Classical nonnegative solutions for a class IVP for nonlinear three-dimensional wave equations

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Abstract. This paper is devoted to investigate classical solutions for a class of three-dimensional nonlinear wave equations. The existence of classical nonegative solutions of the considered IVP is proved via new fixed point approach. The results in the manuscript are provided with examples.

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1 Introduction

Global existence for nonlinear wave equations is an important mathematical topic. Mathematicians, including F. John, S. Kleinerman, L. Hörmander, etc., have made investigations to this subject. The first non-trivial long-time existence result was established by F. John and S. Kleinerman in [8], where it is proved the almost global existence for a class $3D$ quasilinear scalar wave equations. Global existence for $3D$ quasilinear wave equations was established firstly by S. Kleinerman in [6] and by D. Christodoulou, independently by S. Kleinerman, in [4]. Here we propose a new approach for investigations for classical solutions of a class $3D$ nonlinear wave equations. We obtain the classical solutions by using the so-called fixed point approach, using not strict mathematical models, but the most general considerations about the nature of dispersion, dissipation, and nonlinearity.
In the present paper, we consider the following class of three-dimensional nonlinear wave equations. Namely, we study the following IVP for $(x, y, z) \in \mathbb{R}^3$

$$(1.1) \qquad \begin{cases} u_{tt} - u_{xx} - u_{yy} - u_{zz} = f(t, x, y, z, u, u_t, u_x, u_y, u_z), \ t \geq 0 \\ u(0, x, y, z) = u_0(x, y, z) \\ u_t(0, x, y, z) = u_1(x, y, z), \end{cases}$$

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where $u_0, u_1 \in \mathcal{C}^2(\mathbb{R}^3, \mathbb{R}_+)$ and $f \in \mathcal{C}([0,\infty)\times\mathbb{R}^8)$ satisfies a general growth condition. Let $E = \mathcal{C}^2([0,\infty)\times\mathbb{R}^3)$ be the space endowed with the norm

$$\begin{aligned}
\|u\| &= \max\Bigg\{ \sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} |u(t,x,y,z)|, \sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} \left|\frac{\partial}{\partial t}u(t,x,y,z)\right|, \\
&\sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} \left|\frac{\partial^2}{\partial t^2}u(t,x,y,z)\right|, \sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} \left|\frac{\partial}{\partial x}u(t,x,y,z)\right|, \\
&\sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} \left|\frac{\partial^2}{\partial x^2}u(t,x,y,z)\right|, \sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} \left|\frac{\partial}{\partial y}u(t,x,y,z)\right|, \\
&\sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} \left|\frac{\partial^2}{\partial y^2}u(t,x,y,z)\right|, \sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} \left|\frac{\partial}{\partial z}u(t,x,y,z)\right|, \\
&\sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} \left|\frac{\partial^2}{\partial z^2}u(t,x,y,z)\right| \Bigg\},
\end{aligned}$$

provided it exists.
Our main assumptions in this paper are as follows.

(H1) $f \in \mathcal{C}([0,\infty)\times\mathbb{R}^8)$ be such that

$$0 \le f(t,x,y,z,u_1,u_2,u_3,u_4,u_5) \le \sum_{i=1}^{5}\sum_{j=1}^{l_i} c_j^i(t,x)|u_i|^{p_j^i},$$

$p_j^i > 0$, $l_i \in \mathbb{N}$, $c_j^i \in \mathcal{C}([0,\infty)\times\mathbb{R}^3)$, $0 \le c_j^i \le a$ on $[0,\infty)\times\mathbb{R}^3$, $j \in \{1,\dots,l\}$, $i \in \{1,\dots,5\}$, for some positive constant a.

Our main result is as follows.

Theorem 1.1. *Under the assumptions* (**H1**), *there exists $r > 0$ such that if $u_0, u_1 \in E$, $0 \le u_0, u_1$ on $\mathbb{R}^3$ and $\|u_0\| \le r$, $\|u_1\| \le r$, then the IVP* (1.1) *has at least one nonnegative solution $u \in \mathcal{C}^2([0,\infty)\times\mathbb{R}^3)$.*

The set up of the paper is as follows. In the next section we give some preliminary and auxiliary results which will be used to prove our main result. In section 3, we prove our main result. In section 4, we give an example.

2 Preliminary and auxiliary results

In what follows, $\mathcal{Q}$ will refer to a cone in a Banach space $(X, \|.\|)$.
The following Proposition 2.1 will be mainely used to prove Theorem 1.1.

Proposition 2.1. *[9, 7, 5] Let Ω be a subset of $\mathcal{Q}$ and U be a bounded open subset of $\mathcal{Q}$ with $0 \in U$. Assume that the mapping $T : \Omega \subset \mathcal{Q} \to X$ be such that $(I - T)$ is Lipschitz invertible with constant $\gamma > 0$, $S : \overline{U} \to X$ is a k-set contraction with $0 \le k < \gamma^{-1}$, and $S(\overline{U}) \subset (I - T)(\Omega)$.*
If

$$Sx \ne (I-T)(\lambda x) \quad \text{for all } x \in \partial U \bigcap \Omega,\ \lambda \ge 1 \quad \text{and} \quad \lambda x \in \Omega,$$

then the fixed point index $i_*\left(T + S, U \bigcap \Omega, \mathcal{Q}\right) = 1$.

Let $0 < B < A < 1$ and $q > 1$. Set

$$\begin{aligned} l &= \max\{1, p_j^i : j \in \{1, \dots, l_i\}, \quad i \in \{1, \dots, 5\}\}, \\ r_1 &= \max\left\{ r, a \sum_{i=1}^{5} \sum_{j=1}^{l_i} r^{p_j^i} \right\}. \end{aligned}$$

We choose $r > 1$ large enough so that

$$B(1-A)r^{6l+1} > (1+A)q\left(r + 5r_1 A\right). \tag{2.1}$$

We have a need of the following technical Lemma.

Lemma 2.2. *We have*

$$\begin{aligned} I(x) &= \left| \int_0^x (x - x_1)^2 dx_1 \right| \leq |x|^3, \\ &\textit{and} \\ J(x) &= \left| \int_0^x (x - x_1)^4 dx_1 \right| \leq |x|^5, \quad x \in \mathbb{R}. \end{aligned}$$

Proof. We distinguish two cases.

1. Let $x \geq 0$. Then, it is not hard to see

$$\begin{aligned} I(x) &= \int_0^x (x - x_1)^2 dx_1 \leq x^3, \\ &\textit{and} \\ J(x) &= \int_0^x (x - x_1)^4 dx1 \leq x^5. \end{aligned}$$

2. Let $x<0$. Then

$$\begin{aligned} I(x) &= -\int_0^x (x - x_1)^2 dx_1 = -\int_0^x (x^2 - 2xx_1 + x_1^2) dx_1 \\ &= -\int_0^x x^2 dx_1 + 2x \int_0^x x_1 dx_1 - \int_0^x x_1^2 dx_1 = -x^3 + x^3 - \frac{x^3}{3} \\ &= -\frac{x^3}{3} \leq -x^3 = |x|^3, \end{aligned}$$

and

$$\begin{aligned}
J(x) &= -\int_0^x (x-x_1)^4 dx_1 \\
&= -\int_0^x (x^4+4x^2x_1^2+x_1^4-4x^3x_1+2x^2x_1^2-4xx_1^3)dx_1 \\
&= -\int_0^x x^4 dx_1 - 4x^2\int_0^x x_1^2 dx_1 - \int_0^x x_1^4 dx_1 + 4x^3\int_0^x x_1 dx_1 - 2x^2\int_0^x x_1^2 dx_1 \\
&\quad +4x\int_0^x x_1^3 dx_1 = -x^5 - \frac{4}{3}x^5 - \frac{x^5}{5} + 2x^5 - \frac{2}{3}x^5 + x^5 \\
&= -\frac{x^5}{5} \le -x^5 = |x|^5.
\end{aligned}$$

This completes the proof.

□

Below, we suppose the follwing condition.

(H2) There exists a function $g \in \mathcal{C}([0,\infty)\times\mathbb{R}^3)$ such that, $g > 0$ on $[0,\infty)\times\mathbb{R}^3$, and

$$F(t,x,y,z) = 24(1+t+t^2)^2(1+|x|+x^2)(1+|y|+y^2)(1+|z|+z^2)$$

$$\times\Big|\int_0^t\int_0^x\int_0^y\int_0^z (1+I(x_1)+J(x_1))(1+I(y_1)+J(y_1))(1+I(z_1)+J(z_1))$$

$$\times g(t_1,x_1,y_1,z_1)dz_1dy_1dx_1dt_1\Big| \le A,$$

$(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3$, and

$$\int_2^3\int_2^3\int_2^3\int_2^3 g(t_1,x_1,y_1,z_1)dz_1dy_1dx_1dt_1 \ge B.$$

In the last section we will give an example for such function g. For $u \in E$, define

$$\begin{aligned}
h_1(t,x,y,z,u) &= \int_0^x\int_0^y\int_0^z (x-x_1)^4(y-y_1)^4(z-z_1)^4 u(t,x_1,y_1,z_1)dz_1dy_1dx_1, \\
h_2(t,x,y,z,u) &= \int_0^x\int_0^y\int_0^z (x-x_1)^4(y-y_1)^4(z-z_1)^4(u_0(x_1,y_1,z_1)+tu_1(x_1,y_1,z_1)) \\
&\quad dz_1dy_1dx_1, \\
h_3(t,x,y,z,u) &= \int_0^t\int_0^x\int_0^y\int_0^z (t-t_1)(x-x_1)^2(y-y_1)^4(z-z_1)^4 u(t_1,x_1,y_1,z_1) \\
&\quad dz_1dy_1dx_1dt_1,
\end{aligned}$$

$$h_4(t,x,y,z,u) = \int_0^t\int_0^x\int_0^y\int_0^z (t-t_1)(x-x_1)^4(y-y_1)^2(z-z_1)^4u(t_1,x_1,y_1,z_1)\, dz_1dy_1dx_1dt_1,$$

$$h_5(t,x,y,z,u) = \int_0^t\int_0^x\int_0^y\int_0^z (t-t_1)(x-x_1)^4(y-y_1)^4(z-z_1)^2u(t_1,x_1,y_1,z_1)\, dz_1dy_1dx_1dt_1,$$

$$\begin{aligned} h_6(t,x,y,z,u) &= \int_0^t\int_0^x\int_0^y\int_0^z (t-t_1)(x-x_1)^4(y-y_1)^4(z-z_1)^4 \\ &\quad \times f(t_1,x_1,y_1,z_1,u(t_1,x_1,y_1,z_1),u_t(t_1,x_1,y_1,z_1), \\ &\quad u_x(t_1,x_1,y_1,z_1),u_y(t_1,x_1,y_1,z_1) \\ &\quad u_z(t_1,x_1,y_1,z_1))dz_1dy_1dx_1dt_1, \quad (t,x,y,z)\in[0,\infty)\times\mathbb{R}^3. \end{aligned}$$

Lemma 2.3. *If $u\in E$ satisfies the equation*

$$\begin{aligned} H(t,x,y,z,u) &= h_1(t,x,y,z,u)-h_2(t,x,y,z,u)-12h_3(t,x,y,z,u) \\ &\quad -12h_4(t,x,y,z,u)-12h_5(t,x,y,z,u)-h_6(t,x,y,z,u)=0, \end{aligned} \tag{2.2}$$

$(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3$, then u is a solution of the IVP (1.1).

Proof. Let $u\in E$ satisfies (2.2). We differentiate the equation (2.2) twice in t, trice in x, y, z, and we obtain

$$\begin{aligned} & 24^3\int_0^x\int_0^y\int_0^z (x-x_1)(y-y_1)(z-z_1)u_{tt}(t_1,x_1,y_1,z_1)dz_1dy_1dx_1 \\ &\quad -24^3\int_0^y\int_0^z (y-y_1)(z-z_1)u(t,x,y_1,z_1)dz_1dy_1 \\ &\quad -24^3\int_0^x\int_0^z (x-x_1)(z-z_1)u(t,x_1,y,z_1)dz_1dx_1 \\ &\quad -24^3\int_0^x\int_0^y (x-x_1)(y-y_1)u(t,x_1,y_1,z)dy_1dx_1 \\ &= 24^3\int_0^x\int_0^y\int_0^z (x-x_1)(y-y_1)(z-z_1)f(t,z_1,y_1,z_1,u(t,x_1,y_1,z_1), \\ &\quad u_t(t,x_1,y_1,z_1),u_x(t,x_1,y_1,z_1),u_y(t,x_1,y_1,z_1),u_z(t,x,y,z))dz_1dy_1dx_1, \end{aligned}$$

$(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3$, which we differentiate twice in x, y, z and we arrive at

$$\begin{aligned} & u_{tt}(t,x,y,z)-u_{xx}(t,x,y,z)-u_{yy}(t,x,y,z)-u_{zz}(t,x,y,z) \\ = \quad & f(t,x,y,z,u(t,x,y,z),u_t(t,x,y,z),u_x(t,x,y,z), \\ & u_y(t,x,y,z),u_z(t,x,y,z)), \quad (t,x,y,z)\in[0,\infty)\times\mathbb{R}^3. \end{aligned}$$

Now, we put $t=0$ in (2.2) and we find

$$\int_0^x\int_0^y\int_0^z (x-x_1)^4(y-y_1)^4(z-z_1)^4(u(0,x_1,y_1,z_1)-u_0(x_1,y_1,z_1))dz_1dy_1dx_1=0,$$

$(x,y,z)\in\mathbb{R}^3$. We differentiate five times in x, y, z and we obtain

$$u(0,x,y,z)=u_0(x,y,z), \quad (x,y,z)\in\mathbb{R}^3.$$

Now, we differentiate (2.2) once in t, then we put $t=0$, and we get

$$\int_0^x\int_0^y\int_0^z (x-x_1)^4(y-y_1)^4(z-z_1)^4(u_t(0,x_1,y_1,z_1)-u_1(x_1,y_1,z_1))dz_1dy_1dx_1=0,$$

$(x,y,z)\in\mathbb{R}^3$, which we differentiate five times in x, y, z and we obtain

$$u_t(0,x,y,z)=u_1(x,y,z), \quad (x,y,z)\in\mathbb{R}^3.$$

Therefore u satisfies the IVP (1.1). This completes the proof. □

Lemma 2.4. *If $u\in E$ is a solution to the equation*

$$\begin{aligned} &\int_0^t\int_0^x\int_0^y\int_0^z (t-t_1)^2(x-x_1)^2(y-y_1)^2(z-z_1)^2 g(t_1,x_1,y_1,z_1) \\ &\times H(t_1,x_1,y_1,z_1,u(t_1,x_1,y_1,z_1))dz_1dy_1dx_1dt_1=0, \end{aligned} \tag{2.3}$$

$(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3$*, then u is a solution to the IVP* (1.1).

Proof. Let $u\in E$ be a solution to (2.3). We differentiate (2.3) trice in t, x, y, z, we obtain

$$g(t,x,y,z)H(t,x,y,z,u(t,x,y,z))=0, \quad (t,x,y,z)\in[0,\infty)\times\mathbb{R}^3,$$

whereupon

$$H(t,x,y,z,u(t,x,y,z))=0, \quad (t,x,y,z)\in[0,\infty)\times\mathbb{R}^3.$$

Hence, applying Lemma 2.3, we get the desired result. □

Lemma 2.5. *For $u \in E$, $\|u\| \leq r$, we have the following estimates*

$$\begin{aligned} |h_1(t,x,y,z,u)| &\leq rJ(x)J(y)J(z), \\ |h_2(t,x,y,z,u)| &\leq r(1+t)J(x)J(y)J(z), \\ |h_3(t,x,y,z,u)| &\leq rt^2 I(x)J(y)J(z), \\ |h_4(t,x,y,z,u)| &\leq rt^2 J(x)I(y)J(z), \\ |h_5(t,x,y,z,u)| &\leq rt^2 J(x)J(y)I(z), \\ |h_6(t,x,y,z,u)| &\leq a\sum_{i=1}^{5}\sum_{j=1}^{l_i} r^{p_j^i} t^2 J(x)J(y)J(z), \end{aligned}$$

$(t,x,y,z) \in [0,\infty) \times \mathbb{R}^3$.

Proof. Let $u \in E$ and $\|u\| \leq r$. Then, applying Lemma 2.2, we get

$$\begin{aligned} |h_1(t,x,y,z,u)| &= \left| \int_0^x \int_0^y \int_0^z (x-x_1)^4 (y-y_1)^4 (z-z_1)^4 u(t,x_1,y_1,z_1) dz_1 dy_1 dx_1 \right| \\ &\leq \left| \int_0^x \int_0^y \int_0^z (x-x_1)^4 (y-y_1)^4 (z-z_1)^4 |u(t,x_1,y_1,z_1)| dz_1 dy_1 dx_1 \right| \leq \\ &\leq r \left| \int_0^x (x-x_1)^4 dx_1 \right| \left| \int_0^y (y-y_1)^4 dy_1 \right| \left| \int_0^z (z-z_1)^4 dz_1 \right| \\ &= rJ(x)J(y)J(z), \quad (t,x,y,z) \in [0,\infty) \times \mathbb{R}^3. \end{aligned}$$

Now, applying (**H1**) and Lemma 2.2, we get

$$\begin{aligned}
|h_6(t,x,y,z,u)| &= \Big|\int_0^t\int_0^x\int_0^y\int_0^z (t-t_1)(x-x_1)^4(y-y_1)^4(z-z_1)^4 \\
&\quad \times f(t_1,x_1,y_1,z_1,u(t_1,x_1,y_1,z_1),u_t(t_1,x_1,y_1,z_1), \\
&\quad u_x(t_1,x_1,y_1,z_1),u_y(t_1,x_1,y_1,z_1) \\
&\quad u_z(t_1,x_1,y_1,z_1))dz_1dy_1dx_1dt_1\Big| \\
&\le \Big|\int_0^t\int_0^x\int_0^y\int_0^z (t-t_1)(x-x_1)^4(y-y_1)^4(z-z_1)^4 \\
&\quad \times |f(t_1,x_1,y_1,z_1,u(t_1,x_1,y_1,z_1),u_t(t_1,x_1,y_1,z_1), \\
&\quad u_x(t_1,x_1,y_1,z_1),u_y(t_1,x_1,y_1,z_1) \\
&\quad u_z(t_1,x_1,y_1,z_1))|dz_1dy_1dx_1dt_1\Big| \\
&\le \Big|\int_0^t\int_0^x\int_0^y\int_0^z (t-t_1)(x-x_1)^4(y-y_1)^4(z-z_1)^4 \\
&\quad \times\left(\sum_{i=1}^5\sum_{j=1}^{l_i} c_j^i(t_1,x_1,y_1,z_1)|u(t_1,x_1,y_1,z_1)|^{p_j^i}\right)dz_1dy_1dx_1dt_1 \\
&\le a\sum_{i=1}^5\sum_{j=1}^{l_i} r^{p_j^i}\int_0^t (t-t_1)dt_1\left|\int_0^x (x-x_1)^4dx_1\right| \\
&\quad \times\left|\int_0^y (y-y_1)^4dy_1\right|\left|\int_0^z (z-z_1)^4dz_1\right| \\
&\le a\sum_{i=1}^5\sum_{j=1}^{l_i} r^{p_j}t^2J(x)J(y)J(z),
\end{aligned}$$

$(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3$. This completes the proof. □

For $u \in E$, define the operators

$$\begin{aligned} H_i u(t,x,y,z) &= \int_0^t \int_0^x \int_0^y \int_0^z (t-t_1)^2 (x-x_1)^2 (y-y_1)^2 (z-z_1)^2 g(t_1,x_1,y_1,z_1) \\ &\quad \times h_i(t_1,x_1,y_1,z_1,u) dz_1 dy_1 dx_1 dt_1, \quad i \in \{1,2,6\}, \\ H_i u(t,x,y,z) &= 12 \int_0^t \int_0^x \int_0^y \int_0^z (t-t_1)^2 (x-x_1)^2 (y-y_1)^2 (z-z_1)^2 g(t_1,x_1,y_1,z_1) \\ &\quad \times h_i(t_1,x_1,y_1,z_1,u) dz_1 dy_1 dx_1 dt_1, \quad i \in \{3,4,5\}, \\ S_1 u(t,x,y,z) &= \sum_{j=2}^{5} H_j u(t,x,y,z), \end{aligned}$$

$(t,x,y,z) \in [0,\infty) \times \mathbb{R}^3$.

Lemma 2.6. *For $u \in E$ and $\|u\| \le r$, we have*

$$\|H_j u\| \le r_1 A,$$

$(t,x,y,z) \in [0,\infty) \times \mathbb{R}^3$, $i \in \{1,\dots,6\}$. *Moreover,* $\|H_1 u\| \le A\|u\|$.

Proof. Let $i \in \{1,\dots,6\}$ be fixed. By Lemma 2.5, we get

$$|h_i(t,x,y,z)| \le r_1(1+t+t^2)(1+I(x)+J(x))(1+I(y)+J(y))(1+I(z)+J(z)),$$

$(t,x,y,z) \in [0,\infty) \times \mathbb{R}^3$, $i \in \{1,\dots,6\}$. Hence, applying the condition (**H2**), we get

$$\begin{aligned} |H_i u(t,x,y,z)| &\le 12 r_1 \Big| \int_0^t \int_0^x \int_0^y \int_0^z (t-t_1)^2 (x-x_1)^2 (y-y_1)^2 (z-z_1)^2 \\ &\quad \times (1+t_1+t_1^2)(1+I(x_1)+J(x_1))(1+I(y_1)+J(y_1)) \\ &\quad \times (1+I(z_1)+J(z_1)) g(t_1,x_1,y_1,z_1) dz_1 dy_1 dx_1 dt_1 \Big| \\ &\le r_1 A, \quad (t,x,y,z) \in [0,\infty) \times \mathbb{R}^3, \end{aligned}$$

and

$$\begin{aligned} \left| \frac{\partial}{\partial t} H_i u(t,x,y,z) \right| &\le 24 r_1 \Big| \int_0^t \int_0^x \int_0^y \int_0^z (t-t_1)(x-x_1)^2 (y-y_1)^2 (z-z_1)^2 \\ &\quad \times (1+t_1+t_1^2)(1+I(x_1)+J(x_1))(1+I(y_1)+J(y_1)) \\ &\quad \times (1+I(z_1)+J(z_1)) g(t_1,x_1,y_1,z_1) dz_1 dy_1 dx_1 dt_1 \Big| \\ &\le r_1 A, \quad (t,x,y,z) \in [0,\infty) \times \mathbb{R}^3, \end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial^2}{\partial t^2}H_i u(t,x,y,z)\right| &\leq 24r_1\left|\int_0^t\int_0^x\int_0^y\int_0^z(x-x_1)^2(y-y_1)^2(z-z_1)^2\right.\\ &\quad\times(1+t_1+t_1^2)(1+I(x_1)+J(x_1))(1+I(y_1)+J(y_1))\\ &\quad\left.\times(1+I(z_1)+J(z_1))g(t_1,x_1,y_1,z_1)dz_1dy_1dx_1dt_1\right|\\ &\leq r_1A,\quad (t,x,y,z)\in[0,\infty)\times\mathbb{R}^3,\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial}{\partial x}H_i u(t,x,y,z)\right| &\leq 24r_1\left|\int_0^t\int_0^x\int_0^y\int_0^z(t-t_1)^2(x-x_1)(y-y_1)^2(z-z_1)^2\right.\\ &\quad\times(1+t_1+t_1^2)(1+I(x_1)+J(x_1))(1+I(y_1)+J(y_1))\\ &\quad\left.\times(1+I(z_1)+J(z_1))g(t_1,x_1,y_1,z_1)dz_1dy_1dx_1dt_1\right|\\ &\leq r_1A,\quad (t,x,y,z)\in[0,\infty)\times\mathbb{R}^3,\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial^2}{\partial x^2}H_i u(t,x,y,z)\right| &\leq 24r_1\left|\int_0^t\int_0^x\int_0^y\int_0^z(t-t_1)^2(y-y_1)^2(z-z_1)^2\right.\\ &\quad\times(1+t_1+t_1^2)(1+I(x_1)+J(x_1))(1+I(y_1)+J(y_1))\\ &\quad\left.\times(1+I(z_1)+J(z_1))g(t_1,x_1,y_1,z_1)dz_1dy_1dx_1dt_1\right|\\ &\leq r_1A,\quad (t,x,y,z)\in[0,\infty)\times\mathbb{R}^3,\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial}{\partial y}H_i u(t,x,y,z)\right| &\leq 24r_1\left|\int_0^t\int_0^x\int_0^y\int_0^z(t-t_1)^2(x-x_1)^2(y-y_1)(z-z_1)^2\right.\\ &\quad\times(1+t_1+t_1^2)(1+I(x_1)+J(x_1))(1+I(y_1)+J(y_1))\\ &\quad\left.\times(1+I(z_1)+J(z_1))g(t_1,x_1,y_1,z_1)dz_1dy_1dx_1dt_1\right|\\ &\leq r_1A,\quad (t,x,y,z)\in[0,\infty)\times\mathbb{R}^3,\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial^2}{\partial y^2}H_iu(t,x,y,z)\right| &\leq 24r_1\left|\int_0^t\int_0^x\int_0^y\int_0^z(t-t_1)^2(x-x_1)^2(z-z_1)^2\right.\\ &\quad\times(1+t_1+t_1^2)(1+I(x_1)+J(x_1))(1+I(y_1)+J(y_1))\\ &\quad\left.\times(1+I(z_1)+J(z_1))g(t_1,x_1,y_1,z_1)dz_1dy_1dx_1dt_1\right|\\ &\leq r_1A,\quad (t,x,y,z)\in[0,\infty)\times\mathbb{R}^3,\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial}{\partial z}H_iu(t,x,y,z)\right| &\leq 24r_1\left|\int_0^t\int_0^x\int_0^y\int_0^z(t-t_1)^2(x-x_1)^2(y-y_1)^2(z-z_1)\right.\\ &\quad\times(1+t_1+t_1^2)(1+I(x_1)+J(x_1))(1+I(y_1)+J(y_1))\\ &\quad\left.\times(1+I(z_1)+J(z_1))g(t_1,x_1,y_1,z_1)dz_1dy_1dx_1dt_1\right|\\ &\leq r_1A,\quad (t,x,y,z)\in[0,\infty)\times\mathbb{R}^3,\end{aligned}$$

and

$$\begin{aligned}\left|\frac{\partial^2}{\partial z^2}H_iu(t,x,y,z)\right| &\leq 24r_1\left|\int_0^t\int_0^x\int_0^y\int_0^z(t-t_1)^2(x-x_1)^2(y-y_1)^2\right.\\ &\quad\times(1+t_1+t_1^2)(1+I(x_1)+J(x_1))(1+I(y_1)+J(y_1))\\ &\quad\left.\times(1+I(z_1)+J(z_1))g(t_1,x_1,y_1,z_1)dz_1dy_1dx_1dt_1\right|\\ &\leq r_1A,\quad (t,x,y,z)\in[0,\infty)\times\mathbb{R}^3.\end{aligned}$$

Consequently

$$\|H_iu\|\leq r_1A.$$

This completes the pfroof. □

By Lemma 2.4, it follows that if $u\in E$ is a solution of the equation $(H_1-S_1)u=0$, then u is a solution to the IVP (1.1). Moreover, if $u\in E$ and $u\geq 0$ on $[0,\infty)\times\mathbb{R}^3$, then $S_1u\geq 0$ on $[0,\infty)\times\mathbb{R}^3$. Next, if $u\in E$ and $\|u\|\leq r$, by Lemma 2.6, we have

$$\|S_1u\|\leq 5r_1A.$$

Lemma 2.7. *If $u\in E$, $u\geq 0$ on $[0,\infty)\times\mathbb{R}^3$ and*

$$\inf_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3}u(t,x,y,z)\geq\frac{1}{q}\|u\|,$$

then

$$\sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} H_3u(t,x,y,z) \geq \frac{r^{6l}B}{q}\|u\|.$$

Proof. We have

$$\begin{aligned}
h_3(2,2,2,2,u) &= \int_0^2\int_0^2\int_0^2\int_0^2 (2-t_1)(2-x_1)^2(2-y_1)^4(2-z_1)^4 \\
&\quad \times u(t_1,x_1,y_1,z_1)dz_1dy_1dx_1dt_1 \\
&\geq \frac{1}{q}\|u\|\int_0^1\int_0^1\int_0^1\int_0^1 (2-t_1)(2-x_1)^2(2-y_1)^4(2-z_1)^4 \\
&\quad dz_1dy_1dx_1dt_1 \\
&\geq \frac{1}{q}\|u\|.
\end{aligned}$$

Hence,

$$\begin{aligned}
\sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} H_3u(t,x,y,z) &\geq \int_2^{r^l+3}\int_2^{r^l+3}\int_2^{r^l+3}\int_2^{r^l+3} (r^l+3-t_1)^2(r^l+3-x_1)^2(r^l+3-y_1)^2 \\
&\quad \times(r^l+3-z_1)^2 g(t_1,x_1,y_1,z_1)h_3(t_1,x_1,y_1,z_1,u) \\
&\quad dz_1dy_1dx_1dt_1 \\
&\geq h_3(2,2,2,2,u) \\
&\quad \times\int_2^3\int_2^3\int_2^3\int_2^3 (r^l+3-t_1)^2(r^l+3-x_1)^2(r^l+3-y_1)^2(r^l+3-z_1)^2 \\
&\quad \times g(t_1,x_1,y_1,z_1)dz_1dy_1dx_1dt_1 \\
&\geq \frac{1}{q}\|u\|\int_2^3\int_2^3\int_2^3\int_2^3 (r^l+3-t_1)^2(r^l+3-x_1)^2(r^l+3-y_1)^2 \\
&\quad \times(r^l+3-z_1)^2 g(t_1,x_1,y_1,z_1)dz_1dy_1dx_1dt_1 \\
&\geq \frac{r^{6l}}{q}\|u\|\int_2^3\int_2^3\int_2^3\int_2^3 g(t_1,x_1,y_1,z_1) \\
&\quad dz_1dy_1dx_1dt_1 \\
&\geq \frac{B}{q}r^{6l}\|u\|.
\end{aligned}$$

This completes the proof. □

3 Proof of the Main Result

For $u \in E$, define the operators

$$Tu = (1-\epsilon)u - \epsilon H_1 u, \quad Su = \epsilon u + \epsilon S_1 u.$$

Let

$$R = \frac{r + 5r_1 A}{1 - A}, \quad \widetilde{\mathcal{P}} = \{u \in E : u \geq 0\}.$$

With $\mathcal{P}$, we denote the set of all equi-continuous families in $\widetilde{\mathcal{P}}$. Set

$$\begin{aligned} \Omega &= \{u \in \mathcal{P} : \|u\| \leq R\}, \\ U &= \{u \in \mathcal{P} : \|u\| \leq r, \inf_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} u(t,x,y,z) \geq \frac{1}{q}\|u\|\}. \end{aligned}$$

Note that any fixed point $u \in \mathcal{P}$ of the operator $T + S$ is a solution of the IVP (1.1).

1. For $u, v \in \Omega$, we have

$$\begin{aligned} \|(I-T)(u-v)\| &\geq \|\epsilon(u-v) + \epsilon H_1(u-v)\| \\ &\geq \epsilon\|u-v\| - \epsilon\|H_1(u-v)\| \\ &\geq \epsilon\|u-v\| - \epsilon A\|u-v\| \\ &= \epsilon(1-A)\|u-v\|, \end{aligned}$$

   and

$$\begin{aligned} \|(I-T)(u-v)\| &\leq \epsilon\|u-v\| + \epsilon\|H_1(u-v)\| \\ &\leq \epsilon(1+A)\|u-v\|. \end{aligned}$$

   Thus, $I - T : \Omega \to E$ is Lipschitz invertible with constant $\gamma = \frac{1}{\epsilon(1-A)}$.

2. For $u \in \overline{U}$, we have

$$\begin{aligned} \|Su\| &= \|\epsilon u + \epsilon S_1 u\| \\ &\leq \epsilon\|u\| + \epsilon\|S_1 u\| \\ &\leq \epsilon r + 5\epsilon r_1 A. \end{aligned}$$

   Therefore $S : \overline{U} \to E$ is uniformly bounded. Since $S : \overline{U} \to E$ is a continuous operator, we conclude that $S(\overline{U})$ is equi-continuous and $S : \overline{U} \to E$ is relatively compact. Consequently $S : \overline{U} \to E$ is a 0-set contraction.

3. Let $v \in \overline{U}$ be arbitrarily chosen, $v \not\equiv 0$. Set $u = (I-T)^{-1}Sv$. Then

$$\begin{aligned} \|(I-T)^{-1}Sv\| &\leq \frac{1}{\epsilon(1-A)}\|Sv\| \\ &\leq \frac{1}{1-A}\left(\|u\| + \|S_1 v\|\right) \\ &\leq \frac{1}{1-A}(r + 5r_1 A). \end{aligned}$$

   Assume that $u \leq 0$ on $[0,\infty)(\times\mathbb{R}^3$. Then

$$0 \geq (I-T)u = \epsilon u + \epsilon H_1 u = Sv \geq 0,$$

   which is a contradiction. Therefore $u \in \Omega$ and $S(\overline{U}) \subset (I-T)(\Omega)$.

4. Assume that there are $v \in \partial U$ and $\lambda \geq 1$ so that

$$Sv = (I-T)(\lambda v) \quad \text{and} \quad \lambda v \in \Omega.$$

   Then

$$\begin{aligned} R &\geq \|\lambda v\| \\ &= \|(I-T)^{-1}Sv\| \\ &\geq \frac{1}{\epsilon(1+A)}\|Sv\| \\ &\geq \frac{1}{1+A}\sup_{(t,x,y,z)\in[0,\infty)\times\mathbb{R}^3} H_3 v(t,x,y,z) \\ &\geq \frac{B}{(1+A)q} r^{6l}\|v\| \\ &= \frac{B}{(1+A)q} r^{6l+1}. \end{aligned}$$

   Hence,

$$\frac{r + 5r_1 A}{1-A} \geq \frac{Br^{6l+1}}{(1+A)q},$$

   or

$$B(1-A)r^{6l+1} \leq (1+A)q(r + 5r_1 A).$$

   This is a contradiction.

Hence and Proposition 2.1, we conclude that the operator $T + S$ has a fixed point in $\overline{U}$. This completes the proof.

4 Example

Below, we will illustrate our main result. Let

$$h(x) = \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}}, \quad l(s) = \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}}, \quad s \in \mathbb{R}.$$

Then

$$\begin{aligned} h'(s) &= \frac{22\sqrt{2}s^{10}(1 - s^{22})}{(1 - s^{11}\sqrt{2} + s^{22})(1 + s^{11}\sqrt{2} + s^{22})}, \\ l'(s) &= \frac{11\sqrt{2}s^{10}(1 + s^{20})}{1 + s^{40}}, \quad s \in \mathbb{R}. \end{aligned}$$

Therefore

$$\begin{aligned} -\infty &< \lim_{s \to \pm\infty} (1 + s + s^2)^2 h(s) < \infty, \\ -\infty &< \lim_{s \to \pm\infty} (1 + s + s^2)^2 l(s) < \infty. \end{aligned}$$

Hence, there exists a positive constant C_1 so that

$$\begin{aligned} (1 + s + s^2)^2 \left(\frac{1}{44\sqrt{2}} \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}} + \frac{1}{22\sqrt{2}} \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}} \right) &\leq C_1, \\ (1 + s + s^2)^2 \left(\frac{1}{44\sqrt{2}} \log \frac{1 + s^{11}\sqrt{2} + s^{22}}{1 - s^{11}\sqrt{2} + s^{22}} + \frac{1}{22\sqrt{2}} \arctan \frac{s^{11}\sqrt{2}}{1 - s^{22}} \right) &\leq C_1, \end{aligned}$$

$t \in [0, \infty)$, $s \in \mathbb{R}$. Note that by [3](pp. 707, Integral 79), we have

$$\int \frac{dz}{1 + z^4} = \frac{1}{4\sqrt{2}} \log \frac{1 + z\sqrt{2} + z^2}{1 - z\sqrt{2} + z^2} + \frac{1}{2\sqrt{2}} \arctan \frac{z\sqrt{2}}{1 - z^2}.$$

Let

$$Q(s) = \frac{s^{10}}{(1 + s^{44})(1 + s + s^2)^2(1 + I(s) + J(s))^2}, \quad s \in \mathbb{R},$$

and

$$g(t, x, y, z) = Q(t)Q(x)Q(y)Q(z), \quad t \in [0, \infty), \quad x, y, z \in \mathbb{R}.$$

Then there exist constants $A > B > 0$ so that

$$\begin{aligned} & 24(1 + t + t^2)^2(1 + |x| + x^2)(1 + |y| + y^2)(1 + |z| + z^2) \\ & \times \left| \int_0^t \int_0^x \int_0^y \int_0^z (1 + I(x_1) + J(x_1))(1 + I(y_1) + J(y_1))(1 + I(z_1) + J(z_1)) \right. \\ & \left. \times g(t_1, x_1, y_1, z_1) dz_1 dy_1 dx_1 dt_1 \right| \leq A, \end{aligned}$$

$(t, x, y, z) \in [0, \infty) \times \mathbb{R}^3$, and

$$\int_2^3 \int_2^3 \int_2^3 \int_2^3 g(t_1, x_1, y_1, z_1) dz_1 dy_1 dx_1 dt_1 \geq B.$$

Therefore $(H3)$ holds. Now, consider the IVP

$$(4.1) \quad \begin{aligned} u_{tt} - u_{xx} - u_{yy} - u_{zz} &= \tfrac{1}{100(1+x^2+y^2+z^2)} u^{12}, \quad (t, x_1, x_2) \in (0, \infty) \times \mathbb{R}^3, \\ u(0, x, y, z) &= \tfrac{r}{150(1+x^2+y^2)}, \quad (x, y, z) \in \mathbb{R}^3, \\ u_t(0, x, y, z) &= \tfrac{r}{200(1+x^2)^4(1+y^4)^8}, \quad (x, y, z) \in \mathbb{R}^3, \end{aligned}$$

where $r > 1$ so that (2.1) holds. We have (**H1**) and (**H2**) hold.

Hence, by Theorem 1.1, it follows that the IVP (4.1) has at least one nonnegative solution $u \in \mathcal{C}^2([0, \infty) \times \mathbb{R}^2)$.

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