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THE ORDER OF CONVEXITY OF SOME INTEGRAL OPERATORS

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Abstract. In this paper we consider the classes of starlike functions of order α, convex functions of order α and we study the convexity and α-order convexity for some general integral operators. Several corollaries of the main results are also considered.

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1. Introduction

We consider the unit open disk of the complex plane denoted by U, $U = \{z : |z| < 1\}$ and let $\mathcal{A}$ be the class of holomorphic functions in U of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in U. We denote by S the class of univalent functions in the unit disk.

A function $f(z) \in S$ is a starlike of order α if it satisfies

$$\text{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in U) \tag{2}$$

for some α $(0 \leq \alpha < 1)$. We denote by $S^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of the functions which are starlike of order α in U. For $\alpha = 0$ we obtain the class of starlike functions, denoted by S^*.

A function $f(z) \in S$ is convex of order α if it satisfies

$$\mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, \quad (z \in U) \tag{3}$$

for some α $(0 \leq \alpha < 1)$. We denote by $K(\alpha)$ the subclass of $\mathcal{A}$ consisting of the functions which are convex of order α in U. For $\alpha = 0$ we obtain the class of convex functions, denoted by K.

A function $f \in \mathcal{A}$ is in the class $R(\alpha)$ if $\mathrm{Re}(f'(z)) > \alpha$, $(z \in U)$.

Recently, Frasin and Jahangiri in [3] define the family $B(\mu, \alpha),\ \mu \geq 0,\ 0 \leq \alpha < 1$ so that it consists of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| f'(z) \left(\frac{z}{f(z)}\right)^{\mu} - 1 \right| < 1 - \alpha, \quad (z \in U). \tag{4}$$

In this paper we will obtain the order of convexity of the following integral operators:

$$G_{\gamma}(z) = \int_0^z \left(te^{f(t)}\right)^{\frac{1}{\gamma}} \mathrm{d}t, \tag{5}$$

$$G_{n,\gamma}(z) = \int_0^z \prod_{i=1}^{n} \left(te^{f_i(t)}\right)^{\gamma} \mathrm{d}t, \tag{6}$$

$$H_{n,\gamma}(z) = \int_0^z \prod_{i=1}^{n} \left(te^{f_i(t)}\right)^{\frac{1}{\gamma}} \mathrm{d}t, \tag{7}$$

and

$$H_n(z) = \int_0^z \prod_{i=1}^{n} \left(te^{f_i(t)}\right)^{\gamma_i} \mathrm{d}t, \tag{8}$$

where the functions f_i for all $i = 1, 2, ..., n$ and f are in $B(\mu, \alpha)$.

Lemma 1 (5). *(General Schwarz Lemma). Let the function f be regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m} \cdot |z|^m \quad (z \in U_R).$$

The equality can hold only if

$$f(z) = e^{i\theta} \cdot \frac{M}{R^m} \cdot z^m,$$

where θ is constant.

Theorem 1. *[4]. Let $f \in \mathcal{A}$ be in the class $B(\mu,\alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$. If $|f(z)| \leq M$ $(M \geq 1,\ z \in U)$ then the integral operator*

$$G(z) = \int_0^z \left(te^{f(t)}\right)^{\gamma} \mathrm{d}t \tag{9}$$

is in $K(\delta)$, where

$$\delta = 1 - |\gamma| \left[(2-\alpha)M^{\mu} + 1\right] \tag{10}$$

and $|\gamma| < \dfrac{1}{(2-\alpha)M^{\mu}+1}$, $\gamma \in \mathbb{C}$.

2. Main results

Theorem 2. *Let $f \in \mathcal{A}$ be in the class $B(\mu,\alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$. If $|f(z)| \leq M$ $(M \geq 1,\ z \in U)$ then the integral operator*

$$G_{\gamma}(z) = \int_0^z \left(te^{f(t)}\right)^{\frac{1}{\gamma}} \mathrm{d}t \tag{11}$$

is in $K(\delta)$, where

$$\delta = 1 - \frac{1}{|\gamma|} \left[(2-\alpha)M^{\mu} + 1\right] \tag{12}$$

and $\dfrac{1}{|\gamma|} < \dfrac{1}{(2-\alpha)M^{\mu}+1}$, $\gamma \in \mathbb{C} \setminus \{0\}$.

Proof. Let $f \in \mathcal{A}$ be in the class $B(\mu,\alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$. It follows from (11) that

$$G'_{\gamma}(z) = \left(ze^{f(z)}\right)^{\frac{1}{\gamma}}$$

and

$$G''_{\gamma}(z) = \frac{1}{\gamma}\left(ze^{f(z)}\right)^{\frac{1}{\gamma}-1}\left(e^{f(z)} + ze^{f(z)}f'(z)\right).$$

Then $\dfrac{G''_{\gamma}(z)}{G'_{\gamma}(z)} = \dfrac{1}{\gamma}\left(\dfrac{1}{z} + f'(z)\right)$ and, hence

$$\left|\frac{zG''_{\gamma}(z)}{G'_{\gamma}(z)}\right| = \frac{1}{|\gamma|}\left(\left|1 + zf'(z)\right|\right) \leq \frac{1}{|\gamma|}\left(1 + \left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu}\right| \cdot \left|\left(\frac{f(z)}{z}\right)^{\mu}\right| \cdot |z|\right). \tag{13}$$

Applying the General Schwarz lemma, we have $\left|\dfrac{f(z)}{z}\right| \leq M$, $(z \in U)$. Therefore, from (13), we obtain

$$\left|\frac{zG''_{\gamma}(z)}{G'_{\gamma}(z)}\right| \leq \frac{1}{|\gamma|}\left(1 + \left|f'(z)\left(\frac{z}{f(z)}\right)^{\mu}\right| \cdot M^{\mu}\right),\ z \in U. \tag{14}$$

From (4) and (14), we see that

$$\left|\frac{zG''_{\gamma}(z)}{G'_{\gamma}(z)}\right| \leq \frac{1}{|\gamma|}\left[(2-\alpha)M^{\mu}+1\right] = 1-\delta.$$

□

Letting $\mu = 0$ in Theorem 2, we have $B(0,\alpha) \equiv R(\alpha)$ and we obtain next corollary.

Corollary 1. *Let $f \in \mathcal{A}$ be in the class $R(\alpha)$, $0 \leq \alpha < 1$. Then the integral operator*

$$\int_0^z \left(te^{f(t)}\right)^{\frac{1}{\gamma}} \mathrm{d}t \in K(\delta),$$

where

$$\delta = 1 - \frac{1}{|\gamma|}(3-\alpha) \tag{15}$$

and $\dfrac{1}{|\gamma|} < \dfrac{1}{3-\alpha}$, $\gamma \in \mathbb{C}\setminus\{0\}$.

Letting $\mu = 1$ in Theorem 2, we have $B(1,\alpha) \equiv S^*(\alpha)$ and we obtain next corollary.

Corollary 2. *Let $f \in \mathcal{A}$ be in the class $S^*(\alpha)$, $0 \leq \alpha < 1$. If $|f(z)| \leq M$ $(M \geq 1,\ z \in U)$ then the integral operator*

$$\int_0^z \left(te^{f(t)}\right)^{\frac{1}{\gamma}} \mathrm{d}t \in K(\delta),$$

where

$$\delta = 1 - \frac{1}{|\gamma|}[(2-\alpha)M+1] \tag{16}$$

and $\dfrac{1}{|\gamma|} < \dfrac{1}{(2-\alpha)M+1}$, $\gamma \in \mathbb{C}\setminus\{0\}$.

Letting $\alpha = \delta = 0$ in Corollary 2, we have

Corollary 3. *Let $f \in \mathcal{A}$ be a starlike function in U. If $|f(z)| \leq M$ $(M \geq 1,\ z \in U)$ then the integral operator $\int_0^z \left(te^{f(t)}\right)^{\frac{1}{\gamma}} \mathrm{d}t$ is convex in U, where $\dfrac{1}{|\gamma|} = \dfrac{1}{2M+1}$, $\gamma \in \mathbb{C}\setminus\{0\}$.*

Theorem 3. *Let* $f_i(z) \in \mathcal{A}$ *be in the class* $B(\mu, \alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$ *for all* $i = 1, 2, ..., n$. *If* $|f_i(z)| \leq M_i$ $(M_i \geq 1,\ z \in U)$ *for all* $i = 1, 2, ..., n$, *then the integral operator*

$$G_{n,\gamma}(z) = \int_0^z \prod_{i=1}^{n} \left(te^{f_i(t)}\right)^{\gamma} \mathrm{d}t$$

is in $K(\delta)$, *where*

$$\delta = 1 - |\gamma| \left[n + (2-\alpha) \sum_{i=1}^{n} M_i^{\mu}\right] \tag{17}$$

and $|\gamma| < \dfrac{1}{n + (2-\alpha)\sum\limits_{i=1}^{n} M_i^{\mu}}$, $\gamma \in \mathbb{C}$.

Proof. Let $f_i \in \mathcal{A}$ be in the class $B(\mu, \alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$. It follows from (6) that

$$G_{n,\gamma}(z) = \int_0^z t^{n\gamma} e^{\gamma \sum\limits_{i=1}^{n} f_i(t)} \mathrm{d}t \quad \text{and} \quad G'_{n,\gamma}(z) = z^{n\gamma} e^{\gamma \sum\limits_{i=1}^{n} f_i(z)}.$$

Also

$$G''_{n,\gamma}(z) = \gamma \left(z^n e^{\sum\limits_{i=1}^{n} f_i(z)}\right)^{\gamma-1} \cdot z^{n-1} \cdot e^{\sum\limits_{i=1}^{n} f_i(z)} \left(n + z\sum_{i=1}^{n} f'_i(z)\right)$$

Then

$$\frac{G''_{n,\gamma}(z)}{G'_{n,\gamma}(z)} = \gamma \left(\frac{n}{z} + \sum_{i=1}^{n} f'_i(z)\right)$$

and, hence

$$\begin{aligned} \left|\frac{zG''_{n,\gamma}(z)}{G'_{n,\gamma}(z)}\right| &= |\gamma| \left|n + z\sum_{i=1}^{n} f'_i(z)\right| \leq |\gamma| \sum_{i=1}^{n} \left|1 + zf'_i(z)\right| \\ &\leq |\gamma| \sum_{i=1}^{n} \left[1 + \left|f'_i(z)\left(\frac{z}{f_i(z)}\right)^{\mu}\right| \cdot \left|\left(\frac{f_i(z)}{z}\right)^{\mu}\right| \cdot |z|\right] \end{aligned} \tag{18}$$

Applying the General Schwarz lemma, we have $\left|\dfrac{f_i(z)}{z}\right| \leq M_i$, for all $i = 1, 2, ..., n$. Therefore, from (18), we obtain

$$\left|\frac{zG''_{n,\gamma}(z)}{G'_{n,\gamma}(z)}\right| \leq |\gamma| \sum_{i=1}^{n} \left[1 + \left|f'_i(z)\left(\frac{z}{f_i(z)}\right)^{\mu}\right| \cdot M_i^{\mu}\right],\ (z \in U). \tag{19}$$

From (4) and (19), we see that

$$\left|\frac{zG''_{n,\gamma}(z)}{G'_{n,\gamma}(z)}\right| \leq |\gamma| \left[n + (2-\alpha)\sum_{i=1}^{n} M_i^{\mu}\right] = 1 - \delta.$$

This completes the proof. □

For $M_1 = M_2 = ... = M_n = M$ we have

Corollary 4. *Let $f_i(z) \in \mathcal{A}$ be in the class $B(\mu, \alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$ for all $i = 1, 2, ..., n$. If $|f_i(z)| \leq M$ ($M \geq 1$, $z \in U$) for all $i = 1, 2, ..., n$, then the integral operator*

$$G_{n,\gamma}(z) = \int_0^z \prod_{i=1}^{n} \left(te^{f_i(t)}\right)^{\gamma} \mathrm{d}t$$

is in $K(\delta)$, where

$$\delta = 1 - |\gamma| \left[n(1 + (2-\alpha)M^{\mu})\right] \tag{20}$$

and $|\gamma| < \dfrac{1}{n[1 + (2-\alpha)M^{\mu}]}$, $\gamma \in \mathbb{C}$.

Letting $\mu = 0$ in Corollary 4, we have

Corollary 5. *Let $f_i(z) \in \mathcal{A}$ be in the class $R(\alpha)$, $0 \leq \alpha < 1$ for all $i = 1, 2, ..., n$. Then the integral operator defined in (6) is in $K(\delta)$, where*

$$\delta = 1 - |\gamma| n(3-\alpha) \tag{21}$$

and $|\gamma| < \dfrac{1}{n(3-\alpha)}$, $\gamma \in \mathbb{C}$.

Letting $\mu = 1$ in Corollary 4, we have

Corollary 6. *Let $f_i \in \mathcal{A}$ be in the class $S^*(\alpha)$, $0 \leq \alpha < 1$ for all $i = 1, 2, ..., n$. If $|f_i(z)| \leq M$ ($M \geq 1$, $z \in U$) for all $i = 1, 2, ..., n$, then the integral operator defined in (6) is in $K(\delta)$, where*

$$\delta = 1 - |\gamma|[n(1 + (2-\alpha)M] \tag{22}$$

and $|\gamma| < \dfrac{1}{n[1 + (2-\alpha)M]}$, $\gamma \in \mathbb{C}$.

Letting $\alpha = \delta = 0$ in Corollary 6, we have

Corollary 7. *Let $f_i \in \mathcal{A}$ be starlike functions in U for all $i = 1, 2, ..., n$. If $|f_i(z)| \leq M$ $(M \geq 1,\ z \in U)$ for all $i = 1, 2, ..., n$ then the integral operator defined in (6) is convex in U, where $|\gamma| = \dfrac{1}{n(2M+1)}$, $\gamma \in \mathbb{C}$.*

Letting $n = 1$ in Corollary 4, we obtain Theorem 1 from paper [4].

Theorem 4. *Let $f_i(z) \in \mathcal{A}$ be in the class $B(\mu, \alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$ for all $i = 1, 2, ..., n$. If $|f_i(z)| \leq M_i$ $(M_i \geq 1,\ z \in U)$ for all $i = 1, 2, ..., n$, then the integral operator*

$$H_{n,\gamma}(z) = \int_0^z \prod_{i=1}^n \left(te^{f_i(t)}\right)^{\frac{1}{\gamma}} \mathrm{d}t$$

is in $K(\delta)$, where

$$\delta = 1 - \frac{1}{|\gamma|}\left[n + (2-\alpha)\sum_{i=1}^n M_i^\mu\right] \tag{23}$$

and $\dfrac{1}{|\gamma|} < \dfrac{1}{n + (2-\alpha)\sum\limits_{i=1}^n M_i^\mu}$, $\gamma \in \mathbb{C} \setminus \{0\}$.

Proof. Let $f_i \in \mathcal{A}$ be in the class $B(\mu, \alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$. We have from (7) that

$$H_{n,\gamma}(z) = \int_0^z t^{\frac{n}{\gamma}} e^{\frac{1}{\gamma}\sum\limits_{i=1}^n f_i(t)} \mathrm{d}t \quad \text{and} \quad H'_{n,\gamma}(z) = z^{\frac{n}{\gamma}} e^{\frac{1}{\gamma}\sum\limits_{i=1}^n f_i(z)}.$$

Also

$$H''_{n,\gamma}(z) = \frac{1}{\gamma}\left(z^n e^{\sum\limits_{i=1}^n f_i(z)}\right)^{\frac{1}{\gamma}-1} \cdot z^{n-1} \cdot e^{\sum\limits_{i=1}^n f_i(z)}\left(n + z\sum_{i=1}^n f'_i(z)\right)$$

Then

$$\frac{H''_{n,\gamma}(z)}{H'_{n,\gamma}(z)} = \frac{1}{\gamma}\left(\frac{n}{z} + \sum_{i=1}^n f'_i(z)\right)$$

and, hence

$$\begin{aligned} \left|\frac{zH''_{n,\gamma}(z)}{H'_{n,\gamma}(z)}\right| &= \frac{1}{|\gamma|}\left|n + z\sum_{i=1}^n f'_i(z)\right| \leq \frac{1}{|\gamma|}\left(\sum_{i=1}^n \left|1 + zf'_i(z)\right|\right) \\ &\leq \frac{1}{|\gamma|}\sum_{i=1}^n\left[1 + \left|f'_i(z)\left(\frac{z}{f_i(z)}\right)^\mu\right| \cdot \left|\left(\frac{f_i(z)}{z}\right)^\mu\right| \cdot |z|\right] \end{aligned} \tag{24}$$

Applying the General Schwarz lemma, we have $\left|\frac{f_i(z)}{z}\right| \leq M_i$, for all $i = 1, 2, ..., n$. Therefore, from (24), we obtain

$$\left|\frac{zH''_{n,\gamma}(z)}{H'_{n,\gamma}(z)}\right| \leq \frac{1}{|\gamma|}\sum_{i=1}^{n}\left[1 + \left|f'_i(z)\left(\frac{z}{f_i(z)}\right)^{\mu}\right| \cdot M_i^{\mu}\right], \ (z \in U). \tag{25}$$

From (4) and (25), we see that

$$\left|\frac{zH''_{n,\gamma}(z)}{H'_{n,\gamma}(z)}\right| \leq \frac{1}{|\gamma|}\left[n + (2-\alpha)\sum_{i=1}^{n} M_i^{\mu}\right] = 1 - \delta.$$

□

For $M_1 = M_2 = ... = M_n = M$ we have

Corollary 8. *Let $f_i(z) \in \mathcal{A}$ be in the class $B(\mu, \alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$ for all $i = 1, 2, ..., n$. If $|f_i(z)| \leq M$ $(M \geq 1,\ z \in U)$ for all $i = 1, 2, ..., n$, then the integral operator*

$$H_{n,\gamma}(z) = \int_0^z \prod_{i=1}^{n}\left(te^{f_i(t)}\right)^{\frac{1}{\gamma}} \mathrm{d}t$$

is in $K(\delta)$, where

$$\delta = 1 - \frac{n}{|\gamma|}[(2-\alpha)M^{\mu} + 1] \tag{26}$$

and $\frac{1}{|\gamma|} < \frac{1}{n[(2-\alpha)M^{\mu}+1]}$, $\gamma \in \mathbb{C} \setminus \{0\}$.

Letting $\mu = 0$ in Corollary 8, we have

Corollary 9. *Let $f_i(z) \in \mathcal{A}$ be in the class $R(\alpha)$, $0 \leq \alpha < 1$ for all $i = 1, 2, ..., n$. Then the integral operator defined in (7) is in $K(\delta)$, where*

$$\delta = 1 - \frac{n}{|\gamma|}(3-\alpha) \tag{27}$$

and $\frac{1}{|\gamma|} < \frac{1}{n(3-\alpha)}$, $\gamma \in \mathbb{C} \setminus \{0\}$.

Letting $\mu = 1$ in Corollary 8, we have

Corollary 10. *Let $f_i \in \mathcal{A}$ be in the class $S^*(\alpha)$, $0 \le \alpha < 1$ for all $i = 1, 2, ..., n$. If $|f_i(z)| \le M$ $(M \ge 1,\ z \in U)$ for all $i = 1, 2, ..., n$, then the integral operator defined in (7) is in $K(\delta)$, where*

$$\delta = 1 - \frac{n}{|\gamma|}[1 + (2 - \alpha)M] \tag{28}$$

and $\dfrac{1}{|\gamma|} < \dfrac{1}{n[1 + (2 - \alpha)M]}$, $\gamma \in \mathbb{C} \setminus \{0\}$.

Letting $\alpha = \delta = 0$ in Corollary 10, we have

Corollary 11. *Let $f_i(z) \in \mathcal{A}$ be starlike functions in U for all $i = 1, 2, ..., n$. If $|f_i(z)| \le M$ $(M \ge 1,\ z \in U)$ for all $i = 1, 2, ..., n$ then the integral operator defined in (7) is convex in U, where* $\dfrac{1}{|\gamma|} = \dfrac{1}{n(2M + 1)}$, $\gamma \in \mathbb{C} \setminus \{0\}$.

Letting $n = 1$ in Corollary 8, we obtain Theorem 2.

Theorem 5. *Let $f_i(z) \in \mathcal{A}$ be in the class $B(\mu, \alpha)$, $\mu \ge 0$, $0 \le \alpha < 1$ for all $i = 1, 2, ..., n$. If $|f_i(z)| \le M_i$ $(M_i \ge 1,\ z \in U)$ for all $i = 1, 2, ..., n$, then the integral operator*

$$H_n(z) = \int_0^z \prod_{i=1}^{n} \left(te^{f_i(t)}\right)^{\gamma_i} \mathrm{d}t$$

is in $K(\delta)$, where

$$\delta = 1 - \sum_{i=1}^{n} |\gamma_i| \cdot [1 + (2 - \alpha)M_i^{\mu}] \tag{29}$$

and $\sum\limits_{i=1}^{n} |\gamma_i| \cdot [1 + (2 - \alpha)M_i^{\mu}] < 1$, $\gamma_i \in \mathbb{C}$ for all $i = 1, 2, ...n.$.

Proof. Let $f_i \in \mathcal{A}$ be in the class $B(\mu, \alpha)$, $\mu \ge 0$, $0 \le \alpha < 1$. It follows from (8) that

$$H_n(z) = \int_0^z t^{\sum\limits_{i=1}^{n} \gamma_i} e^{\sum\limits_{i=1}^{n} \gamma_i f_i(t)} \mathrm{d}t \quad \text{and} \quad H_n'(z) = z^{\sum\limits_{i=1}^{n} \gamma_i} e^{\sum\limits_{i=1}^{n} \gamma_i f_i(z)}.$$

Also

$$H_n''(z) = z^{\sum\limits_{i=1}^{n} \gamma_i - 1} \cdot e^{\sum\limits_{i=1}^{n} \gamma_i f_i(z)} \left[\sum_{i=1}^{n} \gamma_i + z \sum_{i=1}^{n} \gamma_i f_i'(z) \right]$$

Then

$$\frac{H_n''(z)}{H_n'(z)} = \frac{\sum\limits_{i=1}^{n} \gamma_i + z \sum\limits_{i=1}^{n} \gamma_i f_i'(z)}{z}$$

and, hence

$$\begin{aligned}\left|\frac{zH_n''(z)}{H_n'(z)}\right| &= \left|\sum_{i=1}^{n}\gamma_i + z\sum_{i=1}^{n}\gamma_i f_i'(z)\right| \le \sum_{i=1}^{n}|\gamma_i| + |z|\sum_{i=1}^{n}|\gamma_i|\cdot|f_i'(z)| \\ &\le \sum_{i=1}^{n}|\gamma_i| + |z|\cdot\sum_{i=1}^{n}|\gamma_i|\cdot\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu}\right|\cdot\left|\left(\frac{f_i(z)}{z}\right)^{\mu}\right| \qquad (30)\end{aligned}$$

Applying the General Schwarz lemma, we have $\left|\frac{f_i(z)}{z}\right| \le M_i$, for all $i = 1, 2, ..., n$. Therefore, from (30), we obtain

$$\left|\frac{zH_n''(z)}{H_n'(z)}\right| \le \sum_{i=1}^{n}|\gamma_i| + \sum_{i=1}^{n}|\gamma_i|\cdot\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu}\right|\cdot M_i^{\mu},\ (z\in U). \qquad (31)$$

From (4) and (31), we see that

$$\left|\frac{zH_n''(z)}{H_n'(z)}\right| \le \sum_{i=1}^{n}|\gamma_i|\cdot[1+(2-\alpha)M_i^{\mu}] = 1-\delta.$$

This completes the proof. □

For $M_1 = M_2 = ... = M_n$ we have

Corollary 12. *Let $f_i(z)\in\mathcal{A}$ be in the class $B(\mu,\alpha)$, $\mu \ge 0$, $0 \le \alpha < 1$ for all $i = 1, 2, ..., n$. If $|f_i(z)| \le M$ $(M \ge 1,\ z \in U)$ for all $i = 1, 2, ..., n$, then the integral operator*

$$H_n(z) = \int_0^z \prod_{i=1}^{n}\left(te^{f_i(t)}\right)^{\gamma_i}\mathrm{d}t$$

is in $K(\delta)$, where

$$\delta = 1 - \sum_{i=1}^{n}|\gamma_i|\cdot[(2-\alpha)M^{\mu}+1] \qquad (32)$$

and $\sum_{i=1}^{n}|\gamma_i| < \frac{1}{(2-\alpha)M^{\mu}+1}$, $\gamma_i\in\mathbb{C}$ for all $i = 1, 2, ...n.$.

Letting $\mu = 0$ in Corollary 12, we have

Corollary 13. *Let $f_i(z)\in\mathcal{A}$ be in the class $R(\alpha)$, $0 \le \alpha < 1$ for all $i = 1, 2, ..., n$. Then the integral operator defined in (8) is in $K(\delta)$, where*

$$\delta = 1 - \sum_{i=1}^{n}|\gamma_i|(3-\alpha) \qquad (33)$$

and $\sum_{i=1}^{n}|\gamma_i| < \frac{1}{3-\alpha}$, $\gamma_i \in \mathbb{C}$ for all $i = 1, 2, ..., n$.

Letting $\mu = 1$ in Corollary 12, we have

Corollary 14. *Let $f_i \in \mathcal{A}$ be in the class $S^*(\alpha)$, $0 \le \alpha < 1$ for all $i = 1, 2, ..., n$. If $|f_i(z)| \le M$ $(M \ge 1,\ z \in U)$ for all $i = 1, 2, ..., n$, then the integral operator defined in (8) is in $K(\delta)$, where*

$$\delta = 1 - \sum_{i=1}^{n} |\gamma_i|[1 + (2-\alpha)M] \tag{34}$$

and $\sum_{i=1}^{n}|\gamma_i| < \frac{1}{1+(2-\alpha)M}$, $\gamma_i \in \mathbb{C}$ for all $i = 1, 2, ..., n$.

Letting $\alpha = \delta = 0$ in Corollary 14, we have

Corollary 15. *Let $f_i \in \mathcal{A}$ be starlike functions in U for all $i = 1, 2, ..., n$. If $|f(z)| \le M$ $(M \ge 1,\ z \in U)$ for all $i = 1, 2, ..., n$ then the integral operator defined in (8) is convex in U, where* $\sum_{i=1}^{n}|\gamma_i| = \frac{1}{2M+1}$, $\gamma_i \in \mathbb{C}$ for all $i = 1, 2, ..., n$.

Letting $n = 1$ in Corollary 12, we obtain Theorem 1.

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