Analytic weighted rough statistical convergence with rate of rough convergence and Voronovskaya theorem of triple difference sequences

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Abstract. In this paper, we study a new concepts these are rough statistically summable triple difference sequences, rough statistically convergent difference triple sequences and rough strongly summable triple difference sequences. Also, we study the rate of the weighted–rough statistical convergence of a triple sequence of positive linear operators. In addition, we study Voronovskaya type theorem.

M.S.C. 2010: 41A36, 40G15.
Key words: weighted statistical convergence; rate of convergence; Voronovskaya type theorem.

1 Introduction

Let K be a subset of the set of positive integers $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$, and let us denote the set $\{(m,n,k) \in K : m \leq u, n \leq v, k \leq w\}$ by K_{uvw}. Then the natural density of K is given by $\delta(K) = lim_{uvw \to \infty} \frac{|K_{uvw}|}{uvw}$, if the limit exists where the vertical bars indicate the number of elements in the closed set, where $|K_{uvw}|$ denotes the number of elements in K_{uvw}. Clearly, a finite subset has natural density zero, and we have $\delta(K^c) = 1 - \delta(K)$ where $K^c = \mathbb{N} \backslash K$ is the complement of K. If $K_1 \subseteq K_2$, then $\delta(K_1) \leq \delta(K_2)$.

The triple sequence $x = (x_{mnk})$ is said to be rough statistically convergent to l if for every $\epsilon > 0$, the set $K_\epsilon = \{(m,n,k) \in \mathbb{N} : |x_{mnk} - l| \geq \beta + \epsilon\}$ has natural density zero, for each $\epsilon > 0, lim_{rst \to \infty} \frac{1}{rst} |\{(m,n,k) \leq (r,s,t) : |x_{mnk} - l| \geq \beta + \epsilon\}| = 0$. In this case, we write $l = st - limx$. Throughout the paper, $\mathbb{R}$ denotes the real of three dimensional space with metric (X,d). Consider a triple sequence $x = (x_{mnk})$ such that $x_{mnk} \in \mathbb{R}, m,n,k \in \mathbb{N}$.

A triple sequence $x = (x_{mnk})$ is said to be statistically convergent to $0 \in \mathbb{R}$, written as st–$lim\ x = 0$, provided that the set

$$\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk}| \geq \epsilon\right\}$$

Applied Sciences, Vol. 22, 2020, pp. 157-168.

has natural density zero for any $\epsilon > 0$. In this case, 0 is called the statistical limit of the triple sequence x.

If a triple sequence is statistically convergent, then for every $\epsilon > 0$, infinitely many terms of the sequence may remain outside the $\epsilon-$ neighbourhood of the statistical limit, provided that the natural density of the set consisting of the indices of these terms is zero. This is an important property that distinguishes statistical convergence from ordinary convergence. Because the natural density of a finite set is zero, we can say that every ordinary convergent sequence is statistically convergent.

If a triple sequence $x = (x_{mnk})$ satisfies some property P for all m, n, k except a set of natural density zero, then we say that the triple sequence x satisfies P for almost all (m, n, k) and we abbreviate this by a.a. (m, n, k).

Let $\left(x_{m_i n_j k_\ell}\right)$ be a subsequence of $x = (x_{mnk})$. If the natural density of the set $K = \left\{(m_i, n_j, k_\ell) \in \mathbb{N}^3 : (i, j, \ell) \in \mathbb{N}^3\right\}$ is different from zero, then $\left(x_{m_i n_j k_\ell}\right)$ is called a non thin subsequence of a triple sequence x.
$c \in \mathbb{R}$ is called a statistical cluster point of a triple sequence $x = (x_{mnk})$ provided that the natural density of the set

$$\left\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk} - c| < \epsilon\right\},$$

is different from zero for every $\epsilon > 0$. We denote the set of all statistical cluster points of the sequence x by Γ_x.

A triple sequence $x = (x_{mnk})$ is said to be statistically analytic if there exists a positive number M such that

$$\delta\left(\left\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk}|^{1/m+n+k} \geq M\right\}\right) = 0.$$

The theory of statistical convergence has been discussed in trigonometric series, summability theory, measure theory, turnpike theory, approximation theory, fuzzy set theory and so on.

The idea of rough convergence was introduced by Phu [19], who also introduced the concepts of rough limit points and roughness degree. The idea of rough convergence occurs very naturally in numerical analysis and has interesting applications. Aytar [2] extended the idea of rough convergence into rough statistical convergence using the notion of natural density just as usual convergence was extended to statistical convergence. Pal et al. [18] extended the notion of rough convergence using the concept of ideals which automatically extends the earlier notions of rough convergence and rough statistical convergence and Bernstein and operational methods was introduced by Dattoli et al. [3].

Let (X, ρ) be a metric space. For any non empty closed subsets $A, A_{mnk} \subset X\,(m, n, k \in \mathbb{N})$, we say that the triple sequence (A_{mnk}) is Wijsman statistical convergent to A is the triple sequence $(d\,(x, A_{mnk}))$ is statistically convergent to $d\,(x, A)$, i.e., for $\epsilon > 0$ and for each $x \in X$

$$lim_{rst} \frac{1}{rst} \left|\{m \leq r, n \leq s, k \leq t : |d\,(x, A_{mnk}) - d\,(x, A)| \geq \epsilon\}\right| = 0.$$

In this case, we write

$$st\text{–}lim_{mnk} A_{mnk} = A \text{ or } A_{mnk} \longrightarrow A\,(WS)\,.$$

The triple sequence (A_{mnk}) is bounded if $sup_{mnk} d\,(x, A_{mnk}) < \infty$ for each $x \in X$.

A triple sequence (real or complex) can be defined as a function $x : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \to \mathbb{R}\,(\mathbb{C})$, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, real numbers and complex numbers respectively. The different types of notions of triple sequence was introduced and investigated at the initial by Sahiner et al. [20, 21], Esi et al. [6]-[8], Dutta et al. [5], Subramanian et al. [10, 11, 13],[23]-[30], Debnath et al. [4], Aiyub et al. [1] and Zweier sequence was introduced and investigated at the initial by Fadile Karababa et al. [12], Sharma et al. [22], Khan et al. [15] many others.

Throughout the paper let β be a nonnegative real number.

Definition 1.1. An Orlicz function (see [14]) is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing and convex with $M\,(0) = 0$, $M\,(x) > 0$, for $x > 0$ and $M\,(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M\,(x + y) \leq M\,(x) + M\,(y)$, then this function is called modulus function.

Lindenstrauss and Tzafriri ([16]) used the idea of Orlicz function to construct Orlicz sequence space.

A sequence $g = (g_{mn})$ defined by

$$g_{mn}\,(v) = sup\,\{|v|\,u - (f_{mnk})\,(u) : u \geq 0\}\,, m, n, k = 1, 2, \ldots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak-Orlicz function f, (see [17]) the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{x \in w^3 : I_f\,(|x_{mnk}|)^{1/m+n+k} \to 0\, as\, m, n, k \to \infty\right\},$$

where I_f is a convex modular defined by

$$I_f\,(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{mnk}\,(|x_{mnk}|)^{1/m+n+k}\,, x = (x_{mnk}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric define as follows

$$d\,(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_{mnk}\left(\frac{|x_{mnk}|^{1/m+n+k}}{mnk}\right)$$

on an extended real number.

Definition 1.2. A triple sequence $x = (x_{mnk})$ of real numbers is said to be statistically convergent to $l \in \mathbb{R}^3$ if for any $\epsilon > 0$ we have $d\,(A\,(\epsilon)) = 0$, where

$$A\,(\epsilon) = \left\{(m, n, k) \in \mathbb{N}^3 : |x_{mnk} - l| \geq \epsilon\right\}.$$

Definition 1.3. A triple sequence $x = (x_{mnk})$ is said to be statistically convergent to $l \in \mathbb{R}^3$, written as $st\text{–}lim x = l$, provided that the set

$$\left\{(m,n,k) \in \mathbb{N}^3 : |x_{mnk} - l| \geq \epsilon\right\},$$

has natural density zero for every $\epsilon > 0$. In this case, l is called the statistical limit of the sequence x.

Definition 1.4. Let $x = (x_{mnk})_{m,n,k \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ be a triple sequence in a metric space $(X, |.,.|)$ and r be a non-negative real number. A triple sequence $x = (x_{mnk})$ is said to be $r-$convergent to $l \in X$, denoted by $x \to^r l$, if for any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that for all $m,n,k \geq N_\epsilon$ we have

$$|x_{mnk} - l| < r + \epsilon$$

In this case l is called an r–*limit* of x.

Remark 1.5. We consider the r–*limit* set x which is denoted by LIM_x^r and is defined by

$$LIM_x^r = \{l \in X : x \to^r l\}.$$

Definition 1.6. A triple sequence $x = (x_{mnk})$ is said to be $r-$ convergent if $LIM_x^r \neq \phi$ and r is called a rough convergence degree of x. If $r = 0$, then, it is ordinary convergence of triple sequence.

Definition 1.7. Let $x = (x_{mnk})$ be a triple sequence in a metric space $(X, |.,.|)$ and r be a non-negative real number is said to be $r-$ statistically convergent to l, denoted by $x \to^{r-st_3} l$, if for any $\epsilon > 0$ we have $d\,(A\,(\epsilon)) = 0$, where

$$A\,(\epsilon) = \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : |x_{mnk} - l| \geq r + \epsilon\}.$$

In this case l is called $r-$ statistical limit of x. If $r = 0$ then it is ordinary statistical convergent of triple sequence.

The difference triple sequence space was introduced by Debnath et al. (see [4]) and is defined as

$$\Delta x_{mnk} = x_{mnk} - x_{m,n+1,k} - x_{m,n,k+1} + x_{m,n+1,k+1} - x_{m+1,n,k} + x_{m+1,n+1,k}$$
$$+ x_{m+1,n,k+1} - x_{m+1,n+1,k+1}$$

and

$$\Delta^0 x_{mnk} = \langle x_{mnk}\rangle.$$

Let $\Lambda = \{\lambda_{mnk} : m,n,k = 0,1,2,\ldots\}$ be a nondecreasing sequence of positive numbers tending to ∞, as $(m,n,k) \to \infty$ and $\Delta^3 \lambda_{rst} \geq 0$.

Let $x = (x_{mnk})$ be a triple sequence of complex numbers, then, by Λ_Δ^3- will denote the following

$$\Lambda_\Delta^3\,(x) = \frac{1}{\lambda_{rst} - \lambda_{r-1s-1t-1}} \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} \lambda_{mnk}x_{mnk} - \lambda_{mn+1k}x_{m,n+1,k} - \lambda_{mnk+1}x_{m,n,k+1}$$
$$+ \lambda_{mn+1k+1}x_{m,n+1,k+1} - \lambda_{m+1nk}x_{m+1,n,k} + \lambda_{m+1n+1k}x_{m+1,n+1,k}$$
$$+ \lambda_{m+1nk+1}x_{m+1,n,k+1} - \lambda_{m+1n+1k+1}x_{m+1,n+1,k+1}.$$

We say that the triple difference sequence (x_{rst}) is $\Lambda^3_\Delta-$ summable to l if $lim_{rst}\Lambda^3_\Delta = l$, and we say that the triple difference sequence (x_{rst}) is rough statistically summable to l by the weighted method determined by the triple difference sequence $\Lambda^3_\Delta-$ if $st\text{–}lim_{rst}\Lambda^3_\Delta = l$. We denote by $\Lambda^3_\Delta(st)$ the set of all triple difference sequences which are rough statistically summable Λ^3_Δ.

Definition 1.8. The triple difference sequence $x = (x_{rst})$ is weighted $\Lambda^3_\Delta-$ rough statistically convergent to L if for every $\beta, \epsilon > 0$,

$$lim_{rst\to\infty} \left|\left\{\Lambda^3_\Delta(x) - L\right\} \geq \beta + \epsilon\right| = 0$$

In this case, we write $L = st_{\Lambda^3_\Delta}\text{–}lim_{rst}x_{rst}$.

Definition 1.9. The triple difference sequence $x = (x_{rst})$ is said to be rough strongly $\Lambda^3_{\Delta_r}-$ summable $(0 < r < \infty)$ to the limit L if

$$lim_{rst\to\infty} \left|\Lambda^3_\Delta(x) - L\right|^r = 0.$$

and we write it as $x_{mnk} \to \Lambda^3_{\Delta_r}$. In this case L is called rough strongly $\Lambda^3_{\Delta_r}-$ limit of the sequence $x = (x_{rst})$.

2 Relation between the $\Lambda^3_\Delta(x)-$ rough statistically summable triple difference sequences, $\Lambda^3_\Delta(x)-$ rough statistically convergent difference triple sequences and rough strongly $\Lambda^3_{\Delta_r}(x)-$ summable triple difference sequences

In this section we denoted rs for *rough statistically*

Theorem 2.1. *Let $\left|\Lambda^3_\Delta(x) - L\right| \leq M$ for all $m, n, k \in \mathbb{N}$. If a triple difference sequence $x = (\Delta x_{mnk})$ is $\Lambda^3_\Delta(x)-$ rough statistically convergent to L then it is $\Lambda^3_\Delta(x)-$ rough statistically summable to L, but not conversely.*

Proof. Let $x = (x_{mnk})$ is $\Lambda^3_\Delta(x)-$ rs–convergent to L, it means that

$$lim_{rst} \left|\left\{\left\{\Lambda^3_\Delta(x) - L\right\} \geq \beta + \epsilon\right\}\right| = 0.$$

Let us denote $K = \left\{\left|\Lambda^3_\Delta(x) - L\right| \geq \beta + \epsilon\right\}$ and $K^c = \left\{\left|\Lambda^3_\Delta(x) - L\right| < \beta + \epsilon\right\}$. Then, we have:

$$\begin{aligned}
\left|\Lambda^3_\Delta(x) - L\right| &= \left|\sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t}\Lambda^3_\Delta(x) - L\right| \\
&\leq \left|\sum_{m=0,m\in K}^{r}\sum_{n=0,n\in K}^{s}\sum_{k=0,k\in K}^{t}\Lambda^3_\Delta(x) - L\right| \\
&+ \left|\sum_{m=0,m\in K^c}^{r}\sum_{n=0,n\in K^c}^{s}\sum_{k=0,k\in K^c}^{t}\Lambda^3_\Delta(x) - L\right| \\
&\leq \sum_{m=0,m\in K}^{r}\sum_{n=0,n\in K}^{s}\sum_{k=0,k\in K}^{t}\left|\Lambda^3_\Delta(x) - L\right| \\
&+ \sum_{m=0,m\in K^c}^{r}\sum_{n=0,n\in K^c}^{s}\sum_{k=0,k\in K^c}^{t}\left|\Lambda^3_\Delta(x) - L\right| \\
&\leq M\cdot|K| + \sum_{m=0,m\in K^c}^{r}\sum_{n=0,n\in K^c}^{s}\sum_{k=0,k\in K^c}^{t}(\beta+\epsilon) \to 0 \\
&+ (\beta+\epsilon)\cdot 1 = (\beta+\epsilon)\, as\, (r,s,t)\to\infty
\end{aligned}$$

Which implies that (x_{rst}) is $\Lambda^3_\Delta(x)-$ r summable to L. That is, $x = (x_{rst})$ is $\Lambda^3_\Delta(x)-$ r summable to L. So x is $\Lambda^3_\Delta(x)-$ rs summable to L. $\square$

To prove that converse is not true, we discuss the following example:

Example 2.1. Let us consider that $\lambda_{rst} = (r,s,t)^3$. We will define

$$x_{mnk} = \begin{cases} \frac{1}{(abc)^{4+\alpha}}\ , \text{ for } \alpha > 0 \; and \; m,n,k = (abc)^3 - abc, \ldots, (abc)^3 - 1 \\ -\frac{1}{(abc)^9}\ , \text{ for } \; m,n,k = (abc)^3, a,b,c = 3,4,\ldots \\ 0, \text{ otherwise} \end{cases}$$

we get $lim_{rst}\Lambda^3_\Delta(x) - 0$, and hence $st-lim_{rst\to\infty}\Lambda^3_\Delta(x) - 0$, i.e., $x = (x_{rst})$ is $\Lambda^3_\Delta(x)-$ rs summable to 0. On the other hand, the triple sequence $\left((a,b,c)^3 : a,b,c = 3,4,\cdots,\right)$ is rst–convergent to 0, it is clear that $st-liminf_{rst}x_{rst} = 0$ and $st-limsup_{rst}x_{rst} = 1$. Thus $x = (x_{rst})$ is not rs–convergent, nor $\Lambda^3_\Delta(x)-$ rs–convergent.

Proposition 2.2. *Let us suppose that $x = (x_{rst})$ is $\Lambda^3_{\Delta_r}(x)-$ summable to l if*

(i) $0 < r < 1$ *and* $0 \leq \left|\Lambda^3_\Delta(x) - L\right| < 1$,

(ii) $1 \leq r < \infty$ *and* $1 \leq \left|\Lambda^3_\Delta(x) - L\right| < \infty$, *then, x is $\Lambda^3_\Delta(x)-$ rs–convergent to l.*

Proof. Let us suppose that $x = (x_{rst})$ is $\Lambda^3_{\Delta_r}(x)-$ summable to l, we get

$$\left|\Lambda^3_\Delta(x) - l\right|^r \geq \left|\Lambda^3_\Delta(x) - l\right|,$$

$$|K| = \frac{1}{\epsilon} \sum_{m=1, m \in K}^{r} \sum_{n=1, n \in K}^{s} \sum_{k=1, k \in K}^{t} \epsilon \leq \frac{1}{\epsilon} \sum_{m=1, m \in K}^{r} \sum_{n=1, n \in K}^{s} \sum_{k=1, k \in K}^{t} \left| \Lambda_{\Delta}^{3}(x) - l \right|$$

$$\leq \frac{1}{\epsilon} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} \left| \Lambda_{\Delta}^{3}(x) - l \right| \leq \frac{1}{\epsilon} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} \left| \Lambda_{\Delta}^{3}(x) - l \right|^{r}$$

$$\to 0, \text{ as } (r, s, t) \to \infty.$$

Hence $x = (x_{rst})$ is $\Lambda_{\Delta}^{3}(x)-$ rs–convergent to l. □

Proposition 2.3. *Let us suppose that $x = (x_{rst})$ is $\Lambda_{\Delta}^{3}(x)-$ rs–convergent to l and $\left| \Lambda_{\Delta}^{3}(x) - l \right| \leq M \, (m, n, k \in \mathbb{N})$. If*

(i) $0 < r < 1$ *and* $1 \leq M < \infty$,

(ii) $1 \leq r < \infty$ *and* $0 \leq M < 1$, *then x is $\Lambda_{\Delta_r}^{3}(x) -$ r summable to l.*

Proposition 2.4. **(i)** *If $x = (x_{mnk}) \to l$ rs–convergent, it is $\Lambda_{\Delta}^{3}(x)$–rs–convergent to l, but not conversely.*

(ii) *If $\left(\frac{\lambda_{rst} - \lambda_{r-1,s-1,t-1}}{rst} \right)$ is a triple bounded sequence, then rs–convergent is equivalent to $\Lambda_{\Delta}^{3}(x)-$ rs–convergent.*

3 Rate of rough convergence

In this section, we study the rate of the weighted–rough statistical convergence of a triple sequence of positive linear operators $\Lambda_{\Delta}^{3}(x)$ defined on $C[a, b]$.

Definition 3.1. Let (a_{rst}) be any positive, non increasing triple sequence of positive numbers. We say that triple sequence $x = (x_{rst})$ is $\Lambda_{\Delta}^{3}(x)-$ rs–convergent to number l with rate of convergence $C(a_{rst})$ if for every $\beta, \epsilon > 0$.

$$lim_{rst \to \infty} \frac{1}{a_{rst}} \left| \left\{ \left| \Lambda_{\Delta}^{3}(x) - l \right| \geq \beta + \epsilon \right\} \right| = 0.$$

In this case, we write $x_{mnk} - l =$ rough $st_{\Lambda_{\Delta}^{3}(x)} - o(a_{rst})$.

Proposition 3.1. *Let (a_{rst}) and (b_{rst}) be two positive non increasing numeric sequences. Let $x = (x_{rst})$ and $y = (y_{rst})$ be two triple sequences such that $x_{rst} - l_1 =$ rough $st_{\Lambda_{\Delta}^{3}(x)} - o(a_{rst})$ and $y_{rst} - l_2 =$ rough $st_{\Lambda_{\Delta}^{3}(x)} - o(b_{rst})$. Then,*

(i) $(x_{rst} - l_1) \pm (y_{rst} - l_2) =$ *rough* $st_{\Lambda_{\Delta}^{3}(x)} - o(c_{rst})$.

(ii) $\alpha (x_{rst} - l) =$ *rough* $st_{\Lambda_{\Delta}^{3}(x)} - o(a_{rst})$, *for any scalar* α,

(iii) $(x_{rst} - l_1)(y_{rst} - l_2) =$ *rough* $st_{\Lambda_{\Delta}^{3}(x)} - o(a_{rst} \cdot b_{rst})$, *where,* $c_{rst} = max\{a_{rst}, b_{rst}\}$.

Proof. For $\beta, \epsilon > 0$, let us denote by

$$A_1 = \left| \Lambda_{\Delta}^{3}(x) + \Lambda_{\Delta}^{3}(x) - (l_1 + l_2) \right| \geq \beta + \epsilon,$$

$$A_2 = \left| \Lambda_{\Delta}^{3}(x) - l_1 \right| \geq \frac{\beta + \epsilon}{2}, A_3 = \left| \Lambda_{\Delta}^{3}(x) - l_2 \right| \geq \frac{\beta + \epsilon}{2}.$$

Then, observe that $A_1 \subset A_2 \bigcup A_3$. Since, $c_{rst} = max\{a_{rst}, b_{rst}\}$, we get

$$\frac{|A_1|}{(\lambda_{rst} - \lambda_{r-1s-1t-1}) \cdot c_{rst}} \leq \frac{|A_2|}{(\lambda_{rst} - \lambda_{r-1s-1t-1}) \cdot c_{rst}} + \frac{|A_3|}{(\lambda_{rst} - \lambda_{r-1s-1t-1}) \cdot c_{rst}}$$
$$\implies lim_{rst} \frac{|A_1|}{(\lambda_{rst} - \lambda_{r-1s-1t-1}) \cdot c_{rst}} = 0.$$

□

Remark 3.2. The modulus of continuity for function $f(x) \in C_{2\pi}(\mathbb{R})$ is defined as follows:

$$w(f, \delta) = sup_{|h|<\delta} |f(x+h) - f(x)|.$$

We will consider these two cases:

(i) If $|x-y| < \delta$, then we have: $|f(x) - f(y)| \leq w(f, \delta)$ and

(ii) If $|x-y| > \delta$, then we have: $|f(x) - f(y)| \leq w(f, \delta) \cdot \frac{|x-y|}{\delta}$.

From the last two relations, for any value of the $|x-y|$, we get

$$|f(x) - f(y)| \leq w(f, \delta) \cdot \left(\frac{|x-y|}{2} + 1 \right).$$

Theorem 3.2. *Let (B_{rst}) be a triple sequence of positive linear operators from $C[a,b]$ into $C[a,b]$. Suppose that*

(i) $\|B_{rst}(1,x) - 1\|_\infty =$ *rough* $st_{\Lambda^3_\Delta(x)} - o(a_{rst})$,

(ii) $w(f, \lambda_{rst}) =$ *rough* $st_{\Lambda^3_\Delta(x)} - o(b_{rst})$, *where* $\lambda_{rst} = \sqrt{B_{rst}(\psi, x)}$ *and* $\psi \equiv \psi(t,x) = (e^{-t} - e^{-x})^3$. *Then for all* $f \in C[a,b]$ *and* $x \in [a,b]$, *we have* $\|B_{rst}(f,x) - f(x)\|_\infty =$ *rough* $st_{\Lambda^3_\Delta(x)} - o(c_{rst})$, *where* $c_{rst} = max\{a_{rst}, b_{rst}\}$.

Proof. Let $f \in C[a,b]$ and $x \in [a,b]$, we get

$$\begin{aligned}
|B_{rst}(f,x) - f(x)| &\leq |B_{rst}(|f(y) - f(x)|), x| + |f(x)| \cdot |B_{rst}(1,x) - 1| \\
&\leq B_{rst}\left(\frac{|x-y|}{\delta} + 1, x \right) w(f, \delta) + |f(x)| \cdot |B_{rst}(1,x) - 1| \\
&\leq B_{rst}\left(1 + \frac{M}{\delta} (e^{-t} - e^{-x})^3, x \right) w(f, \delta) + |f(x)| \cdot |B_{rst}(1,x) - 1| \\
&\leq \left(B_{rst}(1,x) + \frac{M}{\delta} B_{rst}(\psi, x) \right) w(f, \delta) + |f(x)| \cdot |B_{rst}(1,x) - 1|
\end{aligned}$$

Put $\delta = \frac{\lambda^3_{rst}}{M} = \frac{B_{rst}(\psi,x)}{M}$, we obtain:

$$\begin{aligned}
\|B_{rst}(f,x) - f(x)\|_\infty &\leq \|f\|_\infty \|B_{rst}(1,x) - 1\|_\infty + w(f, \lambda_{rst}) + w(f, \lambda_{rst}) \|B_{rst}(1,x) - 1\|_\infty \\
&\leq C \{ \|B_{rst}(f,x) - f(x)\|_\infty + w(f, \lambda_{rst}) + w(f, \lambda_{rst}) \|B_{rst}(f,x) - f(x)\|_\infty \}.
\end{aligned}$$

Where, $C = max\left\{\|f\|_{\infty}, 1\right\}$. Now replacing $B_{mnk}(\cdot, x)$ by

$$\Lambda^3_{\Delta}(x) = \frac{1}{\lambda_{rst} - \lambda_{r-1s-1t-1}} \sum_{m=0}^{r}\sum_{n=0}^{s}\sum_{k=0}^{t} \lambda_{mnk}B_{mnk}(\cdot, x) - \lambda_{mn+1k}B_{mn+1k}(\cdot, x)$$
$$- \lambda_{mnk+1}B_{m,n,k+1}(\cdot, x) + \lambda_{mn+1k+1}B_{m,n+1,k+1}(\cdot, x) - \lambda_{m+1nk}B_{m+1,n,k}(\cdot, x)$$
$$+ \lambda_{m+1n+1k}B_{m+1,n+1,k}(\cdot, x) + \lambda_{m+1nk+1}B_{m+1,n,k+1}(\cdot, x)$$
$$- \lambda_{m+1n+1k+1}B_{m+1,n+1,k+1}(\cdot, x),$$

we get

$$\left\|\Lambda^3_{\Delta}(f, x) - f(x)\right\|_{\infty} \leq C\left\{\left\|\Lambda^3_{\Delta}(1, x) - 1\right\|_{\infty} + w(f, \lambda_{rst}) + w(f, \lambda_{rst})\left\|\Lambda^3_{\Delta}(1, x) - 1\right\|_{\infty}\right\}.$$

The proof follows from the conditions (i),(ii) and Proposition (3.2.) □

4 Voronovskaya type theorem

Theorem 4.1. *For every $f \in C[0,1]$ such that, $f^{'}, f^{"} \in C[0,1]$, then,* $(rst)(L_{rst}(f) - f) \approx \frac{1}{2}\left(x + x^2\right) f^{"}(x)\left(\Lambda^3_{\Delta}(\text{ rough } st)\right)$ *on* $[0,1]$.

Proof. Let us suppose that $f^{'}, f^{"} \in C[0,1]$ and $x \in [0,1]$. Define

$$\psi_x(y) = \begin{cases} \frac{f(y) - f(x) - (y-x)f^{'}(x) - \frac{1}{2}(y-x)^2 f^{"}(x)}{(y-x)^2}, & \text{for } x \neq y \\ 0, & \text{for } x = y \end{cases}$$

Then, $\psi_x(x) = 0$ and $\psi_x \in C[0,1]$. By Taylors formula, we get

$$f(y) = f(x) + (y-x)f^{'}(x) + \frac{1}{2}(y-x)f^{"}(x) + (y-x)^2\psi_x(y). \tag{4.1}$$

Knowing that

$$L_{rst}(1, x) = (1 + x_{rst}); L_{rst}((y-x), x) = 0$$

and

$$L_{rst}\left((y-x)^2, x\right) = (1 + x_{rst})\frac{x(1+x)}{rst},$$

Both sides of relation (4.1) by operator L_{rst}, we obtain:

$$L_{rst}(f) = f(x) + x_{rst}f(x) + \frac{f^{"}(x)}{2}\frac{x(1+x)}{n}(1 + x_{rst}) + (1 + x_{rst})V_{rst}\left(\phi^3\psi_x, x\right),$$

which yields

$$\left|rst\left[L_{rst}(f) - f(x)\right] - \frac{1}{2}\left(x + x^2\right)f^{"}(x)\right| \leq (rst)x_{rst}|f(x)| + x_{rst}\left|f^{"}(x)\right|$$
$$+ rst(1 + x_{rst})\left|V_{rst}\left(\phi^3\psi_x, x\right)\right|,$$

respectively

(4.2)
$$\left|rst\left[L_{rst}(f) - f(x)\right] - \frac{1}{2}\left(x + x^2\right)f^{"}(x)\right| \leq (rst)x_{rst}M + rst(1 + x_{rst})\left|V_{rst}\left(\phi^3\psi_x, x\right)\right|.$$

Where $\phi(y) = y - x$ and $M = \|f\|$ By Cauchy-Schwartz inequality in (4.2), we obtain:

$$rst\left|V_{rst}\left(\phi^3\psi_x, x\right)\right| \leq \left[(rst)^2 V_{rst}\left(\phi^4, x\right)\right]^{1/2}\left[V_{rst}\left(\psi_x, x\right)\right]^{1/2}. \tag{4.3}$$

Putting $\eta_x(y) = (\psi_x(y))^2$, we get that, $\eta_x(x) = 0$ and $\eta_x(\cdot) \in C[0,1]$. It follows that

$$L_{rst}(\eta_x) \to 0\left(\Lambda^3_\Delta(roughst)\right) \ on \ [0,1]. \tag{4.4}$$

Now, from relations (4.3) and (4.4), we have,

$$L_{rst}\left(\phi^3\psi_x, x\right) \to 0\,(roughst) \ on \ [0,1]. \tag{4.5}$$

For a given $\beta, \epsilon > 0$, we define the following sets:
$A_{rst}(x,\beta,\epsilon) = \left|\left\{\left|(mnk)\left(V_{mnk}(f,x) - f(x)\right) - \frac{1}{2}\left(x + x^2\right) f"(x)\right| \geq \beta + \epsilon\right\}\right|$.
$A_{1,rst}(x,\beta,\epsilon) = \left|\left\{|(mnk)\,x_{rst}| \geq \frac{\beta+\epsilon}{2M}\right\}\right|$, $A_{2,rst}(x,\beta,\epsilon) = \left|\left\{|(mnk)\,V_{rst}(\psi_x,x)| \geq \frac{\beta+\epsilon}{2}\right\}\right|$.
From last relation we have $A_{rst}(x,\beta,\epsilon) \leq A_{1,rst}(x,\beta,\epsilon) + A_{2,rst}(x,\beta,\epsilon)$, respectively.

$$\Lambda^3_\Delta\left(A_{rst}(\cdot,\beta,\epsilon)\right) \leq \Lambda^3_\Delta\left(A_{1,rst}(\cdot,\beta,\epsilon)\right) + \Lambda^3_\Delta\left(A_{2,rst}(\cdot,\beta,\epsilon)\right). \tag{4.6}$$

From definition of the triple sequence (x_{rst}), we get

$$((rst)\,x_{rst}) \to 0\left(\Lambda^3_\Delta(\text{ rough } st)\right) \ on \ [0,1]. \tag{4.7}$$

Now from relations (4.5),(4.6) and (4.7) , the RHS of the relation (4.6) tends to zero as $(r,s,t) \to \infty$. Therefore we have

$$\text{rough } st - lim_{(rst)\to\infty}\Lambda^3_\Delta\left(A_{rst}(\cdot,\beta,\epsilon)\right) = 0,$$

which proves that

$$(rst)\left(L_{rst}(f) - f\right) \approx \frac{1}{2}\left(x + x^2\right) f"(x)\left(\Lambda^3_\Delta(\text{ rough } st)\right) \ \text{on} \ [0,1].$$

□

Acknowledgements. The authors are extremely grateful to the anonymous referees for their valuable suggestion and constructive comments for the improvement of the manuscript.

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