An. Şt. Univ. Ovidius Constanţa Vol. 9(1), 2001, 39–46

CLASSES OF MODULES RELATED TO SERRE SUBCATEGORIES

Septimiu Crivei and Iuliu Crivei

Abstract

Let R be an associative ring with non-zero identity. For a Serre subcategory $\mathcal{C}$ of the category R-mod of left R-modules, we consider the class $\mathcal{A}_{\mathcal{C}}$ of all modules that do not belong to $\mathcal{C}$, but all of their proper submodules belong to $\mathcal{C}$. Alongside of basic properties of such associated classes of modules, we will prove that every uniform module of $\mathcal{A}_{\mathcal{C}}$ has a local endomorphism ring. Moreover, if R is a commutative ring, then every torsionfree faithful R-module of $\mathcal{A}_{\mathcal{C}}$ is isomorphic to the injective hull of R and its endomorphism ring is a division ring.

1 Introduction

In recent years certain classes of modules have been studied due to their importance for ring theory or theory of categories. We mention here hereditary torsion or pretorsion classes (e.g. [6]), Serre subcategories of R-mod (e.g. [4]), natural or prenatural classes (e.g. [1], [9]), open classes (e.g. [3]). All of them are formally defined as classes of modules closed under at least two of the following: submodules, direct sums, direct products, homomorphic images, isomorphic copies, extensions and injective hulls. For instance:

(1) hereditary pretorsion classes are closed under submodules, direct sums and homomorphic images;

(2) hereditary torsion classes are closed under submodules, direct sums, homomorphic images and extensions;

(3) torsionfree classes are closed under submodules, direct products, extensions and injective hulls;

(4) Serre subcategories are closed under submodules, homomorphic images and extensions;

Received: October, 2001.

(5) prenatural classes are closed under submodules, direct sums and homomorphic images;

(6) natural classes are closed under submodules, direct sums, homomorphic images and extensions;

(7) open classes are closed under submodules and homomorphic images.

For a class $\mathcal{A}$ of modules, it might happen the following phenomenon: none of the modules of $\mathcal{A}$ belongs to one of the previously mentioned classes, but all the proper submodules of modules of $\mathcal{A}$ belong to that specific class.

Throughout the present paper we will refer to a class of modules related to Serre subcategories of R-mod, whose introduction is motivated by the situation described above. Thus, for a Serre subcategory $\mathcal{C}$, we will consider the class $\mathcal{A}_{\mathcal{C}}$ of all modules that are not in $\mathcal{C}$, but all of their proper submodules are in $\mathcal{C}$.

For instance, let $\mathcal{C}$ be the class of all noetherian modules, which is obviously a Serre subcategory. Then a module $A \in \mathcal{A}_{\mathcal{C}}$ if and only if A is not finitely generated, but every proper submodule of A is finitely generated. These modules are called a.f.g. (almost finitely generated) and were studied by W.D. Weakley [7]. The injective hull of a discrete valuation ring is always an a.f.g. module [7, p.190].

Now let us mention the notation and some preliminary definitions. Throughout the paper R will denote an associative ring with non-zero identity and all modules will be left unital R-modules.

For a module M we will denote by $Ann_R M$ its annihilator in R, by $End_R(A)$ its endomorphism ring and by $E(M)$ its injective hull.

A submodule N of a module M is called essential if for every non-zero submodule L of M we have $N \cap L \neq 0$ [8, p.137]. A submodule N of a module M is called superfluous if for every submodule L of M, $N + L = M$ implies $L = M$ [8, p.159].

A non-zero module M is said to be hollow if every proper submodule is superfluous in M [8, p.351]. If M has a proper submodule which contains all the other proper submodules, then M is called a local module [8, p.351].

For the torsion-theoretical notions and properties we will sometimes use, the reader is referred to [2].

2 Basic properties

Recall that a non-empty class of modules is called a *Serre subcategory* of the category R-mod of left R-modules if it is closed under submodules, homomorphic images and extensions.

Immediate examples of Serre subcategories are: every hereditary torsion class, every cohereditary torsionfree class [2, p.44], every stable cotorsionfree class associated to a hereditary torsion class [2, Propositions 7.2 and 7.5], the class of all noetherian (artinian) modules.

In what follows $\mathcal{C}$ will denote a Serre subcategory of R-mod.

Now we consider the class that will be studied throughout the present paper. Thus, we will denote by $\mathcal{A}_{\mathcal{C}}$ the class consisting of all modules A with the properties that $A \notin \mathcal{C}$ and $B \in \mathcal{C}$ for every proper submodule B of A.

We collect some first properties of the class $\mathcal{A}_{\mathcal{C}}$ in the following proposition.

Proposition 2.1 *(i) $\mathcal{A}_{\mathcal{C}}$ is closed under non-zero homomorphic images.*

(ii) $\mathcal{A}_{\mathcal{C}}$ is contained in the class of hollow modules.

(iii) Every module of $\mathcal{A}_{\mathcal{C}}$ is indecomposable.

(iv) Every simple module belongs either to $\mathcal{C}$ or to $\mathcal{A}_{\mathcal{C}}$.

(v) Let $A \in \mathcal{A}_{\mathcal{C}}$. Then for every $B \in \mathcal{C}$, $Hom_R(A, B) = 0$.

(vi) If $\mathcal{C}$ is the torsion class for a hereditary torsion theory τ, then either $\mathcal{A}_{\mathcal{C}} = \emptyset$ or every module in $\mathcal{A}_{\mathcal{C}}$ is cyclic.

Proof. (i) Let B be a non-zero proper submodule of A. Then $A/B \notin \mathcal{C}$, because $\mathcal{C}$ is closed under submodules and extensions. Moreover, A/B has every proper submodule in $\mathcal{C}$, because $\mathcal{C}$ is closed under submodules and homomorphic images.

(ii) Let B be a proper submodule of A. Then $A/B \in \mathcal{A}_{\mathcal{C}}$ by (i). Suppose that there exists a proper submodule C of A such that $B + C = A$. Then we have the isomorphism $A/B = (B + C)/B \cong C/(B \cap C)$. But $C/(B \cap C) \in \mathcal{C}$, since $C \in \mathcal{C}$. Hence $A/B \in \mathcal{C}$, which is a contradiction. Therefore every proper submodule B of A is superfluous, i.e. A is hollow.

(iii) It follows by (ii), since every hollow module is indecomposable [8, p.352].

(iv) Clear.

(v) Let $B \in \mathcal{C}$. Suppose that $Hom_R(A, B) \neq 0$ and take a non-zero $f \in Hom_R(A, B)$. Then $Im\, f \in \mathcal{C}$ and $Ker\, f \neq 0$. Now it follows by (i) that $Im\, f \in \mathcal{A}_{\mathcal{C}}$, which is a contradiction.

(vi) Suppose that $\mathcal{A}_{\mathcal{C}} \neq \emptyset$, say $A \in \mathcal{A}_{\mathcal{C}}$. Then A is not τ-torsion and its torsion submodule $T_\tau(A)$ is the unique maximal submodule of A. Hence A is local, so that A is cyclic [8, p.352]. □

Corollary 2.2 *Let $\mathcal{C}$ be a hereditary torsion class. Then $\mathcal{A}_{\mathcal{C}}$ is contained in the corresponding cotorsionfree class, that is the class of all modules A such that $Hom_R(A, B) = 0$ for every $B \in \mathcal{C}$.*

Remarks. (*i*) In general, a class $\mathcal{A}_{\mathcal{C}}$ does not coincide either with the class of hollow modules or with some cotorsionfree class of modules.

(*ii*) There exist Serre subcategories $\mathcal{C}$ such that $\mathcal{A}_{\mathcal{C}} = \emptyset$, others then the trivial case when $\mathcal{C}$ is R-mod.

Example 2.3 (*i*) Let $\mathcal{C}$ be any Serre subcategory and take R to be a local ring, that is clearly hollow, with the maximal ideal M. Suppose that $R \in \mathcal{A}_{\mathcal{C}}$. Then $M \in \mathcal{C}$ and $R/M \in \mathcal{A}_{\mathcal{C}}$ by Proposition 2.1, hence $R/M \notin \mathcal{C}$. Now let a be a non-zero element of M. Then $Ra \in \mathcal{C}$ has a maximal ideal N. It follows that $R/M \cong Ra/N \in \mathcal{C}$, which is a contradiction. Therefore, R is hollow, but $R \notin \mathcal{A}_{\mathcal{C}}$ for any Serre subcategory $\mathcal{C}$.

On the other hand, every cotorsionfree class corresponding to a hereditary torsion class is closed under homomorphic images [2, Proposition 7.2], which is not the case for a class $\mathcal{A}_{\mathcal{C}}$ (see Proposition 2.1). Therefore, $\mathcal{A}_{\mathcal{C}}$ does not coincide with any cotorsionfree class of modules.

(*ii*) Let $\mathcal{C}$ be the torsion class for the Dickson torsion theory, that is generated by all simple modules. Assume that $\mathcal{A}_{\mathcal{C}} \neq \emptyset$. Then by Proposition 2.1, every module in $\mathcal{A}_{\mathcal{C}}$ is cyclic, hence $\mathcal{A}_{\mathcal{C}}$ contains a simple module, which is a contradiction. Therefore $\mathcal{A}_{\mathcal{C}} = \emptyset$.

Now let us consider the extreme case when $\mathcal{C}$ is the least possible class.

Theorem 2.4 $\mathcal{C} = \{0\}$ *if and only if* $\mathcal{A}_{\mathcal{C}}$ *is the class of all simple modules.*

Proof. Suppose first that $\mathcal{C} = \{0\}$. Then every simple module is in $\mathcal{A}_{\mathcal{C}}$. On the other hand, if $A \in \mathcal{A}_{\mathcal{C}}$ and B is a proper submodule of A, then $B \in \mathcal{C}$, i.e. $B = 0$. Hence A is simple.

Conversely, suppose that $\mathcal{A}_{\mathcal{C}}$ is the class of all simple modules. Assume now $\mathcal{C} \neq \{0\}$, say $0 \neq A \in \mathcal{C}$. Let a be a non-zero element of A. Then there exists a left ideal I of R such that $R/I \cong Ra \in \mathcal{C}$. Since I is included in a maximal left ideal M of R, $R/M \in \mathcal{C}$. But this is a contradiction, because R/M is simple. Therefore $\mathcal{C} = \{0\}$. □

Corollary 2.5 *If* $\mathcal{A}_{\mathcal{C}}$ *contains a non-simple module, then* $\mathcal{C}$ *contains a simple module.*

Theorem 2.6 *Let*

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$$

be a short exact sequence of modules such that $A \in \mathcal{C}$, $f(A)$ *is superfluous in* B *and* $C \in \mathcal{A}_{\mathcal{C}}$. *Then* $B \in \mathcal{A}_{\mathcal{C}}$.

Proof. We may assume that A is a submodule of B.

First, if $B \in \mathcal{C}$, then $C = g(B) \in \mathcal{C}$, contradiction. Hence $B \notin \mathcal{C}$.

Now let D be a proper submodule of B. Then $D+A$ is a proper submodule of B, because A is superfluous in B. Hence $D/(D \cap A) \cong (D+A)/A$ is a proper submodule of $B/A \cong C \in \mathcal{A}_{\mathcal{C}}$. Therefore $D/(D \cap A) \in \mathcal{C}$. But $D \cap A \in \mathcal{C}$, hence $D \in \mathcal{C}$. It follows that $B \in \mathcal{A}_{\mathcal{C}}$. □

Theorem 2.7 *Let $A \in \mathcal{A}_{\mathcal{C}}$, B a non-zero proper submodule of A and C a proper submodule of B. Then there exists $D \in \mathcal{A}_{\mathcal{C}}$ such that B/C is a proper essential submodule of D.*

Proof. Let $i : B \to A$ and $j : B/C \to E(B/C)$ be the inclusion homomorphisms and $p : B \to B/C$ the natural homomorphism. By injectivity of $E(B/C)$, there exists a homomorphism f such that $fi = jp$. Denote $D = f(A)$. It follow that $D \in \mathcal{A}_{\mathcal{C}}$ and D is an essential extension of B/C and an essential submodule of $E(B/C)$. Moreover, B/C is a proper submodule of D, because $B/C \in \mathcal{C}$. □

Corollary 2.8 *Let A be a non-simple module in $\mathcal{A}_{\mathcal{C}}$. Then there exists a simple module $S \in \mathcal{C}$ and a uniform module $D \in \mathcal{A}_{\mathcal{C}}$ such that D strictly contains S and is essential in $E(S)$.*

Proof. Let a be a non-zero element of A such that Ra is a proper submodule of A. Now take $B = Ra$ and C a maximal submodule of B in Theorem 2.7. □

Proposition 2.9 *Let $A \notin \mathcal{C}$ be an artinian module. Then there exists a submodule B of A such that $B \in \mathcal{A}_{\mathcal{C}}$.*

Proof. Since A is artinian, the set of all submodules of A which are not in $\mathcal{C}$ has a minimal element B, that is obviously in $\mathcal{A}_{\mathcal{C}}$. □

3 Endomorphism rings of modules of $\mathcal{A}_{\mathcal{C}}$

Throughout this section we will establish a few results on the endomorphism ring of a module of $\mathcal{A}_{\mathcal{C}}$, either arbitrary or more particular, such as uniform or faithful.

Theorem 3.1 *Let $A, B \in \mathcal{A}_{\mathcal{C}}$ and let $f : A \to B$ be a non-zero homomorphism. Then f is an epimorphism.*

Proof. Obviously, $f(A) \neq 0$ and by Proposition 2.1, $f(A) \in \mathcal{A}_{\mathcal{C}}$. If f is not an epimorphism, then $f(A)$ is a proper submodule of B, hence $f(A) \in \mathcal{C}$, contradiction. Therefore f is an epimorphism. □

Corollary 3.2 *If $A \in \mathcal{A}_{\mathcal{C}}$, then $End_R(A)$ is a domain.*

According to Corollary 2.8, if $\mathcal{A}_{\mathcal{C}}$ contains a non-simple module, then $\mathcal{A}_{\mathcal{C}}$ contains a uniform module as well. In the case of such a module, we have more information on its endomorphism ring.

Theorem 3.3 *Let $A \in \mathcal{A}_{\mathcal{C}}$ be a uniform module. Then $End_R(A)$ is a local ring.*

Proof. Since $A \neq 0$, we have $End_R(A) \neq 0$. Now let f and g be two non-zero endomorphisms of A which are not isomorphisms. Then by Theorem 3.1, f and g are epimorphisms, hence $Ker\, f \neq 0$ and $Ker\, g \neq 0$. Since A is uniform, we have $Ker\, f \cap Ker\, g \neq 0$. But $Ker\, f \cap Ker\, g \subseteq Ker\,(f+g)$, so that $f+g$ is not an isomorphism. Therefore $End_R(A)$ is local [5, Chapter 3, Lemma 3.8]. □

For the rest of the paper, the ring R will be assumed to be commutative.

Theorem 3.4 *Let $A \in \mathcal{A}_{\mathcal{C}}$. Then:*
(i) $Ann_R A$ is a prime ideal of R;
(ii) A is divisible over the domain $R/Ann_R A$.

Proof. (i) Let $r, s \in R$ such that $rs \in Ann_R A$. Then $rsA = 0$. Assume that $s \notin Ann_R A$. Then $sA \neq 0$ and by Theorem 3.1 we have $sA = A$, so that $rA = rsA = 0$, i.e. $r \in Ann_R A$. Therefore, $Ann_R A$ is a prime ideal of R.

(ii) The R-module A is also an $R/Ann_R A$-module in the usual way, that is $\bar{r}a = ra$ for all $\bar{r} = r + Ann_R A \in R/Ann_R A$ and $a \in A$. Then by Theorem 3.1, $\bar{r}A = rA = A$. Consequently, A is divisible over the domain $R/Ann_R A$. □

Theorem 3.5 *Let $A \in \mathcal{A}_{\mathcal{C}}$ be a faithful module. Then:*
(i) R is a domain;
(ii) If A is torsionfree, then $A \cong E(R)$ and $End_R(A)$ is a division ring.

Proof. (i) This follows by Theorem 3.4, since $Ann_R A = 0$.

(ii) Since A is a torsionfree divisible R-module by Theorem 3.4, it follows that A is injective [5, Proposition 2.7]. Let a be a non-zero element of A. Then $Ann_R A = 0$, hence $Ra \cong R$. But $A \in \mathcal{A}_{\mathcal{C}}$, so that A is indecomposable, hence $A = E(Ra) \cong E(R)$.

Now let f be a non-zero endomorphism of A and assume that $Ker\, f \neq 0$. Let $b \in Ker\, f$ and let r be a non-zero element of R. By divisibility of A, there exists $a \in A$ such that $ra = b$. Then $rf(a) = f(ra) = f(b) = 0$, whence $f(a) = 0$, i.e. $a \in Ker\, f$. But then $b \in rKer\, f$. Therefore $Ker\, f \subseteq rKer\, f$.

The converse inclusion obviously holds, so that we have $Ker\, f = rKer\, f$. This means that $Ker\, f$ is a torsionfree divisible module, hence $Ker\, f$ is injective. But then $Ker\, f$ is a direct summand of the indecomposable module A, which is a contradiction. Therefore f is a monomorphism, hence an isomorphism by Theorem 3.1. □

Corollary 3.6 *Let R be a Dedekind domain and let $A \in \mathcal{A}_{\mathcal{C}}$ be a faithful module. Then A is injective.*

Proof. By Theorem 3.4, A is a divisible R-module, hence A is injective [5, Proposition 2.10]. □

Remark. Some of the properties proved here when $\mathcal{C}$ is a Serre subcategory, namely Theorem 2.4, Corollary 2.5 and Proposition 2.9, still hold when $\mathcal{C}$ is an open class and consequently when $\mathcal{C}$ is a natural or even prenatural class.

References

[1] Dauns, J., *Direct sums and subdirect products*, Methods in Ring Theory (Colorado Springs, 1991), 39-65, Dekker, New York, 1993.

[2] Golan, J.S., Torsion theories, Longman Scientific and Technical, New York, 1986.

[3] Raggi, F., Rincon, H., Signoret, C., *On some classes of R-modules and congruences in R-tors*, Comm. Algebra **27** (1999), 889-901.

[4] Raggi, F., Signoret, C.J.E., *Serre subcategories of R-mod*, Comm. Algebra **24** (1996), 2877-2886.

[5] Sharpe, D.W., Vámos, P., Injective modules, Cambridge Univ. Press, 1972.

[6] Viola-Prioli, A., Viola-Prioli, J., Wisbauer, R., *Module categories with linearly ordered closed subcategories*, Comm. Algebra **22** (1994), 3613-3627.

[7] Weakley, W.D., *Modules whose proper submodules are finitely generated*, J. Algebra **84** (1983), 189-219.

[8] Wisbauer, R., Foundations of module and ring theory, Gordon and Breach, Reading, 1991.

[9] Zhou, Y., *The lattice of pre-natural classes of modules*, J. Pure Appl. Algebra **140** (1999), 191-207.

Department of Mathematics
and Computer Science,
"Babeş-Bolyai" University,
Cluj-Napoca,
Romania
e-mail: crivei@math.ubbcluj.ro

Department of Mathematics,
Technical University,
Cluj-Napoca,
Romania