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ON HARMONIC UNIFORMLY STARLIKE FUNCTIONS DEFINED BY AN INTEGRAL OPERATOR

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Abstract. Using the integral operator, we define and study a generalized family of complex-valued harmonic functions that are univalent, sense-preserving and are associated with uniformly harmonic functions in the unit disk. We obtain coefficient inequalities, extreme points and distortion bounds for the functions in our class. The results obtained for the our class reduce to the corresponding results for various well-known classes in the literature.

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1. Introduction

A continuous complex-valued function $f = u + iv$ defined in a complex domain D is said to be harmonic in D if both u and v are real harmonic in D. In any simply connected domain we can write $f = h + \bar{g}$, where h and g are analytic in D. A necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|, z \in D$.
Denote by $\mathcal{H}$ the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ with

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, g(z) = \sum_{k=1}^{\infty} b_k z^k.$$

The class $\mathcal{H}$ was defined and studied by Clunie and Sheil-Small in [1]. Let $H(U)$ be the space of holomorphic functions in U. We let:

$$A_n = \{f \in H(U), f(z) = z + a_{n+1}z^{n+1} + \ldots, z \in U\}, with \quad A_1 = A.$$

Let $H[a,n]$ be denote the class of analytic functions in U of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots, z \in U.$$

The integral operator I^n is defined in [2] by

$$(i) \quad I^0 f(z) = f(z);$$

$$(ii) \quad I^1 f(z) = If(z) = \int_0^z f(t) t^{-1} dt;$$

$$(iii) \quad I^n f(z) = I(I^{n-1} f(z)), n \in \mathbb{N} = \{1,2,3,...\} \textit{ and } f \in A.$$

Ahuja and Jahangiri [3] defined the class $H(n)$ $(n \in N)$ consisting of all harmonic univalent functions $f = h + \overline{g}$ that are sense-preserving in U and h and g are of the form

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \qquad g(z) = \sum_{k=1}^{\infty} b_k z^k, |b_1| < 1 \tag{1}$$

For $f = h + \overline{g}$ given by (1) the integral operator I^n of f is defined as

$$I^n f(z) = I^n h(z) + (-1^n)\overline{I^n g(z)}, \tag{2}$$

where

$$I^n h(z) = z + \sum_{k=2}^{\infty} k^{-n} a_k z^k \qquad \textit{and} \qquad I^n g(z) = \sum_{k=1}^{\infty} k^{-n} b_k z^k.$$

For $0 \leq \alpha < 1, n \in \mathbb{N}, z \in U, q \in \mathbb{N}, p \geq 0, \theta \in \mathbb{R}$, let $H_{p,q}(n,\alpha)$ the family of harmonic functions f of the form (1) such that

$$Re\{(1 + pe^{i\theta}) \frac{I^n f(z)}{I^{n+q} f(z)} - pe^{i\theta}\} \geq \alpha. \tag{3}$$

For particular cases of p and q, especially for $p = 0$ and $q = 1$, we can write

$$H_{0,1}(n,\alpha) = H(n,\alpha)$$

which was studied by Cotîrlă in [4].
Let denote the subclass $\overline{H}_{p,q}(n,\alpha)$ consists of harmonic functions $f_n = h + \overline{g_n}$ in $H_{p,q}(n,\alpha)$ so that h and g_n are of the form

$$h(z) = z - \sum_{k=2}^{\infty} |a_k| z^k, \; g_n(z) = (-1)^{n-1} \sum_{k=1}^{\infty} |b_k| z^k, \; |b_1| < 1. \tag{4}$$

In this paper, we investigate coefficient conditions, extreme points, distortion bounds and examine convexity properties for functions in the families $H_{p,q}(n,\alpha)$ and $\overline{H}_{p,q}(n,\alpha)$.

2. Main Results

We first prove sufficient coefficient conditions for harmonic functions in $H_{p,q}(n,\alpha)$.
Theorem 2.1 Let $f = h + \overline{g}$ be so that h and g are given by (1). If

$$\sum_{k=2}^{\infty} \frac{[k^{-n}(1+p) - k^{-(n+q)}(\alpha+p)]}{1-\alpha}|a_k| + \sum_{k=1}^{\infty} \frac{[k^{-n}(1+p) - (-1)^q k^{-(n+q)}(\alpha+p)]}{1-\alpha}|b_k| \leq 1 \tag{5}$$

then $f \in H_{p,q}(n,\alpha)$.
Proof. Suppose that (5) holds. Using the fact that $Rew \geq \alpha$ if and only if $|1-\alpha+w| \geq |1+\alpha-w|$, it suffices to show that

$$|(1-\alpha)I^{n+q}f(z) + I^n f(z)(1+pe^{i\theta}) - pe^{i\theta}I^{n+q}f(z)|$$

$$-|(1+\alpha)I^{n+q}f(z) - I^n f(z)(1+pe^{i\theta}) + pe^{i\theta}I^{n+q}f(z)| \geq 0. \tag{6}$$

Substituting for $I^n f(z)$ *and* $I^{n+q}f(z)$ in (6) yields,

$$= |(1-\alpha-pe^{i\theta})\{z + \sum_{k=2}^{\infty} k^{-(n+q)}a_k z^k + (-1)^{n+q}\sum_{k=1}^{\infty} k^{-(n+q)}\overline{b_k z^k}\}$$

$$+(1+pe^{i\theta})\{z + \sum_{k=2}^{\infty} k^{-n}a_k z^k + (-1)^n \sum_{k=1}^{\infty} k^{-n}\overline{b_k z^k}\}|$$

$$-|(1+\alpha+pe^{i\theta})\{z + \sum_{k=2}^{\infty} k^{-(n+q)}a_k z^k + (-1)^{n+q}\sum_{k=1}^{\infty} k^{-(n+q)}\overline{b_k z^k}\}$$

$$-(1+pe^{i\theta})\{z + \sum_{k=2}^{\infty} k^{-n}a_k z^k + (-1)^n \sum_{k=1}^{\infty} k^{-n}\overline{b_k z^k}\}|$$

$$= |(2-\alpha)z + \sum_{k=2}^{\infty} k^{-n}(1+pe^{i\theta} + k^{-q}(1-\alpha-pe^{i\theta}))a_k z^k$$

$$+(-1)^n \sum_{k=1}^{\infty} k^{-n}(1+pe^{i\theta} + k^{-q}(-1)^q(1-\alpha-pe^{i\theta}))\overline{b_k z^k}|$$

$$-|\alpha(z) + \sum_{k=2}^{\infty} k^{-n}(-1-pe^{i\theta} + k^{-q}(1+\alpha+pe^{i\theta}))a_k z^k$$

$$-(-1)^n \sum_{k=1}^{\infty} k^{-n}(1+pe^{i\theta} - k^{-q}(-1)^q(1+\alpha+pe^{i\theta}))\overline{b_k z^k}|$$

$$\geq 2(1-\alpha)|z| - 2\sum_{k=2}^{\infty} k^{-n}[(1+p) - k^{-q}(\alpha+p)]|a_k||z|^k$$

$$-2\sum_{k=1}^{\infty} k^{-n}[(1+p) - (-1)^q k^{-q}(\alpha+p)]|b_k||z|^k$$

$$= 2(1-\alpha)|z|\{1 - \sum_{k=2}^{\infty} \frac{k^{-n}[(1+p) - k^{-q}(\alpha+p)]}{1-\alpha}|a_k||z|^{k-1}$$

$$-\sum_{k=1}^{\infty} \frac{k^{-n}[(1+p) - (-1)^q k^{-q}(\alpha+p)]}{1-\alpha}|b_k||z|^{k-1}\}.$$

This last expression is non-negative by hypothesis, and so the proof is complete. The functions

$$f(z) = z + \sum_{k=2}^{\infty} \frac{(1-\alpha)}{k^{-n}(1+p) - k^{-(n+q)}(\alpha+p)} x_k z^k$$

$$+\sum_{k=1}^{\infty} \frac{(1-\alpha)}{k^{-n}(1+p) - (-1)^q k^{-(n+q)}(\alpha+p)} \overline{y_k z^k} \tag{7}$$

show that the coefficient bound given by (5) is sharp where $n \in \mathbb{N}$,$p \geq 0$,$q \in \mathbb{N}$ and

$$\sum_{k=2}^{\infty} |x_k| + \sum_{k=1}^{\infty} |y_k| = 1.$$

In the following theorem it is shown that the condition (5) is also necessary for functions $f_n = h + \overline{g_n}$ where h and g_n are of the form (4).

Theorem 2.2 Let $f_n = h + \overline{g_n}$ be given by (4). Then $f_n \in \overline{H}_{p,q}(n,\alpha)$. if and only

$$\sum_{k=2}^{\infty} \frac{k^{-n}(1+p) - k^{-(n+q)}(\alpha+p)}{1-\alpha}|a_k| + \sum_{k=1}^{\infty} \frac{k^{-n}(1+p) - (-1)^q k^{-(n+q)}(\alpha+p)}{1-\alpha}|b_k| \leq 1 \tag{8}$$

where $\alpha_1 = 1, 0 \leq \alpha < 1, n \in \mathbb{N}, p \geq 0, q \in \mathbb{N}, \theta \in \mathbb{R}$.

Proof. Since $\overline{H}_{p,q}(n,\alpha) \subset H_{p,q}(n,\alpha)$ we only need to prove the "only if" part of the theorem. For functions f_n of the form (4), we note that the condition

$$Re\{(1+pe^{i\theta})\frac{I^n f_n(z)}{I^{n+q} f_n(z)} - pe^{i\theta}\} \geq \alpha$$

is equivalent to

$$Re\{\frac{(1+pe^{i\theta})I^n f_n(z) - I^{n+q} f_n(z)(pe^{i\theta}+\alpha)}{I^{n+q} f_n(z)}\}$$

$$= Re\{\frac{(1+pe^{i\theta})[z-\sum_{k=2}^{\infty} k^{-n}|a_k|z^k + (-1)^{2n-1}\sum_{k=1}^{\infty} k^{-n}|b_k|\overline{z}^k]}{z-\sum_{k=2}^{\infty} k^{-(n+q)}|a_k|z^k + (-1)^{2n+q-1}\sum_{k=1}^{\infty} k^{-(n+q)}|b_k|\overline{z}^k}$$

$$-\frac{(pe^{i\theta}+\alpha)[z-\sum_{k=2}^{\infty} k^{-(n+q)}|a_k|z^k + (-1)^{2n+q-1}\sum_{k=1}^{\infty} k^{-(n+q)}|b_k|\overline{z}^k]}{z-\sum_{k=2}^{\infty} k^{-(n+q)}|a_k|z^k + (-1)^{2n+q-1}\sum_{k=1}^{\infty} k^{-(n+q)}|b_k|\overline{z}^k}\}$$

$$= Re\{\frac{(1-\alpha)z-\sum_{k=2}^{\infty}[k^{-n}(1+pe^{i\theta}) - k^{-(n+q)}(pe^{i\theta}+\alpha)]|a_k|z^k}{z-\sum_{k=2}^{\infty} k^{-(n+q)}|a_k|z^k + (-1)^{2n+q-1}\sum_{k=1}^{\infty} k^{-(n+q)}|b_k|\overline{z}^k}$$

$$+\frac{(-1)^{2n-1}\sum_{k=1}^{\infty}[k^{-n}(1+pe^{i\theta}) - k^{-(n+q)}(-1)^q(pe^{i\theta}+\alpha)]|b_k|\overline{z^k}}{z-\sum_{k=2}^{\infty} k^{-(n+q)}|a_k|z^k + (-1)^{2n+q-1}\sum_{k=1}^{\infty} k^{-(n+q)}|b_k|\overline{z}^k}\}$$

$$= Re\{\frac{(1-\alpha)-\sum_{k=2}^{\infty}[k^{-n}(1+pe^{i\theta}) - k^{-(n+q)}(pe^{i\theta}+\alpha)]|a_k|z^{k-1}}{1-\sum_{k=2}^{\infty} k^{-(n+q)}|a_k|z^{k-1} + \frac{\overline{z}}{z}(-1)^{2n+q-1}\sum_{k=1}^{\infty} k^{-(n+q)}|b_k|\overline{z}^{k-1}}$$

$$+\frac{\frac{\overline{z}}{z}(-1)^{2n-1}\sum_{k=1}^{\infty}[k^{-n}(1+pe^{i\theta}) - k^{-(n+q)}(-1)^q(pe^{i\theta}+\alpha)]|b_k|\overline{z}^{k-1}}{1-\sum_{k=2}^{\infty} k^{-(n+q)}|a_k|z^{k-1} + \frac{\overline{z}}{z}(-1)^{2n+q-1}\sum_{k=1}^{\infty} k^{-(n+q)}|b_k|\overline{z}^{k-1}}\} \qquad (9)$$

$$\geq 0.$$

Upon choosing the values of z on the positive real axis and using $Re(-e^{i\theta}) \geq -|e^{i\theta}| = -1$ where $0 \leq z = r < 1$, the above inequalities reduces to

$$\frac{(1-\alpha)-\sum_{k=2}^{\infty}[k^{-n}(1+p) - k^{-(n+q)}(p+\alpha)]|a_k|r^{k-1}}{1-\sum_{k=2}^{\infty} k^{-(n+q)}|a_k|r^{k-1} - (-1)^q\sum_{k=1}^{\infty} k^{-(n+q)}|b_k|r^{k-1}}$$

$$-\frac{\sum_{k=1}^{\infty}[k^{-n}(1+p) - k^{-(n+q)}(-1)^q(p+\alpha)]|b_k|r^{k-1}}{1-\sum_{k=2}^{\infty} k^{-(n+q)}|a_k|r^{k-1} - (-1)^q\sum_{k=1}^{\infty} k^{-(n+q)}|b_k|r^{k-1}} \geq 0. \qquad (10)$$

If the condition (8) does not hold, then the numerator (10) is negative for sufficiently close to 1. Thus there exist $z_0 = r_0$ in $(0,1)$ for which the quotient in (10) is negative. This contradicts the condition for $f_n \in \overline{H}_{p,q}(n,\alpha)$. So the proof is complete.

Next we determine the extreme points of closed convex hulls of $\overline{H}_{p,q}(n,\alpha)$.

Theorem 2.3 Let f_n be given by (4). Then $f_n \in \overline{H}_{p,q}(n,\alpha)$ if and only if

$$f_n(z) = \sum_{k=1}^{\infty}[X_k h_k(z) + Y_k g_{n_k}(z)]$$

where

$$h(z) = z, \qquad h_k(z) = z - \frac{1-\alpha}{k^{-n}(1+p) - k^{-(n+q)}(\alpha+p)} z^k \qquad (k = 2, 3, ...,)$$

$$g_{n_k}(z) = z + (-1)^{n-1} \frac{1-\alpha}{k^{-n}(1+p) - (-1)^q k^{-(n+q)}(\alpha+p)} \overline{z}^k \qquad (k = 1, 2, 3, ...,)$$

$$X_k \geq 0, Y_k \geq 0, \sum_{k=1}^{\infty}(X_k + Y_k) = 1.$$

In particular, the extreme points of $\overline{H}_{p,q}(n,\alpha)$ are $\{h_k\}$ and $\{g_{n_k}\}$.
Proof. For functions f_n of the form (5), we have

$$f_n(z) = \sum_{k=2}^{\infty}[X_k h_k(z) + Y_k g_{n_k}(z)] = \sum_{k=1}^{\infty}[X_k + Y_k]z - \sum_{k=2}^{\infty} \frac{1-\alpha}{k^{-n}(1+p) - k^{-(n+q)}(\alpha+p)} X_k z^k$$

$$+(-1)^{n-1} \sum_{k=1}^{\infty} \frac{1-\alpha}{k^{-n}(1+p) - (-1)^q k^{-(n+q)}(\alpha+p)} Y_k \overline{z}^k$$

Then

$$\sum_{k=2}^{\infty} \frac{[k^{-n}(1+p) - k^{-(n+q)}(\alpha+p)]}{1-\alpha} \frac{1-\alpha}{[k^{-n}(1+p) - k^{-(n+q)}(\alpha+p)]} X_k$$

$$+\sum_{k=1}^{\infty} \frac{[k^{-n}(1+p) - (-1)^q k^{-(n+q)}(\alpha+p)]}{1-\alpha} \frac{1-\alpha}{[k^{-n}(1+p) - (-1)^q k^{-(n+q)}(\alpha+p)]} Y_k$$

$$= \sum_{k=2}^{\infty} X_k + \sum_{k=1}^{\infty} Y_k = 1 - X_1 \leq 1$$

and so

$f_n(z) \in \overline{H}_{p,q}(n,\alpha)$.

Conversely, suppose $f_n(z) \in \overline{H}_{p,q}(n,\alpha)$. Letting
$X_1 = 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k$,

$$X_k = \frac{k^{-n}(1+p) - k^{-(n+q)}(\alpha+p)}{1-\alpha}|a_k| \qquad (k = 2, 3, ...,)$$

and

$$Y_k = \frac{k^{-n}(1+p) - (-1)^q k^{-(n+q)}(\alpha+p)}{1-\alpha}|b_k| \qquad (k = 1, 2, 3, ...,)$$

we obtain the required representation, since

$$f_n(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + (-1)^{n-1} \sum_{k=1}^{\infty} |b_k| \overline{z}^k$$

$$= z - \sum_{k=2}^{\infty} \frac{1-\alpha}{k^{-n}(1+p) - k^{-(n+q)}(\alpha+p)} X_k z^k$$

$$+(-1)^{n-1} \sum_{k=1}^{\infty} \frac{1-\alpha}{k^{-n}(1+p) - (-1)^q k^{-(n+q)}(\alpha+p)} Y_k \overline{z}^k$$

$$= z - \sum_{k=2}^{\infty} [z - h_k(z)] X_k - \sum_{k=1}^{\infty} [z - g_{n_k}(z)] Y_k$$

$$= [1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k] z + \sum_{k=2}^{\infty} X_k h_k(z) + \sum_{k=1}^{\infty} Y_k g_{n_k}(z)$$

$$= \sum_{k=1}^{\infty} [X_k h_k(z) + Y_k g_{n_k}(z)].$$

The following theorem gives the distortion bounds for functions in $\overline{H}_{p,q}(n,\alpha)$ which yields a covering results for this class.

Theorem 2.4 Let $f_n \in \overline{H}_{p,q}(n,\alpha)$. Then for $|z| = r < 1$ we have

$$|f_n(z)| \leq (1+|b_1|)r + [\frac{1-\alpha}{2^{-n}(1+p) - 2^{-(n+q)}(\alpha+p)} - \frac{(1+p) - (-1)^q(\alpha+p)}{2^{-n}(1+p) - 2^{-(n+q)}(\alpha+p)} |b_1|] r^2$$

and

$$|f_n(z)| \geq (1-|b_1|)r - [\frac{1-\alpha}{2^{-n}(1+p) - 2^{-(n+q)}(\alpha+p)} - \frac{(1+p) - (-1)^q(\alpha+p)}{2^{-n}(1+p) - 2^{-(n+q)}(\alpha+p)} |b_1|] r^2$$

where q is an odd positive integer.

Proof We only prove the right hand inequality. The proof for the left hand inequality is smilar and will be omitted. Let $f_n \in \overline{H}_{p,q}(n,\alpha)$. Taking the absolute value of f_n, we obtain

$$|f_n(z)| \leq (1+|b_1|)r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^k$$

$$\leq (1+|b_1|)r + \sum_{k=2}^{\infty} (|a_k| + |b_k|) r^2$$

$$\leq (1+|b_1|)r+\frac{1-\alpha}{2^{-n}(1+p)-2^{-(n+q)}(\alpha+p)}\sum_{k=2}^{\infty}\frac{2^{-n}(1+p)-2^{-(n+q)}(\alpha+p)}{1-\alpha}(|a_k|+|b_k|)r^2$$

$$\leq (1+|b_1|)r+\frac{1-\alpha}{2^{-n}(1+p)-2^{-(n+q)}(\alpha+p)}\sum_{k=2}^{\infty}[\frac{k^{-n}(1+p)-k^{-(n+q)}(\alpha+p)}{1-\alpha}|a_k|$$

$$+\frac{k^{-n}(1+p)-(-1)^q k^{-(n+q)}(\alpha+p)}{1-\alpha}|b_k|]r^2$$

$$\leq (1+|b_1|)r+\frac{1-\alpha}{2^{-n}(1+p)-2^{-(n+q)}(\alpha+p)}[1-\frac{(1+p)-(-1)^q(\alpha+p)}{1-\alpha}|b_1|]r^2$$

$$\leq (1+|b_1|)r+[\frac{1-\alpha}{2^{-n}(1+p)-2^{-(n+q)}(\alpha+p)}-\frac{(1+p)-(-1)^q(\alpha+p)}{2^{-n}(1+p)-2^{-(n+q)}(\alpha+p)}|b_1|]r^2.$$

The following covering result follows from the left hand inequality in Theorem 2.4.

Corollary 2.5 Let f_n of the form (4) be so that $f_n \in \overline{H}_{p,q}(n,\alpha)$. Then

$$\{w:|w|<\frac{2^{-n}[(1+p)-2^{-q}(\alpha+p)]-(1-\alpha)}{2^{-n}[(1+p)-2^{-q}(\alpha+p)]}$$

$$-\frac{2^{-n}[(1+p)-2^{-q}(\alpha+p)]-(1+p)+(-1)^{-q}(\alpha+p)}{2^{-n}[(1+p)-2^{-q}(\alpha+p)]}|b_1|\subset f_n(U)\}.$$

Now we show that $\overline{H}_{p,q}(n,\alpha)$ is closed under convex combination of its members.

Theorem 2.6 The family $\overline{H}_{p,q}(n,\alpha)$ is closed under convex combination.

Proof For $i=1,2,...$, suppose that $f_n^i \in \overline{H}_{p,q}(n,\alpha)$, where

$$f_n^i(z)=z-\sum_{k=2}^{\infty}|a_k^i|z^k+(-1)^{n-1}\sum_{k=1}^{\infty}|b_k^i|\overline{z}^k$$

then by Theorem 2.2,

$$\sum_{k=1}^{\infty}\frac{k^{-n}(1+p)-k^{-(n+q)}(\alpha+p)}{1-\alpha}|a_k^i|+\sum_{k=1}^{\infty}\frac{k^{-n}(1+p)-(-1)^q k^{-(n+q)}(\alpha+p)}{1-\alpha}|b_k^i|\leq 2 \tag{11}$$

for $\sum_{i=1}^{\infty}t_i=1, 0\leq t_i\leq 1$, the convex combination of f_n^i may be written as

$$\sum_{i=1}^{\infty}t_i f_n^i(z)=z-\sum_{k=2}^{\infty}(\sum_{i=1}^{\infty}t_i|a_k^i|)z^k+(-1)^{n-1}\sum_{k=1}^{\infty}(\sum_{i=1}^{\infty}t_i|b_k^i|)\overline{z}^k.$$

Then by (11)

$$\sum_{k=1}^{\infty}\frac{k^{-n}(1+p)-k^{-(n+q)}(\alpha+p)}{1-\alpha}(\sum_{i=1}^{\infty}t_i|a_k^i|)$$

$$+\sum_{k=1}^{\infty}\frac{k^{-n}(1+p)-(-1)^q k^{-(n+q)}(\alpha+p)}{1-\alpha}(\sum_{i=1}^{\infty}t_i|b_k^i|)$$

$$=\sum_{i=1}^{\infty}t_i[\sum_{k=1}^{\infty}\frac{k^{-n}(1+p)-k^{-(n+q)}(\alpha+p)}{1-\alpha}|a_k^i|$$

$$+\sum_{k=1}^{\infty}\frac{k^{-n}(1+p)-(-1)^q k^{-(n+q)}(\alpha+p)}{1-\alpha}|b_k^i|]\leq 2\sum_{i=1}^{\infty}t_i=2$$

and therefore $\sum_{i=1}^{\infty}t_i f_n^i(z)\in\overline{H}_{p,q}(n,\alpha)$.

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