ON AN ALGORITHM FOR A DOUBLE-RESOLVABILITY TEST

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Abstract. To find all doubly resolvable designs with definite parameters, we construct the design resolutions in lexicographic order point by point. To partial solutions with more than 2/3 of the points we apply a double-resolvability test, namely we try to construct an orthogonal resolution. The test reduces significantly the number of partial solutions, and is very important for the efficiency of the whole computation. We develop and compare two algorithms for a double-resolvability test. The first one implies construction of the orthogonal resolution block by block (BB), while the second one does it class by class (CC). We present experimental results on the performance of both algorithms. Applying them, we also classify for the first time doubly resolvable 2-(16,4,2) and 2-(8,4,18) designs.

1.Introduction

For the basic concepts and notations concerning combinatorial designs and their resolvability refer, for instance, to [1], [2], [4], [7], [13].

A 2-(v, k, λ) design is a collection of k-element subsets (*blocks*) of a set of v elements (*points*), such that each pair of points is contained in exactly λ blocks.

Let b denote the number of the blocks of the design, and r − the number of blocks in which a given point is contained. An incidence matrix of the design is a matrix of v rows and b columns which contains a 1 in the ith row and jth column iff the ith point is contained in the jth block, and 0 if not. The design is completely determined by its incidence matrix.

Two designs are *isomorphic* if there exists a one-to-one correspondence between the point and block sets of the first design and the point and block sets of the second design, and if this one-to-one correspondence does not change the incidence, i.e. if the incidence matrix of the first design can be obtained from the incidence matrix of the second one by permuting rows and columns.

An *automorphism* is an isomorphism of the design to itself, i.e. a permutation of the points that transforms the blocks into blocks.

One of the most important properties of a design is its resolvability. The design is *resolvable* if it has at least one resolution.

A *resolution* is a partition of the blocks into *parallel classes* such that each point is in exactly one block of each parallel class. A parallel class contains v/k blocks and a resolution $\mathcal{R}$ consists of $r = (b * k/v)$ parallel classes $\mathcal{R}_1, ..., \mathcal{R}_r$.

Two resolutions are isomorphic if there exists an automorphism of the design transforming each parallel class of the first resolution into a parallel class of the second one.

Two resolutions are *orthogonal* if any two parallel classes, one from the first, and the other from the second resolution, have at most one common block. If a design has at least two orthogonal resolutions, it is *doubly resolvable.* The resolutions, which are orthogonal to the resolution $\mathcal{R}$, are called its *orthogonal partners* or *partner resolutions.*

A Kirkman square with index λ, latinicity μ, block size k, and v points, $KS_k(v;\mu,\lambda)$ is a $t \times t$ array ($t = \lambda(v-1)/\mu(k-1)$) defined on a set V such that: every point of V is contained in precisely μ cells of each row and column; each cell of the array is either empty or contains a k-subset of V; the collection of blocks obtained from the non-empty cells of the array is a 2-(v,k,λ) design. For μ=1, the existence of a $KS_k(v;\mu,\lambda)$ is equivalent to the existence of a doubly resolvable 2-(v,k,λ) design. In this case the size of the square array t is equivalent to the number of parallel classes of the doubly resolvable design and any two orthogonal resolutions determine a Kirkman square and vice versa.

The existence question for $KS_k(v;\mu,\lambda)$ has been completely settled for $k = 2$ and $\mu = 1$ [9]. The existence of $KS_3(v;1,2)$ for all $v \equiv 3(mod)12$ is proved in [10]. There are some particular results for $k \geq 3$, $\mu = 1$ in [3], [5], [6], [8], [11], [12], [14].

Our final aim is to construct and classify up to isomorphism all doubly resolvable designs with definite parameters. For this purpose we construct the design resolutions in lexicographic order point by point [11], [12], [14]. If some of the points of the design are missing, we can find resolutions and mutually orthogonal resolutions of the corresponding structure. We shall call this structure partial solution.

After adding each point we apply a test for equivalence of the partial solution to a previously generated one, and a double-resolvability test. The double-resolvability test is actually an attempt to construct one orthogonal to the current one resolution. As this is often not possible, the test reduces significantly the number of partial solutions, and thus makes it possible to classify doubly resolvable designs with parameters for which the classification of all non isomorphic resolutions is a difficult task, and has not been done yet. If the design is not doubly resolvable, and we take less than 2/3 of its points, we usually obtain a doubly resolvable partial solution. So we apply the test to partial solutions with more than 2/3 of the points. The efficiency of the double-resolvability test is very important for the efficiency of the whole task.

In the present work we describe and compare two algorithms - one, which constructs the orthogonal resolution block by block (BB) and one which does it class by class (CC). We present experimental data on their performance.

Using these algorithms, we classify 2-(16,4,2) and 2-(8,4,18) doubly resolvable designs.

2.Description of the BB and CC algorithms

Both algorithms realize a backtrack search for a resolution of the design, which is orthogonal to the current one. The search stops if one such resolution is constructed, or if all possibilities have been tested and no orthogonal resolution can be constructed. The search is less efficient on partial solutions, because some of the blocks contain less than k points, and some may contain no points at all, and thus there are more ways to combine the blocks in orthogonal resolution classes.

2.1.The Block by Block construction (BB)

We sort the blocks of the design in lexicographic order. The first point is in the first r blocks. Thus without loss of generality we assume that the ith block is in the ith parallel class of the orthogonal resolution for $i = 1, 2, ..., r$.

Next we start adding the missing blocks to the first class of the orthogonal resolution, then to the second,..., and finally to the rth one.

Since an orthogonal parallel class should contain all points, at each step we try to add only blocks containing the first missing point in the class, and we check that the blocks in each orthogonal class are disjoint and from different classes of the initial resolution [11].

A partial solution might have empty blocks, so we stop adding blocks to an orthogonal parallel class when all points of the partial solution are already in it.

2.2.The Class by Class construction (CC)

The initial resolution of the design has $n = b/r$ blocks in each of the r parallel classes. At first we find all possibilities for an orthogonal class by choosing disjoint blocks from different parallel classes of the initial resolution until all points are covered. An orthogonal class has a block from n of the initial parallel classes and no block from the other $r - n$, so we obtain it in the following format: $(a_1, a_2, ..., a_r)$, where $a_i \in \{1, 2, 3, \ldots, n\}$ is the number of a block within the ith parallel class $(i = 1, 2, ..., r)$ of the initial resolution, or $a_i = 0$ if the orthogonal class has no block of the ith parallel class of the initial resolution. Equality of two elements a_i and a_j, $i \neq j$ is possible, because blocks can be in the same position in different parallel classes.

When we apply this on partial solutions, there might be empty blocks, so instead of the number of an empty block within a parallel class, we also write 0. Thus there might be more than $r - n$ zeros - at most $r - n +$ *the number of the empty blocks*. The zero already has two meanings here: no block from this parallel class, or an empty block in this class. If all points are covered, but there are more than $r - n$ zero entries, all of them from classes without empty blocks, this orthogonal resolution class possibility is invalid and should be rejected.

After finding all possibilities for an orthogonal to the initial resolution parallel class, we try to choose r of them to form the orthogonal resolution, or respectively the corresponding Kirkman square.

To restrict the search we sort the blocks of the design with respect to the number of the possible parallel classes, in which they appear. Then we construct the orthogonal resolution class by class, adding at each step classes with the still not covered block, which has the smallest frequency of appearance.

3. Comparison of the BB and CC algorithms

We experimented both algorithms in the classification of doubly resolvable designs with definite parameters. The results are presented in the next table. The first column shows the design parameters, the second the number of parallel classes in the resolution, and the third the number of blocks. The fourth column presents the classification results, i.e. the number of non isomorphic resolutions, which have orthogonal partners, followed by $*$ if the result is published here for the first time. The last two columns show the time, which is needed to complete the classification using the BB and, respectively CC algorithm on a 2.6 GHz CPU.

Table 1: Classification results and the computation time by both algorithms

design	r	b	resolutions with orthogonal partners	new	CC	BB
2-(16,4,2)	10	40	1	*	6h	10h
2-(9,3,3)	12	36	5		2s	6s
2-(8,4,6)	14	28	1		1s	1s
2-(9,3,4)	16	48	83		9h43min	5h40min
2-(6,3,8)	20	40	1		1s	1s
2-(8,4,9)	21	42	1		1s	1s
2-(12,6,10)	22	44	1		30s	30s
2-(8,4,12)	28	56	4		1s	1s
2-(8,4,15)	35	70	4		11s	7s
2-(10,5,16)	36	72	5		12s	11s
2-(8,4,18)	42	84	13	*	1h29min	1h10min

The initial construction of all possibilities for an orthogonal resolution class takes some time before CC actually starts constructing the orthogonal resolution, so if we only check for the existence of one orthogonal resolution, the results above show that CC works faster for relatively small numbers r of the parallel classes, i.e. small size of the corresponding Kirkman square.

Both algorithms can also be used to construct all orthogonal partners of a resolution. In this case CC is much faster.

4.New classification results

Using these algorithms, we classify up to isomorphism 2-(16,4,2) and 2-(8,4,18) doubly resolvable designs. The results are presented in Table 2, where the number of all non isomorphic doubly resolvable designs with these parameters is given in the last column.

Table 2: New results

design	r	b	resolutions with orth. partners	doubly resolvable designs
2-(16,4,2)	10	40	1	1
2-(8,4,18)	42	84	13	13

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