0 M}\Big(a^{\bar{m}i}-R\eta^i\bar{\eta}^m-\frac{\mu_0''}{\rho\gamma}(b^i-R\bar{\beta}\eta^i)(b^{\bar{m}}-R\beta\bar{\eta}^m)\Big)^2\Bigg]\frac{\partial\bar{\eta}_m}{\partial z^j}\\ &(\frac{1}{\tilde{F}}(l_j+\epsilon\bar{\beta}b_j))\Bigg]\eta^k+\theta^{*i}(l_i+\epsilon\beta b_i)\Bigg\},\end{aligned} \tag{37}$$

for any $r=\overline{1,n}$. Moreover, the projective change is $\tilde{G}^i=G^i+\frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k\eta^i$ and $\tilde{F}$ is also generalized Berwald.

where

$$\begin{aligned}G^i &= \overset{a}{G}{}^i_j+\Bigg\{\frac{1}{2\rho_0}a^{\bar{m}i}(\frac{\partial\bar{\eta}_m}{\partial z^j}-\rho_0\frac{\partial a_{\bar{m}}}{\partial z^j}\eta^l)-\frac{1}{2\rho_0}\Bigg[R\eta^i\bar{\eta}^m-\frac{\rho_0''}{\rho\gamma}(b^i-R\bar{\beta}\eta^i)(b^{\bar{m}}-R\beta\bar{\eta}^m)\\ &-\frac{\mu_{-2}''}{\rho_0 M}\Big(a^{\bar{m}i}-R\eta^i\bar{\eta}^m-\frac{\mu_0''}{\rho\gamma}(b^i-R\bar{\beta}\eta^i)(b^{\bar{m}}-R\beta\bar{\eta}^m)\Big)^2\Bigg]\frac{\partial\bar{\eta}_m}{\partial z^j}\Bigg\}\eta^j.\end{aligned} \tag{38}$$

Proof. The equivalance results by theorem (4) which $\dot{\partial}_{\bar{r}} G^l = 0$, because F is a generalized Berwald metric. In order to show that $\tilde{F}$ is generalized Berwald, we compute

$$\begin{aligned}
\frac{\partial}{\partial \bar{\eta}^r}\left(\frac{1}{\tilde{F}}(\delta_k \tilde{F})\eta^k\right) &= -\frac{1}{\alpha^2+\epsilon|\beta|^2}\left(\frac{1}{2\tilde{F}}(l_{\bar{r}}+\epsilon\beta b_{\bar{r}})\right)\Bigg\{\frac{1}{2\tilde{F}}\bigg(\frac{\partial a_{lk}}{\partial z^k}\eta^l\eta^k \\
&+\epsilon(\bar{\beta}\frac{\partial b_l}{\partial z^k}\eta^l+\beta\frac{\partial b_{\bar{l}}}{\partial z^k}\bar{\eta}^l)\bigg) - \overset{a}{N}{}^i_j + \frac{1}{\rho_0}a^{\bar{m}i}(\frac{\partial \bar{\eta}_m}{\partial z^j} - \rho_0\frac{\partial a_{\bar{m}}}{\partial z^j}\eta^l) \\
&-\frac{1}{\rho_0}\bigg[R\eta^i\bar{\eta}^m - \frac{\rho_0^{''}}{\rho\gamma}(b^i-R\bar{\beta}\eta^i)(b^{\bar{m}}-R\beta\bar{\eta}^m) - \frac{\mu_{-2}^{''}}{\rho_0 M} \\
&\left(a^{\bar{m}i} - R\eta^i\bar{\eta}^m - \frac{\mu_0^{''}}{\rho\gamma}(b^i-R\bar{\beta}\eta^i)(b^{\bar{m}}-R\beta\bar{\eta}^m)\right)^2\bigg]\frac{\partial\bar{\eta}_m}{\partial z^j} \\
&(\frac{1}{\tilde{F}}(l_j+\epsilon\bar{\beta}b_j))\Bigg\}\eta^k + \frac{1}{\sqrt{\alpha^2+\epsilon|\beta|^2}}\Bigg\{\frac{\partial}{\partial\bar{\eta}^r}\bigg[\frac{1}{2\tilde{F}}\bigg(\frac{\partial a_{lk}}{\partial z^k}\eta^l\eta^k \\
&+\epsilon(\bar{\beta}\frac{\partial b_l}{\partial z^k}\eta^l+\beta\frac{\partial b_{\bar{l}}}{\partial z^k}\bar{\eta}^l)\bigg) - \overset{a}{N}{}^i_j + \frac{1}{\rho_0}a^{\bar{m}i}(\frac{\partial \bar{\eta}_m}{\partial z^j} - \rho_0\frac{\partial a_{\bar{m}}}{\partial z^j}\eta^l) \\
&-\frac{1}{\rho_0}\bigg[R\eta^i\bar{\eta}^m - \frac{\rho_0^{''}}{\rho\gamma}(b^i-R\bar{\beta}\eta^i)(b^{\bar{m}}-R\beta\bar{\eta}^m) - \frac{\mu_{-2}^{''}}{\rho_0 M} \\
&\left(a^{\bar{m}i} - R\eta^i\bar{\eta}^m - \frac{\mu_0^{''}}{\rho\gamma}(b^i-R\bar{\beta}\eta^i)(b^{\bar{m}}-R\beta\bar{\eta}^m)\right)^2\bigg]\bigg]\Bigg\}\eta^k = 0,
\end{aligned} \tag{39}$$

by using the first identity from (36). Now, differentiating the projective change $\tilde{G}^i = G^i + \frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k\eta^i$ with respect to $\bar{\eta}^r$ it results $\dot{\partial}_{\bar{r}}\tilde{G}^l = 0$, that is $\tilde{F}$ is generalized Berwald.

Lemma 7. *Let F be a complex Berwald metric on the manifold M and $\tilde{F}$ another complex Finsler metric on M. Then, F and $\tilde{F}$ are projectively related if and only if $\tilde{F}$ is weakly Kähler and*

$$\dot{\partial}_{\bar{r}}(\delta_k\tilde{F})\eta^k = P(\dot{\partial}_{\bar{r}}\tilde{F}); \quad r = \overline{1,n}; \quad P = \frac{1}{\tilde{F}}(\delta_k\tilde{F})\eta^k. \tag{40}$$

Moreover, the projective change is $\tilde{G}^i = G^i + P\eta^i$ and $\tilde{F}$ is generalized berwald.

5. Projectiveness of a purely Hermitian complex Finsler metrics

We consider $\beta(z,\eta) := b_i(z)\eta^i$ a diffrential $(1,0)$-form and $\alpha(z,\eta) := \sqrt{a_{i\bar{j}}(z)\eta^i\bar{\eta}^j}$ a purely Hermitian metric on the manifold M. By these we have defined the purely Hermitian complex Finsler metric $\tilde{F} = \sqrt{\alpha^2 + \epsilon|\beta|^2}$ on $T^{'}M$ with

$$\frac{\partial\alpha}{\partial\eta^i} = \frac{1}{2\alpha}l_i; \quad \frac{\partial|\beta|}{\partial\eta^i} = \frac{\bar{\beta}}{2|\beta|}b_i; \quad \tilde{\eta}_i = l_i + \epsilon\bar{\beta}b_i,$$

$$\begin{aligned}\overset{CF}{N^i_k} &= 2\overset{a}{N^i_k} + \frac{3+\omega}{2+\omega}\left[\left(\frac{\partial b_r}{\partial z^k}\bar{\eta}^r + \frac{\partial b_s}{\partial z^k}\eta^s\right)\frac{\epsilon}{2}b^i + (b_r\bar{\eta}^r + b_s\eta^s)\frac{\partial b^i}{\partial z^k}\right] \\ &+\frac{1}{2+\omega}\left(\frac{\partial a_{r\bar{m}}}{\partial z^k}\bar{\eta}^r + \frac{\partial a_{s\bar{m}}}{\partial z^k}\eta^s\right)b^i b^{\bar{m}},\end{aligned} \tag{41}$$

where $\overset{a}{N^i_k} = a^{\bar{m}i}\frac{\partial a_{l\bar{m}}}{\partial z^k}\eta^l$, $l_i := a_{i\bar{j}}\bar{\eta}^j$, $b^i := a^{\bar{j}i}b_{\bar{j}}$, $b^{\bar{i}} := \bar{b}^i$, and so the spray coefficints are

$$\begin{aligned}\tilde{G}^i &= \left\{G^i + \frac{1}{2}(\frac{3+\omega}{2+\omega})\left[\left(\frac{\partial b_r}{\partial z^k}\bar{\eta}^r + \frac{\partial b_s}{\partial z^k}\eta^s\right)\frac{\epsilon}{2}b^i + (b_r\bar{\eta}^r + b_s\eta^s)\frac{\partial b^i}{\partial z^k}\right]\right. \\ &\left.+\frac{1}{2+\omega}\left(\frac{\partial a_{r\bar{m}}}{\partial z^k}\bar{\eta}^r + \frac{\partial a_{s\bar{m}}}{\partial z^k}\eta^s\right)b^i b^{\bar{m}}\right\}\eta^j.\end{aligned} \tag{42}$$

Where $G^i = \frac{1}{2}\overset{a}{N^i_j}\eta^j$ are the spray coefficients of the purely Hermitian metric α. Moreover, the purely Hermitian complex Finsler metric $\tilde{F}$ is weakly Kähler if and only if

$$\begin{aligned}&\left\{2\left(\frac{\partial a_{lk}}{\partial z^k}\eta^l\eta^k + \epsilon(\bar{\beta}\frac{\partial b_l}{\partial z^k}\eta^l + \beta\frac{\partial b_{\bar{l}}}{\partial z^k}\bar{\eta}^l) - 2\overset{a}{N^l_k} + \frac{3+\omega}{2+\omega}\left[\left(\frac{\partial b_r}{\partial z^k}\bar{\eta}^r + \frac{\partial b_s}{\partial z^k}\eta^s\right)\frac{\epsilon}{2}b^i\right.\right.\right. \\ &\left.\left.+(b_r\bar{\eta}^r + b_s\eta^s)\frac{\partial b^i}{\partial z^k}\right] + \frac{1}{2+\omega}\left(\frac{\partial a_{r\bar{m}}}{\partial z^k}\bar{\eta}^r + \frac{\partial a_{s\bar{m}}}{\partial z^k}\eta^s\right)b^i b^{\bar{m}}(l_l + \bar{\beta}b_l)\right) \\ &\left.\eta^k(l_{\bar{r}} + \epsilon\beta b_{\bar{r}}) + (a_{i\bar{m}} + b_i b_{\bar{m}})\bar{\eta}^m(\frac{\partial}{\partial\bar{\eta}^r}(N^i_k))\right\}\bar{\eta}^r = 0,\end{aligned} \tag{43}$$

where
$2\overset{c}{\delta_k}(\dot{\partial}_{\bar{r}}L)\bar{\eta}^r = -g_{i\bar{m}}\bar{\eta}^m\mathit{\Gamma}^i_{jk}\eta^j - g_{i\bar{m}}\bar{\eta}^m(\dot{\partial}_{\bar{r}}N^i_k)\bar{\eta}^r$ and $\mathit{\Gamma}^i_{jk} = \frac{1}{2}a^{\bar{r}k}(\frac{\partial a_{k\bar{j}}}{\partial z^i} - \frac{\partial a_{i\bar{j}}}{\partial z^k})$.

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