Generalized parallel sums on symmetric cones and series parallel circuits

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Abstract. In this paper, we define power sums, which are generalized parallel sums, on symmetric cones. It is shown that power sums correspond naturally to the synthesized resistance of series parallel circuits. We also discuss the relation of power sums with arithmetic, geometric, harmonic, and α-power means, and compare their monotone functions on symmetric cones, where α represents the dualistic structure on information geometry.

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1 Introduction

Operator means on positive operators have been studied widely. Moreover, arithmetic, geometric, and harmonic means are well known means on positive operators [10, 3, 4]. α-power mean (or power mean) is a generalized geometric mean, and characterizes the mean between the arithmetic and harmonic means[13, 6, 9, 8]. On symmetric cones, the α-power mean is the midpoint on the α-geodesic connecting two points, where α is the parameter of dualistic structure on information geometry [12, 1].

Parallel sum is half of the harmonic mean, and is investigated relative to operator means on positive matrices [11, 2]. However, few literatures appear to have treated the sum as being defined by the normalized geometric mean arising from the difficulty of the preservation of convergence, and the sum being defined by the normalized α-power mean because of the non-continuity of parameterized sums.

The paper is organized as follows. First, we recall the previous results on symmetric cones and define the operator monotone function that generates the α-power sum. Next, we define the α-power sum that corresponds to the arithmetic and the parallel sum for $\alpha = 1, -1$, respectively, and compare its monotone function with the other sums and means on symmetric cones.

Parallel sum for scalars represents the synthesized resistance of a parallel circuit. This study demonstrates that the α-power sum corresponds naturally to an α-series parallel circuit for $-1 \leq \alpha \leq 1$. An α-series parallel circuit is a series and a parallel circuit for $\alpha = 1, -1$, respectively. Finally, we demonstrate that a series circuit

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continuously deforms into a parallel circuit, when the resistance elements have fixed resistivity and cross-sectional areas.

Applications of α-power sum are not restricted to the field of electric circuits. Its application to complex systems and nonextensive statistical mechanics also seem to exist, similar to the applications of α-power mean via information geometry.

2 Symmetric cones and operator monotone functions

First, we will recall a few results on Jordan algebras and symmetric cones [12, 5, 14].

A vector space V is called a Jordan algebra if the product $*$ defined on V satisfies

$$x * y = y * x, \quad x * (x^2 * y) = x^2 * (x * y) \tag{2.1}$$

for all $x, y \in V$ by setting $x^2 = x * x$. Let V be an n-dimensional Jordan algebra over $\mathbf{R}$ with an identity element e, i.e., $x * e = e * x = x$. An element $x \in V$ is said to be invertible if there exists a $y \in \mathbf{R}[x]$ such that $x * y = e$, where $\mathbf{R}[X]$ is a polynomial of X over $\mathbf{R}$. As $\mathbf{R}[x]$ is an associative algebra, y is unique, and is called the inverse of x and denoted by $x^{-1} = y$.

For x in V, let $L(x)$ and $P(x)$ be endomorphisms of V defined by

$$L(x)y = x * y, \quad y \in V \tag{2.2}$$

$$P(x) = 2L(x)^2 - L(x^2). \tag{2.3}$$

The following results about P, the quadratic representation of V, are known.

Proposition 1 *[5] (i) An element x is invertible if and only if $P(x)$ is invertible, and*

$$P(x)x^{-1} = x, \quad P(x)^{-1} = P(x^{-1}). \tag{2.4}$$

(ii) If x and y are invertible, so is $P(x)y$ and

$$(P(x)y)^{-1} = P(x^{-1})y^{-1}. \tag{2.5}$$

(iii) For all x and y,

$$P(P(y)x) = P(y)P(x)P(y). \tag{2.6}$$

Let Ω be an open convex cone on a vector space V. We denote the identity component of the linear automorphism group of Ω by G. Ω is said to be homogeneous if G acts on it transitively. The dual cone of Ω is defined by

$$\Omega^* = \{y \in V \mid (x, y) > 0, \forall x \in \bar{\Omega} \backslash \{0\}\}, \tag{2.7}$$

where (,) is an inner product on V, and $\bar{\Omega}$ is the closure of Ω. If $\Omega = \Omega^*$, a cone Ω is said to be self-dual, whereas it is called symmetric if it is homogeneous and self-dual.

Next, we consider a symmetric cone Ω with a set of positive operators. A binary operation $\sigma : (a,b) \in \bar{\Omega} \times \bar{\Omega} \mapsto a\sigma b \in \bar{\Omega}$ is called an operator connection if the following requirements are fulfilled.

(i) Monotonicity; $a \leq c$ and $b \leq d$ imply $a\sigma b \leq c\sigma d$,

(ii) Transformer inequality; $P(c)(a\sigma b) \leq (P(c)(a))\sigma(P(c)(b))$,

(iii) Semi-continuity; $a_n \downarrow a$ and $b_n \downarrow b$ imply $(a_n \sigma b_n) \downarrow a\sigma b$,

where $a \leq b$ (resp. $a < b$) is $b - a \in \bar{\Omega}$ (resp. in Ω) [10, 12].

with regard to Transformer inequality, it is accepted that $P(c)(a\sigma b) = (P(c)(a))\sigma(P(c)(b))$ for Ω. If normalization $e\sigma e = e$ is satisfied, the operator connection σ is called an operator mean (or a mean).

Let $x = \sum_{i=1}^{r} \lambda_i p_i$ be a spectral decomposition of $x \in V$, where r and $\{p_1, \ldots, p_r\}$ are the rank and a Jordan frame of V, respectively, and $\lambda_1, \ldots, and \lambda_r$ are the eigenvalues of x [5]. For a function $f(t)$ on an interval $\mathbf{I} \subseteq \mathbf{R}$, $f(x)$ is defined by

$$f(x) = \sum_{i=1}^{r} f(\lambda_i) p_i, \tag{2.8}$$

if $\lambda_1, \ldots, \lambda_r \in \mathbf{I}$. A function $f(t)$ on an interval $\mathbf{I} \subseteq \mathbf{R}$ satisfying the Inequation (2.9) is called an operator monotone function on $\mathbf{I}$.

$$a \leq b \Rightarrow f(a) \leq f(b), \tag{2.9}$$

where a and $b \in \Omega$ have eigenvalues on $\mathbf{I}$, respectively.

It is known that α-power mean on Ω is generated by

$$a\sigma^{(\alpha)} b = P(a^{\frac{1}{2}}) f^{(\alpha)}(P(a^{-\frac{1}{2}})b), \quad -1 \leq \alpha \leq 1, \tag{2.10}$$

where f is an operator monotone function defined by

$$f^{(\alpha)}(t) = \left(\frac{1+t^{\alpha}}{2}\right)^{\frac{1}{\alpha}} \quad (\alpha \neq 0), \quad f^{(0)}(t) = \sqrt{t} \tag{2.11}$$

[9, 12]. The arithmetic, geometric, and harmonic means are described by $a\sigma^{(1)}b$, $a\sigma^{(0)}b$, and $a\sigma^{(-1)}b$, respectively. Especially, for positive definite symmetric matrices A and B, they are denoted as:

(i) Arithmetic mean; $A\sigma^{(1)}B = (A+B)/2$,

(ii) Geometric mean; $A\sigma^{(0)}B = A\#B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}A^{\frac{1}{2}}$,

(iii) Harmonic mean; $A\sigma^{(-1)}B = ((A^{-1}+B^{-1})/2)^{-1}$,

(iv) α-power mean; $A\sigma^{(\alpha)}B = A^{\frac{1}{2}}((I + (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha})/2)^{\frac{1}{\alpha}}A^{\frac{1}{2}}$,

where I is the identity matrix. For scalar A and B, the geometric mean is $A\#B = \sqrt{AB}$, and the α-power mean is $A\sigma^{(\alpha)}B = ((A^{\alpha}+B^{\alpha})/2)^{\frac{1}{\alpha}}$.

3 α-power sums

We propose to define the α-power sum via an operator monotone function, which interpolates the generalized sum between the arithmetic and parallel sums.

For $-1 \le \alpha \le 1$, a function

$$f^{(\alpha)}(t) = \frac{(1+t)^{1+\alpha}}{1+t^{\alpha}}, \quad t > 0 \tag{3.1}$$

is an operator monotone function with $t \in \{t | t^{\alpha} - \alpha t^{\alpha-1} + \alpha + 1 > 0\}$. It follows from $df^{(\alpha)}/dt > 0$ that (3.1) is a monotone increasing and an operator monotone function with $t \in \{t | t^{\alpha} - \alpha t^{\alpha-1} + \alpha + 1 > 0\}$.

For example, Function (3.1) is operator monotone with $t > 0$ as $-1 \le \alpha \le 0$, $\alpha = 1$; therefore, $t > 0.0398$ when $\alpha = 0.1$ and similarly, $t > 0.0006$ when $\alpha = 0.9$, respectively.

Hence, Function (3.1) is obviously monotone increasing and operator monotone with α.

Definition 1 *Let $f^{(\alpha)}(t)$ be a function defined by Equation (3.1). For $-1 \le \alpha \le 1$, we define the α-power sum $:^{(\alpha)}$ of a and $b \in \Omega$ by*

$$a :^{(\alpha)} b = P(a^{\frac{1}{2}}) f^{(\alpha)}(P(a^{-\frac{1}{2}})b). \tag{3.2}$$

Theorem 1 *The α-power sum $:^{(\alpha)}$ of a and $b \in \Omega$ corresponds to the arithmetic sum $a + b$ for $\alpha = 1$, and to the parallel sum $a : b = (a^{-1} + b^{-1})^{-1}$ for $\alpha = -1$. Then, 0-power sum $a :^{(0)} b$ is the arithmetic mean $(a + b)/2$.*

proof For $\alpha = 1$, we have $f^{(1)}(t) = 1 + t$. From Equations (2.2), (2.3), and (2.4), we obtain

$$a :^{(1)} b = P(a^{\frac{1}{2}})(e + P(a^{-\frac{1}{2}})b) = a + b. \tag{3.3}$$

For $\alpha = -1$, we have $f^{(-1)}(t) = t/(1 + t)$. From Equations (2.2), (2.3), (2.4), (2.5), and (2.6), we obtain

$$\begin{aligned} a :^{(-1)} b &= P(a^{\frac{1}{2}}) P((P(a^{-\frac{1}{2}})b)^{\frac{1}{2}})(e + P(a^{-\frac{1}{2}})b)^{-1} \\ &= P(a^{\frac{1}{2}})(P((P(a^{-\frac{1}{2}})b)^{-\frac{1}{2}})(e + P(a^{-\frac{1}{2}})b))^{-1} \\ &= P(a^{\frac{1}{2}})((P(a^{-\frac{1}{2}})b)^{-1} + e)^{-1} = (P(a^{-\frac{1}{2}})(P(a^{\frac{1}{2}})b^{-1} + e))^{-1} \\ &= (a^{-1} + b^{-1})^{-1}. \end{aligned} \tag{3.4}$$

For $\alpha = 0$, the operator monotone function is $f^{(0)}(t) = (1 + t)/2$. Thus, 0-power sum $a :^{(0)} b$ is arithmetic mean.

From Theorem 1 we have the following.

Corollary 1 *For positive definite symmetric matrices A and B, α-power sums are*

(i) Arithmetic sum; $A :^{(1)} B = A + B$,

(ii) 0-power sum; $A :^{(0)} B = (A + B)/2$ (arithmetic mean),

(iii) Parallel sum; $A :^{(-1)} B = (A^{-1} + B^{-1})^{-1}$ (half of the harmonic mean),

(iv) α-power sum; $A :^{(\alpha)} B =$
$A^{\frac{1}{2}}(I + (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha})^{-\frac{1}{2}}(I + A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1+\alpha}(I + (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\alpha})^{-\frac{1}{2}}A^{\frac{1}{2}}$.
For scalar A and B, the α-power sum is

$$A :^{(\alpha)} B = \frac{(A+B)^{1+\alpha}}{A^{\alpha}+B^{\alpha}}. \tag{3.5}$$

The generalized sum diverges as $\alpha = 0$ if it is defined by $(A^{\alpha} + B^{\alpha})^{1/\alpha}$, which is the α-power mean without normalization. The α-power sum obtained by using Equations (3.1) and (3.2) possesses continuity at $\alpha = 0$. It satisfies (i) Monotonicity for elements with eigenvalues on the interval $\{t | t^{\alpha} - \alpha t^{\alpha-1} + \alpha + 1 > 0\}$. It also satisfies (ii) Transformer inequality and (iii) Semi-continuity on $\bar{\Omega}$ (resp. Ω).

4 Series parallel circuits realizing α-power sums

It is well known that the resistance of a series circuit realizes the arithmetic sum, whereas that of a parallel circuit realizes the parallel sum. In this section, we describe series parallel circuits that realize the α-power sums.

Let the symbol of a parallel sum $A : B$ also be a circuit connecting two resistances A and B in parallel.

Let $R_1, R_2 > 0$ be the resistances in an electric circuit. We assume that the resistances R_1, R_2 consist of elements with fixed resistivity 1, with cross-sectional areas of 1, and that their lengths are R_1 and $R_2 > 0$, respectively. For each $j = 1, 2$, with common length R_j and cross-sectional areas $R_1/(R_1+R_2)$ and $R_2/(R_1+R_2)$, the two resistances are $(R_1 + R_2)R_1^{-1}R_j$ and $(R_1 + R_2)R_2^{-1}R_j$, respectively. We provide a deformation of the resistances $(R_1 + R_2)R_1^{-1}R_j$ and $(R_1 + R_2)R_2^{-1}R_j$ for $j = 1, 2$ on which increasing rates of cross-sectional areas coincide with the decreasing rates of lengths (See Corollary 4). Then, we obtain the next theorem.

Theorem 2 *Let $R_1, R_2 > 0$ be constant real numbers, and for $-1 \leq \alpha \leq 1$,*

$$R_{ij} = \left(\frac{R_1 + R_2}{R_i}\right)^{\alpha} R_j\ , \quad i, j = 1, 2 \tag{4.1}$$

be the resistances in an electric circuit. Then, the synthetic resistance of the series circuit connecting the parallel circuits, $R_{11} : R_{21}$ and $R_{12} : R_{22}$, which we call the α-series parallel circuit, is the α-power sum of R_1 and R_2, i.e.,

$$R_1 :^{(\alpha)} R_2 = \frac{(R_1 + R_2)^{1+\alpha}}{R_1^{\alpha} + R_2^{\alpha}}. \tag{4.2}$$

Moreover, the α-series parallel circuit is the balanced Wheatstone bridge for each α.

The circuit shown in Figure 1 is the Wheatstone bridge, which is called balanced if an electric current at point **A** is zero for a non-zero voltage input. $R_{11}/R_{12} = R_{21}/R_{22}$ holds true if and only if the Wheatstone bridge is balanced [7].

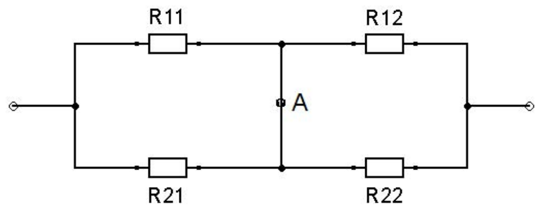


Figure 1: The Wheatstone bridge

Proof of Theorem 2 It follows from Equation (4.1) that the synthetic resistance of the series circuit connecting $R_{11} : R_{21}$ and $R_{12} : R_{22}$ is

$$(4.3)\qquad R_{11} : R_{21} + R_{12} : R_{22} = (R_{11}^{-1} + R_{21}^{-1})^{-1} + (R_{12}^{-1} + R_{22}^{-1})^{-1}$$
$$= ((R_1 + R_2)^{-\alpha} R_1^{\alpha} R_1^{-1} + (R_1 + R_2)^{-\alpha} R_2^{\alpha} R_1^{-1})^{-1}$$
$$+((R_1 + R_2)^{-\alpha} R_1^{\alpha} R_2^{-1} + (R_1 + R_2)^{-\alpha} R_2^{\alpha} R_2^{-1})^{-1}$$
$$= (R_1 + R_2)^{\alpha} (R_1^{\alpha} + R_2^{\alpha})^{-1} (R_1 + R_2) = (R_1 + R_2)^{1+\alpha} (R_1^{\alpha} + R_2^{\alpha})^{-1}.$$

The above corresponds to the right-hand side of Equation (4.2).

For any $-1 \leq \alpha \leq 1$, we have,

$$(4.4)\qquad \frac{R_1}{R_2} = \frac{R_{11}}{R_{12}} = \frac{R_{21}}{R_{22}}.$$

Thus, the α-series parallel circuit in Theorem 2 is a balanced Wheatstone bridge.

Remark 1 For $\alpha = 1$, we have,

$$(4.5)\quad R_{1j} : R_{2j} = (R_{1j}^{-1} + R_{2j}^{-1})^{-1} = (((R_1 + R_2) R_1^{-1} R_j)^{-1} + ((R_1 + R_2) R_2^{-1} R_j)^{-1})^{-1}$$
$$= (R_1 + R_2)(R_1 + R_2)^{-1} R_j = R_j\ , \quad j = 1, 2$$

Then, the 1-series parallel circuit is equivalent to the series circuit connecting the resistances R_1 and R_2.

Remark 2 For $\alpha = 0$, we have,

$$(4.6)\qquad R_{ij} = R_j\ , \quad i, j = 1, 2.$$

Then, the 0-series parallel circuit is equivalent to the Wheatstone bridge that connects two R_1 and two R_2 in parallel.

Remark 3 For $\alpha = -1$, we have,

$$(4.7)\qquad R_{i1} + R_{i2} = (R_1 + R_2)^{-1} R_i R_1 + (R_1 + R_2)^{-1} R_i R_2 = R_i\ , \quad i = 1, 2\ .$$

Then, the (-1)-series parallel circuit is equivalent to a parallel circuit that connects the resistances R_1 and R_2.

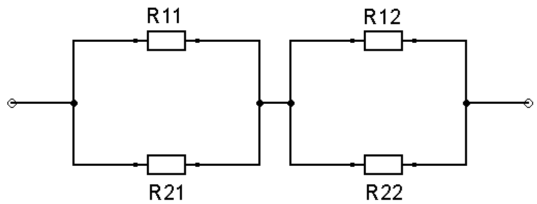


Figure 2: The α-series parallel circuit

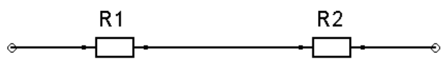


Figure 3: The series circuit ($\alpha = 1$)

The α-series parallel circuit, the series circuit (i.e., 1-series parallel circuit), the 0-series parallel circuit, and the parallel circuit (i.e., (-1)-series parallel circuit) are shown in Figures 2, 3, 4, and 5, respectively.

The next corollary defines the continuous deformation of the circuit into a parallel circuit, both connecting the resistances R_1 and R_2 with fixed resistivities and cross-sectional areas.

Corollary 2 *We suppose that resistances r_{ij}, $i, j = 1, 2$ comprise elements with fixed resistivity* 1 *for all i, j and all $-1 \leq \alpha \leq 1$, i.e., $r_{ij} = L_{ij}/S_{ij}$, where L_{ij} and S_{ij} are the lengths and cross-sectional areas of r_{ij}, respectively. Let the lengths of r_{ij} be*

$$L_{ij} = \left(\frac{R_i}{R_1 + R_2}\right)^{\frac{1-\alpha}{2}} R_j, \quad i, j = 1, 2, \tag{4.8}$$

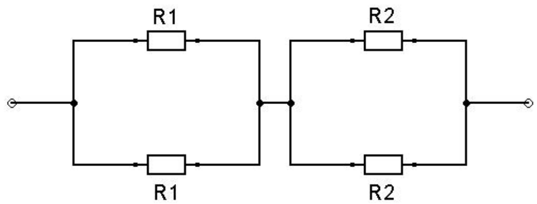
and the cross-sectional areas of r_{ij} be

$$S_{ij} = \left(\frac{R_i}{R_1 + R_2}\right)^{1-\frac{1-\alpha}{2}} = \left(\frac{R_i}{R_1 + R_2}\right)^{\frac{1+\alpha}{2}}, \quad i = 1, 2, \tag{4.9}$$

respectively. Then, the volumes of resistances r_{ij}, $i, j = 1, 2$ are constant values $R_iR_j/(R_1 + R_2)$ for all α, and $r_{ij} = ((R_1 + R_2)/R_i)^{\alpha} R_j$, $i, j = 1, 2$, respectively.

Proof of Corollary 2 The volumes of resistances r_{ij} are $L_{ij}S_{ij} = R_iR_j/(R_1 + R_2)$, $i, j = 1, 2$. The resistance values are $r_{ij} = L_{ij}/S_{ij} = ((R_1 + R_2)/R_i)^{\alpha} R_j$.

In Corollary 2 the sum of the volumes of r_{1j} and r_{2j} is R_j for $j = 1, 2$. The sum of the volumes of r_{i1} and r_{i2} is R_i for $i = 1, 2$. The total volume of r_{ij}, $i, j = 1, 2$ is $R_1 + R_2$. Thus, the deformation of the resistances due to the parameter α is realized as the deformation of the α-series parallel circuits preserving the volume of the resistance elements.



ries parallel circuit

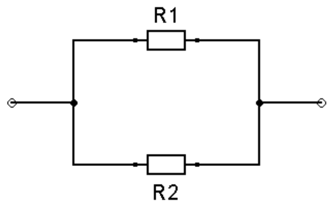


Figure 5: The parallel circuit ($\alpha = -1$)

5 Conclusions

In this paper, we defined α-power sum, which corresponds to the arithmetic and the parallel sum for $\alpha = 1, -1$, respectively, and compared its monotone function with the other sums and means applicable on symmetric cones. We also demonstrated that the α-power sum corresponds naturally to the α-series parallel circuit, which is the series and the parallel circuit for $\alpha = 1, -1$, respectively.

The α-power mean is mostly investigated with respect to the dualistic structure on information geometry. For future work, the remaining topic of the information geometrical characterization of α-power sum can be explored.

References

[1] S. Amari, *Information Geometry and Its Applications*, Springer, Tokyo, Japan, 2016.

[2] P. Berkics, *On parallel sum of matrices*, Linear Multilinear A. 65 (2017), 2114-2123.

[3] D. S. Bernstein, *Matrix Mathematics; Theory, Facts, and Formulas*, Princeton University Press, New Jersey, 2009.

[4] R. Bhatia, *Positive definite symmetric matrices*, Princeton University Press, New Jersey, 2007.

[5] J. Faraut, A. Koranyi, *Analysis on Symmetric Cones*, Clarendon Press, Oxford, 1994.

[6] J. Fujii, E. Kamei, *Uhlmann's interpolational method for operator means*, Math. Jap. 34 (1989), 541-547.

[7] A. R. Hambley, *Electrical Engineering: Principles and Applications* (4th Edition), Prentice Hall, New Jersey, 2007.

[8] G. H. Hardy, J. E. Littlewood, G. Polya, *Inequalities* (2nd ed.), Cambridge University Press: Cambridge, 1952.

[9] E. Kamei, *Paths of operators parameterized by operator means*, Math. Jap. 39 (1994), 395-400.

[10] F. Kubo, T. Ando, *Means of positive linear operators*, Math. Ann. 246 (1980), 205-224.

[11] T. D. Morley, *An alternative approach to the parallel sum*, Adv. Appl. Math. 10 (1989), 358-369.

[12] A. Ohara, *Geodesics for dual connections and means on symmetric cones*, Integr. Eq. Oper. Theory. 50 (2004), 537-548.

[13] A. Uhlmann, *Relative entropy and the Wigner-Yanase-Dyson-Lieb concavity in an interpolation theory*, Commun. Math. Phys. 80 (1977), 21-32.

[14] K. Uohashi, A. Ohara, *Jordan algebras and dual affine connections on symmetric cones*, Positivity, 8 (2004), 369-378.

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