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ON QUASI-HADAMARD PRODUCTS OF P-VALENT FUNCTIONS WITH NEGATIVE COEFFICIENTS DEFINED BY USING A DIFFERENTIAL OPERATOR

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Abstract. In this paper we establish certain results concerning the quasi-Hadamard products of certain p-valent starlike and p-valent convex functions with negative coefficients defined by using a differential operator.

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1. Introduction

Let $T(p)$ denote the class of functions of the form :

$$f(z) = z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \qquad (a_{p+n} \geq 0; p \in N = \{1, 2, ...\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disc $U = \{z : z \in C \text{ and } |z| < 1\}$. In [3], Chen et al. investigated various interesting properties and characteristics of functions belonging to two subclasses $S(p, q, \alpha)$ and $C(p, q, \alpha)$ of the class $T(p)$, where $S(p, q, \alpha)$ and $C(p, q, \alpha)$ are defined as follows:

$$S(p,q,\alpha) = \left\{ f(z) \in T(p) : \text{Re}\left\{ \frac{zf^{(1+q)}(z)}{f^{(q)}(z)} \right\} > \alpha \right. ,$$

$$\left. (z \in U; 0 \leq \alpha < p - q; p \in N; p > q; q \in N_0 = N \cup \{0\}) \right\} \tag{1.2}$$

and

$$C(p,q,\alpha) = \left\{ f(z) \in T(p) : \text{Re}\left\{ 1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \right\} > \alpha \right. ,$$

$$\left.(z \in U; 0 \le \alpha < p - q; p \in N; p > q; q \in N_0)\right\} \ , \tag{1.3}$$

where, for each $f(z) \in T(p)$, we have (see [3])

$$f^{(j)}(z) = \frac{p!}{(p-j)!} z^{p-j} - \sum_{n=1}^{\infty} \frac{(n+p)!}{(n+p-j)!} a_{n+p} z^{n+p-j} \quad (j \in N_0; p > j)\,. \tag{1.4}$$

We note that :

(i) $S(p,0,\alpha) = T^*(p,\alpha)$, is the class of p-valently starlike functions of order $\alpha, 0 \le \alpha < p$;

(ii) C(p,0,α) $= C(p,\alpha)$, is the class of p-valently convex functions of order $\alpha, 0 \le \alpha < p$.

The classes $T^*(p,\alpha)$ and $C(p,\alpha)$ are studied by Owa [13] and Salagean et al. [14].

In [3], Chen et al. obtained the following results.

Lemma 1 [3]. *A function $f(z) \in T(p)$ is in the class $S(p,q,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} (n+p-q-\alpha)\delta(n+p,q) a_{p+n} \le (p-q-\alpha)\delta(p,q) \tag{1.5}$$

$$(0 \le \alpha < p-q; p \in N; p > q; q \in N_0)\,,$$

where

$$\delta(p,q) = \frac{p!}{(p-q)!} = \begin{cases} p(p-1)...(p-q+1) & (q \ne 0) \\ 1 & (q = 0)\,. \end{cases} \tag{1.6}$$

Lemma 2 [3]. *A function $f(z) \in T(p)$ is in the class $C(p,q,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \left(\frac{n+p-q}{p-q}\right)(n+p-q-\alpha)\delta(n+p,q) a_{p+n} \le (p-q-\alpha)\delta(p,q)\,. \tag{1.7}$$

Let $T_0(p)$ denote the class of functions of the form :

$$f(z) = a_p z^p - \sum_{n=1}^{\infty} a_{p+n} z^{p+n} \quad (a_p > 0; a_{p+n} \ge 0; p \in N) \tag{1.8}$$

which are analytic and p-valent in U. Furthermore, let $T_0^*(p,q,\alpha)$ and $C_0(p,q,\alpha)$ be the subclasses of $T_0(p)$ defined as follows :

$$T_0^*(p,q,\alpha) = \left\{ f(z) \in T_0(p) : \text{Re}\left\{ \frac{z f^{(1+q)}(z)}{f^{(q)}(z)} \right\} > \alpha \ ,\right.$$

$$\left.(z \in U; 0 \leq \alpha < p-q; p \in N; p > q; q \in N_0)\right\},$$

and

$$C_0(p,q,\alpha) = \left\{ f(z) \in T_0(p) : \operatorname{Re}\left\{ 1 + \frac{zf^{(2+q)}(z)}{f^{(1+q)}(z)} \right\} > \alpha \right.,$$

$$\left.(z \in U; 0 \leq \alpha < p-q; p \in N; p > q; q \in N_0)\right\} .$$

For these classes, by using Lemma 1 and Lemma 2, we easily obtain the following theorems :

Theorem 1. *A function $f(z) \in T_0(p)$ is in the class $T_0^*(p,q,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} [(n+p-q-\alpha)\delta(n+p,q)a_{p+n}] \leq (p-q-\alpha)\delta(p,q)a_p\,.$$

Theorem 2. *A function $f(z) \in T_0(p)$ is in the class $C_0(p,q,\alpha)$ if and only if*

$$\sum_{n=1}^{\infty} \left[\left(\frac{n+p-q}{p-q} \right) (n+p-q-\alpha)\delta(n+p,q)a_{p+n} \right] \leq (p-q-\alpha)\delta(p,q)a_p\,.$$

We now introduce a subclass $S_0(k,p,q,\alpha)$ of the class $T_0(p)$. We say that a function $f(z)$ belongs to the class $S_0(k,p,q,\alpha)$ if and only if

$$\sum_{n=1}^{\infty} \left[\left(\frac{n+p-q}{p-q} \right)^k (n+p-q-\alpha)\delta(n+p,q)a_{p+n} \right] \leq$$
$$(p-q-\alpha)\delta(p,q)a_p \qquad (0 \leq \alpha < p-q)\,, \tag{1.9}$$

where k is any fixed non-negative real number.

We note that for every nonnegative real number k, the class $S_0(k,p,q,\alpha)$ is nonempty as the functions of the form

$$f(z) = a_p z^p - \sum_{n=1}^{\infty} \frac{(p-q-\alpha)\delta(p,q)a_p}{\left(\frac{n+p-q}{p-q}\right)^k (n+p-q-\alpha)\delta(n+p,q)} \lambda_{p+n} z^{p+n},$$

where $a_p > 0$, $\lambda_{p+n} > 0$ and $\sum\limits_{n=1}^{\infty} \lambda_{p+n} \leq 1$, satisfy the inequality (1.9). Evidently, $S_0(0,p,q,\alpha) \equiv T_0^*(p,q,\alpha)$ and $S_0(1,p,q,\alpha) \equiv C_0(p,q,\alpha)$. Further, $S_0(k,p,q,\alpha) \subset S_0(c,p,q,\alpha)$ if $k > c \geq 0$, the containment being proper. Hence, for any positive integer k, we have the inclusion relation

$$S_0(k,p,q,\alpha) \subset S_0(k-1,p,q,\alpha) .... \subset S_0(2,p,q,\alpha) \subset C_0(p,q,\alpha) \subset T_0^*(p,q,\alpha)\,.$$

Finally, let the functions of the class $T_0(p)$ be of the forms :

$$f_i(z) = a_{p,i}z^p - \sum_{n=1}^{\infty} a_{p+n,i}z^{p+n} \qquad (a_{p,i} > 0; a_{p+n,i} \geq 0)$$

and

$$g_j(z) = b_{p,j}z^p - \sum_{n=1}^{\infty} b_{p+n,j}z^{p+n} \qquad (b_{p,j} > 0; b_{p+n,j} \geq 0)\,,$$

and define the quasi-Hadamard product $f_i * g_j(z)$ of the functions $f_i(z)$ and $g_j(z)$ by

$$f_i * g_j(z) = a_{p,i}b_{p,j}z^p - \sum_{n=1}^{\infty} a_{p+n,i}b_{p+n,j}z^{p+n} \qquad (i, j = 1, 2, 3, ...)\,.$$

Similarly, we can define the quasi-Hadamard product of more than two functions.

The quasi-Hadamard product of two or more functions has recently been defined and used by Owa ([10], [11] and [12]), Kumar ([7], [8] and [9]), Sekine [15], Aouf [1], Aouf et al. [2], Frasin and Aouf [5], Hossen [6] and Darwish [4].

In this paper we establish certain results concerning the quasi-Hadamard product of functions in the classes $S_0(k, p, q, \alpha), T_0(p, q, \alpha)$ and $C_0(p, q, \alpha)$ analgous to the results due to Kumar ([8] and [9]) and Sekine [15].

2. Results involving quasi-Hadamard products

Theorem 3. *Let the functions $f_i(z)$ belong to the classes $T_0^*(p, q, \alpha_i)(i = 1, 2, 3, ..., m)$ and let the functions $g_j(z)$ belong to the classes $C_0(p, q, \beta_j)(j = 1, 2, 3, ..., d)$. Then the quasi-Hadamard product $f_1 * f_2 * f_3 * ... * f_m * g_1 * g_2 * g_3 * ... * g_d(z)$ belongs to the class $S_0(m + 2d - 1, p, q, \gamma)$, where*

$$\gamma = \max\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_m, \beta_1, \beta_2, \beta_3, ..., \beta_d\}\,.$$

Proof. Since $f_i(z) \in T_0^*(p, q, \alpha_i)(i = 1, 2, ..., m)$, by Theorem 1, we have

$$\sum_{n=1}^{\infty}(n + p - q - \alpha_i)\delta(n + p, q)a_{p+n,i} \leq (p - q - \alpha_i)\delta(p, q)a_{p,i}\,. \tag{2.1}$$

which yields

$$a_{p+n,i} \leq \left(\frac{p - q}{n + p - q}\right) a_{p,i} \qquad (1 \leq i \leq m)\,. \tag{2.2}$$

Also, since $g_j(z) \in C_0(p, q, \beta_j)(j = 1, 2, 3, ..., d)$, by Theorem 2, we have

$$\sum_{n=1}^{\infty} \left(\frac{n + p - q}{p - q}\right) (n + p - q - \beta_j)\delta(n + p, q)b_{p+n,j} \leq (p - q - \beta_j)\delta(p, q)b_{p,j}\,. \tag{2.3}$$

which yields

$$b_{p+n,j} \le \left(\frac{p-q}{n+p-q}\right)^2 b_{p,j} \qquad (1 \le j \le d). \tag{2.4}$$

It is sufficient to show that

$$\sum_{n=1}^{\infty}\left\{\left(\frac{n+p-q}{p-q}\right)^{m+2d-1}(n+p-q-\gamma)\delta(n+p,q)\prod_{i=1}^{m}a_{p+n,i}\cdot\prod_{j=1}^{d}b_{p+n,j}\right\}$$
$$\le (p-q-\gamma)\delta(p,q)\prod_{i=1}^{m}a_{p,i}\prod_{j=1}^{d}b_{p,j}\ .$$

The following two cases will arise :

(i) When $\gamma = \max\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_m\}$, we may assume that $\gamma = \alpha_m$. Then, by using (2.2) for $i = 1, 2, ..., m-1$ and (2.4) for $j = 1, 2, ..., d$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{\left(\frac{n+p-q}{p-q}\right)^{m+2d-1}(n+p-q-\gamma)\delta(n+p,q)\prod_{i=1}^{m}a_{p+n,i}\cdot\prod_{j=1}^{d}b_{p+n,j}\right\}\\
\le\ &\sum_{n=1}^{\infty}\left\{\left(\frac{n+p-q}{p-q}\right)^{m+2d-1}(n+p-q-\alpha_m)\delta(n+p,q)\ .\right.\\
&\left.\cdot\left[\left(\frac{p-q}{n+p-q}\right)^{m-1}\prod_{i=1}^{m-1}a_{p,i}\right]\left[\left(\frac{p-q}{n+p-q}\right)^{2d}\prod_{j=1}^{d}b_{p,j}\right]a_{p+n,m}\right\}\\
=\ &\left[\prod_{i=1}^{m-1}a_{p,i}\right]\left[\prod_{j=1}^{d}b_{p,j}\right]\sum_{n=1}^{\infty}(n+p-q-\alpha_m)\delta(n+p,q)a_{p+n,m}\\
\le\ &(p-q-\alpha_m)\delta(p,q)\left[\prod_{i=1}^{m}a_{p,i}\right]\left[\prod_{j=1}^{d}b_{p,j}\right]\\
=\ &(p-q-\gamma)\delta(p,q)\left[\prod_{i=1}^{m}a_{p,i}\right]\left[\prod_{j=1}^{d}b_{p,j}\right]\ .
\end{aligned}$$

(ii) When $\gamma = \max\{\beta_1, \beta_2, \beta_3, ..., \beta_d\}$, we may assume that $\gamma = \beta_d$. Then, by using

(2.2) for $i = 1, 2, ..., m$ and (2.4) for $j = 1, 2, ..., d-1$, we have

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{\left(\frac{n+p-q}{p-q}\right)^{m+2d-1}(n+p-q-\gamma)\delta(n+p,q)\prod_{i=1}^{m}a_{p+n,i}\cdot\prod_{j=1}^{d}b_{p+n,j}\right\}\\
\leq\ &\sum_{n=1}^{\infty}\left\{\left(\frac{n+p-q}{p-q}\right)^{m+2d-1}(n+p-q-\beta_d)\delta(n+p,q)\ .\right.\\
&\left.\cdot\left[\left(\frac{p-q}{n+p-q}\right)^{m}\prod_{i=1}^{m}a_{p,i}\right]\left[\left(\frac{p-q}{n+p-q}\right)^{2(d-1)}\prod_{j=1}^{d-1}b_{p,j}\right]b_{p+n,d}\right\}\\
=\ &\left[\prod_{i=1}^{m}a_{p,i}\right]\left[\prod_{j=1}^{d-1}b_{p,j}\right]\sum_{n=1}^{\infty}\left(\frac{n+p-q}{p-q}\right)(n+p-q-\beta_d)\delta(n+p,q)b_{p+n,d}\\
\leq\ &(p-q-\beta_d)\delta(p,q)\left[\prod_{i=1}^{m}a_{p,i}\right]\left[\prod_{j=1}^{d}b_{p,j}\right]\\
=\ &(p-q-\gamma)\delta(p,q)\left[\prod_{i=1}^{m}a_{p,i}\right]\left[\prod_{j=1}^{d}b_{p,j}\right]\ .
\end{aligned}$$

In both cases we conclude that

$$f_1 * f_2 * f_3 * ... * f_m * g_1 * g_2 * g_3 * ... * g_d(z) \in S_0(m+2d-1, p, q, \gamma)\,.$$

This completes the proof of Theorem 3.

Now we discuss the applications of Theorem 3. Taking into account the quasi-Hadamard product of functions $f_1(z), f_2(z), ..., f_m(z)$ only, in the proof of Theorem 3, and using (2.2) for $i = 1, 2, ..., m-1$, and (2.1) for $i = m$, we are led to
Corollary 1. *Let the functions $f_i(z)$ belong to the classes $T_0^*(p, q, \alpha_i)(i = 1, 2, ..., m)$. Then the quasi-Hadamard product $f_1 * f_2 * f_3 * ... * f_m(z)$ belongs to the class $S_0(m-1, p, q, \beta)$, where $\beta = \max\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_m\}$.*

Next, taking into account the quasi-Hadamard product of the functions $g_1(z), g_2(z), ..., g_d(z)$ only, in the proof of Theorem 3, and using (2.4) for $j = 1, 2, ..., d-1$, and (2.3) for $j = d$, we are led to
Corollary 2. *Let the functions $g_j(z)$ belong to the classes $C_0(p, q, \alpha_j)(j = 1, 2, ..., d)$. Then the quasi-Hadamard product $g_1 * g_1 * g_3 * ... * g_d(z)$ belongs to the class $S_0(2d-1, p, q, \beta)$, where $\beta = \max\{\alpha_1, \alpha_2, \alpha_3, ..., \alpha_d\}$.*

Theorem 4. *Let the functions* $f_i(z)$ *belong to the class* $C_0(p,q,\alpha)(i=1,2,3,...,m)$, *and let* $0 \leq \alpha \leq r_0$, *where* r_0 *is a root of the equation*

$$(p-q+1)^m(p-q-mr)-(p-q)(p-q-r)^m=0$$

in the open interval $\left(0, \frac{p-q}{m}\right)$. Then the quasi-Hadamard product $f_1 * f_2 * f_3 * ... * f_m(z)$ belongs to the class $S_0(m-1,p,q,m\alpha)$.

Proof. Since $f_i(z) \in C_0(p,q,\alpha)(i=1,2,3,...,m)$, by Theorem 2, we have

$$\sum_{n=1}^{\infty}\left(\frac{n+p-q}{p-q}\right)(n+p-q-\alpha)\delta(n+p,q)a_{p+n,i} \leq$$

$$(p-q-\alpha)\delta(p,q)a_{p,i} \quad (1 \leq i \leq m)\,.$$

Therefore

$$\sum_{n=1}^{\infty}(n+p-q-\alpha)\delta(n+p,q)a_{n+p,i} \leq$$

$$\left(\frac{p-q}{1+p-q}\right)(p-q-\alpha)\delta(p,q)a_{p,i} \quad (1 \leq i \leq m)\,, \tag{2.5}$$

which evidently yields

$$(n+p-q-\alpha)\delta(n+p,q)a_{p+n,i} \leq$$

$$\left(\frac{p-q}{1+p-q}\right)(p-q-\alpha)\delta(p,q)a_{p,i} \quad (1 \leq i \leq m)\,. \tag{2.6}$$

By mathematical induction on m, we can get the inequality

$$(n+p-q)^{m-1}(n+p-q-m\alpha) \leq (n+p-q-\alpha)^m, \tag{2.7}$$

where $0 \leq \alpha < p-q$, $m \geq 1$, and $m\alpha < p-q$. Using (2.7), (2.6) for $i=1,2,3,...,m-$

1, and using (2.5) for $i = m$, we also get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{\left(\frac{n+p-q}{p-q}\right)^{m-1}(n+p-q-m\alpha)\delta(n+p,q)\ .\prod_{i=1}^{m}a_{p+n,i}\right\}\\
\leq\ &\sum_{n=1}^{\infty}\left\{\left(\frac{1}{p-q}\right)^{m-1}(n+p-q-\alpha)^{m}\delta(n+p,q)\prod_{i=1}^{m}a_{p+n,i}\right\}\\
\leq\ &\left\{\left(\frac{p-q-\alpha}{1+p-q}\right)^{m-1}\prod_{i=1}^{m-1}a_{p,i}\right\}\sum_{n=1}^{\infty}(n+p-q-\alpha)\delta(n+p,q)a_{p+n,m}\\
\leq\ &(p-q)\left(\frac{p-q-\alpha}{1+p-q}\right)^{m}\delta(p,q)\prod_{i=1}^{m}a_{p,i}\\
\leq\ &(p-q-m\alpha)\delta(p,q)\prod_{i=1}^{m}a_{p,i}\ .
\end{aligned}$$

This proves that

$$f_1 * f_2 * ... * f_m(z) \in S_0(m-1,p,q,m\alpha)\ ,$$

as asserted by Theorem 4.

Remark. Putting $q = 0$ in the above results we obtain the results obtained by Sekine [15].

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