Existence and symmetry results for some overdetermined free boundary problems

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Abstract. In this paper, we prove that a domain which verifies some integral inequality is either (strictly) contained in the solution of some free boundary problem, or it coincides with an N-ball. We also present new overdetermined value problems which have an N-ball as a solution. To reach our results, we use an integral identity which involves the domain derivative of the solution of the corresponding Dirichlet problem.

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1 Introduction

Assuming throughout that: $D \subset \mathbb{R}^N \quad (N \geq 2)$ is a bounded ball which contains all the domains we use. If ω is an open subset of D, let ν be the outward normal to $\partial\omega$ and let $|\partial\omega|$ (respectively $|\omega|$) be the perimeter (respectively the volume) of ω. Let f be a positive function belonging to $L^2(\mathbb{R}^N)$ and having a compact support K with nonempty interior. Denote by C the convex hull of K.

Consider the following overdetermined boundary value problems.

$$\mathcal{QS}(f,k)\begin{cases} -\Delta u_\Omega = f \quad \text{in } \Omega\,, \\ u_\Omega = 0 \text{ on } \partial\Omega\,, \\ -\frac{\partial u_\Omega}{\partial\nu} = k = \text{cst. on } \partial\Omega \end{cases}$$

and

$$\mathcal{P}(f,c)\begin{cases} -\Delta u_\Omega = f \quad \text{in } \Omega\,, u_\Omega = 0 \text{ on } \partial\Omega \quad \}P(\Omega,f)\,, \\ -\Delta v_\Omega = u_\Omega \quad \text{in } \Omega\,, v_\Omega = 0 \text{ on } \partial\Omega \quad \}P(\Omega,u_\Omega)\,, \\ \frac{\partial u_\Omega}{\partial\nu}\frac{\partial v_\Omega}{\partial\nu} = c = \text{cst. on } \partial\Omega. \end{cases}$$

Notice that since u_Ω (resp. v_Ω) vanishes on $\partial\Omega$ then $-\frac{\partial u_\Omega}{\partial\nu} = |\nabla u_\Omega|$ $(-\frac{\partial v_\Omega}{\partial\nu} = |\nabla v_\Omega|)$.

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Remark 1.1. The second problem is equivalent to the following biharmonic one:

$$\begin{cases} \Delta^2 v_\Omega = f & \text{in } \Omega\,, \\ v_\Omega = \Delta v_\Omega = 0 & \text{on } \partial\Omega\,, \\ |\nabla u_\Omega||\nabla v_\Omega| = c & \text{on } \partial\Omega. \end{cases}$$

The problem $\mathcal{QS}(f,k)$ is called the quadrature surfaces free boundary problem and arises in many areas of physics (free streamlines, jets, Hele-show flows, electromagnetic shaping, gravitational problems, etc.) It has been intensively studied from different points of view, by several authors. For more details about the methods used for solving this problem see the [21, Introduction]. Imposing boundary conditions for both u_Ω and $|\nabla u_\Omega|$ on $\partial\Omega$ makes problem $\mathcal{QS}(f,k)$ *overdetermined*, so that, in general without any assumptions on data this problem has no solution. Gustafsson and Shahgholian [21] conclude their paper by giving [21, Theorem 4.7] the following sufficient condition: If Supp$f \subset B_R$ and if $\int_{B_R} f(x)dx > (\frac{6^N Nc}{3R}|B_R|)$ with $B_{3R} \subset \Omega_u$ (B_R being some ball of radius R) then $\mathcal{QS}(f,k)$ has a solution. The method used by Gustafsson and Shahgholian goes back to K. Friedrichs [18], or even to T. Carleman [12], and was considerably developed by H. W. Alt and L. A. Caffarelli [1]. Recently, by combining the maximum principle to the compatibility condition of the Neumann problem, Barkatou et al. [3] gave, $|\nabla u_C| > k$ on ∂C as a sufficient condition of existence for $Q_S(f,k)$. Later, Barkatou [2] showed that this problem admits a solution if and only if the condition $\int_C f(x)dx > k|\partial C|$ is valid.

In 1971, Serrin [34] proved that if Problem $\mathcal{QS}(1,k)$ has a solution $u_\Omega \in C^2(\overline{\Omega})$ then Ω must be an N-ball and u_Ω is radially symmetric. The method used by Serrin combines the maximum principle together with the device of moving planes [20] to a critical position and then showing that the solution is symmetric about the limiting plane. In the same year, Weinberger [36] gave a simplified proof of this problem. His strategy of proof consists first in showing that $|\nabla u|^2 + \frac{2}{N}u = k^2$ in Ω and to derive a radial symmetry from this. A method which does not need the maximum principle was developed by Payne and Schaefer [30]. They developed integral identities which are equivalent to the problems they considered and led to the conclusion that the domain Ω must be an N-ball. Another technique which does not involve the maximum principle was introduced by Brock and Henrot [8] (see also [14] or [5]). It consists of using the domain derivative to get the same conclusion. For more details about the symmetry results see [17, Introduction] and the references therein. Fragalà et al [17], obtained their symmetry result by combining the maximum principle for a suitable P-function with some geometric arguments involving the mean curvature of $\partial\Omega$.
The problem $\mathcal{P}(1,c)$ arises from a variational problem in Probability [19, 25]. Fromm and McDonald [19] related this problem to the fundamental result of Serrin. Then, using the moving plane method combining with Serrin's boundary point Lemma, they showed that if this problem admits a solution Ω then it must be an N-ball. Huang and Miller [23] established the variational formulas for maximizing the functionals (they considered) over C^k domains with a volume constraint and obtained the same symmetry result for their maximizers.

The problem $\mathcal{P}(f,c)$ was first studied in [24]. In general, without any assumptions about data, the problem $\mathcal{P}(f,c)$ has no solution. In [24], by using the maximum principle, the authors showed that if $|\nabla u_C||\nabla v_C| > c$ on ∂C, then this problem has a solution. The aim here is to give a sufficient condition of existence of the problem $\mathcal{P}(f,c)$ better than the earlier (see Section 3.).

In the present paper, we will use some integral inequality on Ω verified by f and k (respectively c) to prove that the domain Ω is either (strictly) contained in a solution of the problem $\mathcal{QS}(f,\mu)$ (respectively $\mathcal{P}(f,\mu)$) for some constant μ or Ω is an N-ball. Next, we will use integral identities involving the domain derivative of the solution of the Dirichlet problem in order to show that the solution of a new overdetermined value problem (if it exists) must be an N-ball. We will also show that stationary point of some functionals of a domain are balls.

2 Preliminaries

Definition 2.1. Let K_1 and K_2 be two compact subsets of D. We call a Hausdorff distance of K_1 and K_2 (or briefly $d_H(K_1,K_2)$), the following positive number:

$$d_H(K_1,K_2) = \max\left[\rho(K_1,K_2),\rho(K_2,K_1)\right],$$

where $\rho(K_i,K_j) = \max_{x\in K_i} d(x,K_j)$, $i,j=1,2$, and $d(x,K_j) = \min_{y\in K_j}|x-y|$.

Definition 2.2. Let ω_n be a sequence of open subsets of D and let ω be an open subset of D. Let K_n and K be their complements in $\bar{D}$. We say that the sequence ω_n converges in the Hausdorff sense, to ω (or briefly $\omega_n \xrightarrow{H} \omega$) if

$$\lim_{n\to+\infty} d_H(K_n,K) = 0.$$

Definition 2.3. Let $\{\omega_n,\omega\}$ be a sequence of open subsets of D. We say that the sequence ω_n converges in the compact sense, to ω (or briefly $\omega_n \xrightarrow{K} \omega$) if

- every compact subset of ω is included in ω_n, for n *large enough*, and
- every compact subset of $\bar{\omega}^c$ is included in $\overline{\omega}_n^c$, for n *large enough.*

Definition 2.4. Let $\{\omega_n,\omega\}$ be a sequence of open subsets of D. We say that the sequence ω_n converges in the sense of characteristic functions, to ω (or briefly $\omega_n \xrightarrow{L} \omega$) if χ_{ω_n} converges to χ_ω in $L^p_{\text{loc}}(\mathbb{R}^N)$, $p \neq \infty$, (χ_ω is the characteristic function of ω).

Definition 2.5. [2] Let C be a compact convex set, the bounded domain ω satisfies C-GNP if

1. $\omega \supset \text{int}(C)$,
2. $\partial\omega \setminus C$ is locally Lipschitz,
3. for any $c \in \partial C$ there is an outward normal ray Δ_c such that $\Delta_c \cap \omega$ is connected, and
4. for every $x \in \partial\omega \setminus C$ the inward normal ray to ω (if exists) meets C.

Remark 2.6. If Ω satisfies the C-GNP and C has a nonempty interior, then Ω is connected.

Put

$$\mathcal{O}_C = \{\omega \subset D : \omega \quad \text{satisfies } C - \text{GNP}\}\,.$$

Theorem 2.1. *If $\omega_n \in \mathcal{O}_C$, then there exist an open subset $\omega \subset D$ and a subsequence (again denoted by ω_n) such that (i) $\omega_n \xrightarrow{H} \omega$, (ii) $\omega_n \xrightarrow{K} \omega$, (iii) χ_{ω_n} converges to χ_ω in $L^1(D)$ and (iv) $\omega \in \mathcal{O}_C$. Furthermore, the assertions (i), (ii) and (iii) are equivalent.*

Barkatou proved this theorem [2, Theorem 3.1] and the equivalence between (i), (ii) and (iii) [2, Propositions 3.4, 3.5, 3.6, 3.7 and 3.8].

Proposition 2.2. *Let $\{\omega_n, \omega\} \subset \mathcal{O}_C$ such that $\omega_n \xrightarrow{H} \omega$. Let u_n and u_ω be respectively the solutions of $P(\omega_n)$ and $P(\omega)$. Then u_n converges strongly in $H_0^1(D)$ to u_ω (u_n and u_ω are extended by zero in D).*

This proposition was proven for $N = 2$ or 3 [2, Theorem 4.3]).

Definition 2.7. Let C be a convex set. We say that an open subset ω has the C-SP, if

1. $\omega \supset \text{int}(C)$,
2. $\partial\omega \setminus C$ is locally Lipschitz,
3. for any $c \in \partial C$ there is an outward normal ray Δ_c such that $\Delta_c \cap \omega$ is connected, and
4. for all $x \in \partial\omega \setminus C \quad K_x \cap \omega = \varnothing$, where K_x is the closed cone defined by

$$\left\{y \in \mathbb{R}^N : (y - x).(z - x) \le 0\,,\ \text{for all } z \in C\right\}.$$

Remark 2.8. K_x is the normal cone to the convex hull of C and $\{x\}$.

Proposition 2.3. *[2, Proposition 2.3] ω has the C-GNP if and only if ω satisfies the C-SP.*

Proposition 2.4. *[10, Theorem 3.5] Let v_n and v_ω be respectively the solutions of the Dirichlet problems $P(\omega_n, g_n)$ and $P(\omega, g)$. If g_n converges strongly in $H^{-1}(D)$ to g then v_n converges strongly in $H_0^1(D)$ to v_ω (v_n and v_ω are extended by zero in D).*

Lemma 2.5. *[9, 32] Let ω_n be a sequence of open and bounded subsets of D. There exist a subsequence (again denoted by ω_n) and some open subset ω of D such that*

1. *ω_n converges to ω in the Hausdorff sense, and*
2. *$|\omega| \le \liminf_{n\to\infty} |\omega_n|$.*

Theorem 2.6. *[2] $\mathcal{QS}(f, k)$ has a solution if and only if $\int_C f dx > k|\partial C|$.*

Lemma 2.7. *Let $u_\Omega \in C^{2,\alpha}(\bar{\Omega})$ be the solution of $P(\Omega, N)$. Then Ω is an N-ball if and only if $|\nabla u_\Omega(x)| = \frac{1}{H_\Omega(x)}$, for every $x \in \partial\Omega$.*

For the proof of this lemma, see Theorem 2.4 [27].

Theorem 2.8. *Let u_Ω (respectively v_Ω) be the solution of $P(\Omega, 1)$ (respectively $P(\Omega, u_\Omega)$). If one of the following conditions is satisfied, then Ω is an N-ball.*

1. $|\nabla v_\Omega| = c$ *on* $\partial\Omega$.
2. $|\nabla v_\Omega| = cx.\nu$ *on* $\partial\Omega$.
3. $|\nabla v_\Omega| = c|\nabla u_\Omega|$ *on* $\partial\Omega$.

For the proof of this theorem, see [31].

As we use the standard tool of the domain derivative [35] to prove many of the propositions we state here, we begin by recalling its definition.

Suppose that the open ω is of class C^2. Consider a deformation field $V \in C^2(\mathbb{R}^N; \mathbb{R}^N)$ and set $\omega_t = \omega + tV(\omega)$, $t > 0$. The application $Id + tV$ (a perturbation of the identity) is a Lipschitz diffeomorphism for t sufficiently small and, by definition, the derivative of J at ω in the direction V is

$$dJ(\omega, V) = \lim_{t \to 0} \frac{J(\omega_t) - J(\omega)}{t}.$$

As the functional J depends on the domain ω through the solution of some Dirichlet problem, we need to define the domain derivative u_ω' of u_ω:

$$u_\omega' = \lim_{t \to 0} \frac{u_{\omega_t} - u_\omega}{t}.$$

$$\begin{cases} -\Delta u_\omega' = 0 & \text{in } \omega \\ u_\omega' = -\frac{\partial u_\omega}{\partial \nu} V.\nu & \text{on } \partial\omega. \end{cases}$$

The domain derivative v_ω' of v_ω (solution of $P(\omega, u_\omega)$) is the solution of:

$$\begin{cases} -\Delta v_\omega' = u_\omega' & \text{in } \omega \\ u_\omega' = -\frac{\partial u_\omega}{\partial \nu} V.\nu & \text{on } \partial\omega. \end{cases}$$

Now, to compute the derivative of the functionals we consider below, recall the following:

1. The domain derivative of the volume is
$$\int_{\partial\omega} V.\nu d\sigma.$$
2. The domain derivative of the perimeter is
$$\int_{\partial\omega} (N-1) H_\omega V.\nu d\sigma,$$
H_ω being the mean curvature of $\partial\omega$.
3. Suppose that $u_\omega \in H_0^1(D)$ and ω is of class C^2, then

(a) If $F(\omega) = \int_\omega u_\omega^2 dx$, then

$$dF(\omega, V) = 2\int_\omega u_\omega u_\omega^{'} dx.$$

But $v_\omega \in H_0^1(D)$ and $-\Delta v_\omega = u_\omega$ in ω, so by Green's formula we obtain

$$dF(\omega, V) = 2\int_{\partial\omega} |\nabla u_\omega||\nabla v_\omega| V.\nu d\sigma.$$

(b) If $G(\omega) = \int_\omega |\nabla u_\omega|^2 dx$, then by Hadamard's formula

$$dG(\omega, V) = \int_{\partial\omega} |\nabla u_\omega|^2 V.\nu d\sigma.$$

Since the set ω satisfies the C-GNP, we ask the deformed set ω_t to satisfy the same property (for t sufficiently small). The aim in the sequel is to prove that the C-GNP is stable by small deformation.
ω having the C-GNP, by Proposition 2.3, it satisfies the C-SP. Then

$$\text{for all } x \in \partial\omega \setminus C : K_x \cap \omega = \varnothing .$$

For t sufficiently small, let $\omega_t = \omega + tV(\omega)$ be the deformation of ω in the direction V. Let $x_t \in \partial\omega_t$. There exists $x \in \partial\omega$ such that $x_t = x + tV(x)$. Using the definition of K_{x_t} and the equation above, we get (for t small enough and for every displacement V):

$$\text{for all } x_t \in \partial\omega_t \setminus C : K_{x_t} \cap \omega_t = \varnothing ,$$

which means that ω_t satisfies the C-SP (and so the C-GNP) for every direction V when t is sufficiently small. In fact, suppose, by contradiction, there exists $x_t \in \partial\omega_t \setminus C$ such that $K_{x_t} \cap \omega_t \neq \varnothing$. Let $y_t \in K_{x_t} \cap \omega_t$, there exists $y \in \omega$, $y = y_t - tV(y)$ such that:

$$\forall c \in C, \quad (y_t - x_t).(c - x_t) \leq 0.$$

Show that $y \in K_x$:

$$\begin{aligned}(y - x).(c - x) &= (y_t - tV(y) - x_t + tV(x)).(c - x_t + tV(x)) \\ &= (y_t - x_t + t(V(y) - V(x))).(c - x_t + tV(x)) \\ &= (y_t - x_t).(c - x_t) + \epsilon(t),\end{aligned}$$

where $\epsilon(t) = t(y_t - x_t).V(x) + t(V(y) - V(x)).(c - x_t) + t^2(V(y) - V(x)).V(x)$ which, as t, tends to 0. Obtaining the contradiction.

3 Existence and symmetry for $\mathcal{QS}(f, k)$ and $\mathcal{P}(f, c)$

Suppose there exists Ω a solution of $\mathcal{P}(f, c)$. Then by Cauchy-Schwarz's inequality

$$\sqrt{c}|\partial\Omega| = \int_{\partial\Omega} \sqrt{|\nabla u_\Omega||\nabla v_\Omega|} \leq (\int_{\partial\Omega} |\nabla u_\Omega|)^{\frac{1}{2}} (\int_{\partial\Omega} |\nabla v_\Omega|)^{\frac{1}{2}}.$$

By using Green's formula,

$$c|\partial\Omega|^2 \le (\int_\Omega f)(\int_\Omega u_\Omega).$$

Now $C \subset \Omega$ and C is convex, then $|\partial C| < |\partial\Omega|$, and so,

$$c|\partial C|^2 < \int_C f \int_\Omega u_\Omega.$$

In the sequel, we will prove

Theorem 3.1. *If $c|\partial C|^2 < \int_C f \int_C u_C$, then there exists $\Omega \supsetneq C$ solution of $\mathcal{P}(f,c)$.*

To prove Theorem 3.1, we proceed as follows.
By using the domain derivative [35], the problem $\mathcal{P}(f,c)$ seems to be the Euler equation of the following optimization problem. Put

$$\mathcal{O}_C = \{\omega \subset D : \omega \quad \text{satisfies } C - \text{GNP}\}.$$

Find $\Omega \in \mathcal{O}_C$ such that

$$J(\Omega) = \text{Min}\{J(\omega),\ \omega \in \mathcal{O}_C\},$$

and

$$J(\omega) = c|\omega| - \frac{1}{2}\int_\omega u_\omega^2.$$

u_ω is the solution of $P(\omega, f)$.

Proposition 3.2. *1. There exists $\Omega \in \mathcal{O}_C$ such that*

$$J(\Omega) = Min\{J(\omega),\ \omega \in \mathcal{O}_C\}.$$

2. If Ω is of class C^2, then

$$\begin{cases} |\nabla u_\Omega||\nabla v_\Omega| \le c \text{ on } \partial\Omega \cap \partial C \\ |\nabla u_\Omega||\nabla v_\Omega| = c \text{ on } \partial\Omega \setminus \partial C. \end{cases} \tag{3.1}$$

Now, put

$$M_C = \frac{1}{|\partial C|^2}\int_C f \int_C u_C,$$

$$F(\omega) = M_C|\omega| - \frac{1}{2}\int_\omega u_\omega^2, \text{ and}$$

$$\mathcal{O}_\Omega = \{\omega \subset \Omega,\ \omega \in \mathcal{O}_C\},$$

Proposition 3.3. *1. There exists $\Omega^* \in \mathcal{O}_\Omega$ such that*

$$F(\Omega) = Min\{F(\omega),\ \omega \in \mathcal{O}_\Omega\}.$$

2. *If Ω^* is of class C^2, then*

$$\left\{\begin{array}{c} |\nabla u_{\Omega^*}||\nabla v_{\Omega^*}| \leq M_C \text{ on } \partial\Omega^* \cap \partial C \\ |\nabla u_{\Omega^*}||\nabla v_{\Omega^*}| \geq M_C \text{ on } \partial\Omega^* \cap \partial\Omega \\ |\nabla u_{\Omega^*}||\nabla v_{\Omega^*}| = M_C \text{ on } \partial\Omega^* \setminus (\partial C \cup \partial\Omega). \end{array}\right. \tag{3.2}$$

The proof of the propositions 3.2 and 3.3 uses Theorem 2.1, Propositions 2.2 and 2.4 and Lemma 2.5.
Next, we prove by contradiction that $\partial\Omega \cap \partial C = \varnothing$. The contradiction is obtained according to (3.1) and (3.2) after applying the maximum principle to Ω and Ω^*. In fact, since $\text{int}(C) \subset \Omega^* \subset \Omega$, one of the following situations occurs.

1. $\partial\Omega \equiv \partial C$;
2. $\partial\Omega \neq \partial C$ and $\partial\Omega^* \equiv \partial C$;
3. $\partial\Omega \neq \partial C$ and $\partial\Omega^* \neq \partial C$;
4. $\partial\Omega \neq \partial C$ and $\partial\Omega \equiv \partial\Omega^*$;
5. $\partial\Omega \neq \partial C$ and $\partial\Omega \neq \partial\Omega^*$.

For any of the above cases, using the maximum principle together with (3.1) and (3.2), we obtain

$$c < M_C = |\nabla u_{\Omega^*}||\nabla v_{\Omega^*}| \leq |\nabla u_\Omega||\nabla v_\Omega| \leq c, \text{ on } \partial\Omega^* \cap \partial\Omega \cap \partial C,$$

which is absurd.

Up to now, we will investigate several situations where an integral inequality on some domain Ω says that

1. either Ω is (strictly) contained in the solution of $\mathcal{QS}(f,\mu)$ (or of $\mathcal{P}(f,\mu)$) (for some constant μ),
2. or, Ω is an N-ball.

In the sequel, we suppose that $int(C) \subset \Omega \subset D$.

Proposition 3.4. *Let u_Ω be the solution of $P(\Omega, f)$. If*

$$|\nabla u_\Omega| \geq k \quad on\ \partial\Omega,$$

then

1. *either, there exists $\Omega^* \supsetneq \bar{\Omega}$ such that Ω^* is solution to $\mathcal{QS}(f,k)$,*
2. *or, $\int_C f = k|\partial\Omega|$, in that case,*
   - *(i) either, there exists $\Omega^{**} \supsetneq C$ such that Ω^{**} is solution to $\mathcal{QS}(f,k)$*
   - *(ii) or, $\Omega = int(C)$ and $|\nabla u_\Omega| = k$ on $\partial\Omega$.*

Proof. By Green's formula,

$$\int_{\Omega} f = \int_{\partial\Omega} |\nabla u_{\Omega}| \geq k|\partial\Omega|.$$

Then

1. either $\int_{\Omega} f > k|\partial\Omega|$, and so, by Theorem 2.6, there exists $\Omega^* \supsetneq \bar{\Omega}$ such that Ω^* is solution to $\mathcal{QS}(f,k)$.
2. or, $\int_{\Omega} f = k|\partial\Omega|$, in that case, since C is convex and $int(C) \subset \Omega$, then $\int_C f \geq k|\partial C|$, and so
   (a) either $\int_C f > k|\partial C|$, and so, $\Omega^{**} \supsetneq C$ such that Ω^{**} is solution to $\mathcal{QS}(f,k)$.
   (b) or, $\int_C f = k|\partial C|$, and so $\Omega = int(C)$, $|\nabla u_{\Omega}| = k$ on $\partial\Omega$.

□

Corollary 3.5. *Let u_{Ω} be the solution of $P(\Omega, \chi_C)$. Suppose that*

$$|\nabla u_{\Omega}| \geq k \quad \text{on } \partial\Omega.$$

Then

1. *either, there exists $\Omega^* \supsetneq \bar{\Omega}$ such that Ω^* is solution to $\mathcal{QS}(\chi_C, k)$,*
2. *or, $|C| = k|\partial\Omega|$, in that case*
   (i) either, there exists $\Omega^{**} \supsetneq C$ such that Ω^{**} is solution to $\mathcal{QS}(\chi_C, k)$.
   (ii) or, Ω is an N-ball.

Proof. If we replace in the previous proof f by χ_C, we obtain item 1. and item 2 (i). For item 2 (ii), we obtain $\int_C f = k|\partial C|$, and $|\nabla u_{\Omega}| = k$ on $\partial\Omega$ which means that Ω is solution to a Serrin's problem and so it coincides with an N-ball. □

Corollary 3.6. *Let u_C be the solution of $P(C,1)$ and let v_{Ω} be the solution of $P(\Omega, u_C)$. Suppose that*

$$|\nabla v_{\Omega}| \geq k \quad \text{on } \partial\Omega.$$

Then

1. *either, there exists $\Omega^* \supsetneq \bar{\Omega}$ such that Ω^* is solution to $\mathcal{QS}(u_C, k)$,*
2. *or, $\int_C u_C = k|\partial\Omega|$, in that case*
   (i) either, there exists $\Omega^{**} \supsetneq C$ such that Ω^{**} is solution to $\mathcal{QS}(u_C, k)$,
   (ii) or, Ω is an N-ball.

Proof. If we replace in the previous proof f by u_C, we obtain item 1. and item 2 (i). For item 2 (ii), we obtain $\int_C f = k|\partial C|$, $u_{\Omega} = u_C$, $v_{\Omega} = v_C$ and $|\nabla v_{\Omega}| = k$ on $\partial\Omega$. But according to Theorem 2.8, this means that Ω is an N-ball. □

Proposition 3.7. *Let u_Ω be the solution of $P(\Omega, f)$. Suppose that*

$$|\nabla u_\Omega| \geq k \quad on\ \partial\Omega.$$

Then

1. *either, there exists $\Omega^* \supsetneq \bar{\Omega}$ such that Ω^* is solution to $\mathcal{QS}(f^2, k^2\frac{|\partial\Omega|}{|\Omega|})$.*
2. *or, Ω is an N-ball.*

Proof. By Green's formula and Cauchy-Schwarz's inequality,

$$k|\partial\Omega| \leq \int_{\partial\Omega} |\nabla u_\Omega| = \int_\Omega f \leq |\Omega|^{\frac{1}{2}}(\int_\Omega f^2)^{\frac{1}{2}}.$$

Then

$$k^2|\partial\Omega|^2 \leq |\Omega| \int_\Omega f^2.$$

As in the proof of Proposition 3.4, we get item 1. For item 2., $k^2|\partial\Omega|^2 = |\Omega|\int_\Omega f^2$, implies that f is constant in Ω, $|\nabla u_\Omega| = k$ on $\partial\Omega$ which means that Ω is solution to a Serrin's problem and so Ω is an N-ball.

□

Remark 3.1. Put $f \equiv 1$. Let u_Ω (respectively v_Ω) be the solution of $P(\Omega, 1)$ (respectively $P(\Omega, u_\Omega)$). Suppose

$$|\nabla v_\Omega| \geq k \ \text{ on } \partial\Omega.$$

Since u_Ω is not constant in Ω, then there exists $\Omega^* \supsetneq \bar{\Omega}$ such that Ω^* is solution to $\mathcal{QS}(u_\Omega^2, k^2\frac{|\partial\Omega|}{|\Omega|})$.

Remark 3.2. When Ω satisfies the uniform interior ball property with radius R, one can replace in the preceding propositions the constant k by NR.

Proposition 3.8. *Let u_Ω be the solution of $P(\Omega, f)$ and let v_Ω be the solution of $P(\Omega, u_\Omega)$. Suppose that*

$$|\nabla u_\Omega||\nabla v_\Omega| \geq c \ on\ \partial\Omega$$

Then,

1. *either, there exists $\Omega^* \supsetneq \bar{\Omega}$ such that Ω^* is solution to $\mathcal{P}(f, c)$,*
2. *or, $|\nabla u_\Omega||\nabla v_\Omega| = c$ on $\partial\Omega$.*

Proof. By Green's formula and Cauchy-Schwarz's inequality,

$$\sqrt{c}|\partial\Omega| = \int_{\partial\Omega} \sqrt{|\nabla u_\Omega||\nabla v_\Omega|} \leq (\int_{\partial\Omega} |\nabla u_\Omega|)^{\frac{1}{2}}(\int_{\partial\Omega} |\nabla v_\Omega|)^{\frac{1}{2}} \leq (\int_\Omega f)^{\frac{1}{2}}(\int_\Omega u_\Omega)^{\frac{1}{2}}.$$

So

$$c|\partial\Omega|^2 \leq (\int_\Omega f)(\int_\Omega u_\Omega).$$

This inequality allows us to get

1. either, there exists $\Omega^* \supsetneq \bar{\Omega}$ such that Ω^* is solution to $\mathcal{P}(f, c)$,
2. or, $|\nabla u_\Omega||\nabla v_\Omega| = c$ on $\partial\Omega$.

□

Remark 3.3. Item 1. of Proposition 3.8 implies the existence of Ω^1 (respectively Ω^2) which (strictly) contains $\bar{\Omega}$ and such that Ω^1 is solution to $\mathcal{QS}(u_\Omega, c\frac{|\partial\Omega|}{\int_\Omega f})$ (respectively Ω^2 is solution to $\mathcal{QS}(f, c\frac{|\partial\Omega|}{\int_\Omega u_\Omega}))$,

Proposition 3.9. *Let u_Ω be the solution of $P(\Omega, 1)$ and let v_Ω be the solution of $P(\Omega, u_\Omega)$. Suppose that*

$$|\nabla u_\Omega||\nabla v_\Omega| \geq c \text{ on } \partial\Omega$$

Then

1. *either, there exists $\Omega^* \supsetneq \bar{\Omega}$ such that Ω^* is solution to $\mathcal{QS}(u_\Omega, c\frac{|\partial\Omega|}{|\Omega|})$,*
2. *or, Ω is an N-ball.*

Proof. When $f \equiv 1$,

1. either, there exists $\Omega^* \supsetneq \bar{\Omega}$ such that Ω^* is solution to $\mathcal{QS}(u_\Omega, c\frac{|\partial\Omega|}{|\Omega|})$,
2. or, $|\nabla u_\Omega||\nabla v_\Omega| = c$ on $\partial\Omega$, and so Ω is an N-ball according to Theorem 2.8.

Notice that, we also get $|\nabla v_\Omega| = \lambda|\nabla u_\Omega|$ on $\partial\Omega$ (where λ is constant). This implies that, in particular, that Ω is solution to Serrin's problem with $\sqrt{\lambda c}$ as constant. □

Remark 3.4. When Ω satisfies the uniform interior ball property with radius R, one can replace in the preceding propositions the constant c by $\frac{N^2}{N+2}R^4$.

Proposition 3.10. *Let u_Ω be the solution of $P(\Omega, 1)$ and let v_Ω be the solution of $P(\Omega, u_\Omega)$. Suppose that*

$$|\nabla v_\Omega| \geq k|\nabla u_\Omega| \text{ on } \partial\Omega$$

Then

1. *either, there exists $\Omega^* \supsetneq \bar{\Omega}$ such that Ω^* is solution to $\mathcal{QS}(u_\Omega, k\frac{|\Omega|}{|\partial\Omega|})$,*
2. *or, Ω is an N-ball.*

Proof. By Green's formula,

$$k\frac{|\Omega|}{|\partial\Omega|}|\partial\Omega| = k\int_{\partial\Omega}|\nabla u_\Omega| \leq \int_{\partial\Omega}|\nabla v_\Omega| = \int_\Omega u_\Omega.$$

Then

1. either, there exists $\Omega^* \supsetneq \bar{\Omega}$ such that Ω^* is solution to $\mathcal{QS}(u_\Omega, k\frac{|\Omega|}{|\partial\Omega|})$,
2. or, $|\nabla v_\Omega| = k|\nabla u_\Omega|$ on $\partial\Omega$ and so Ω is an N-ball.

□

Remark 3.5. When Ω satisfies the uniform interior ball property with radius R, one can replace in the preceding propositions the constant k by $\frac{N}{N+2}R^3$.

Proposition 3.11. *Let u_Ω be the solution of $P(\Omega, 1)$ and let v_Ω be the solution of $P(\Omega, u_\Omega)$. Suppose that*

$$|\nabla v_\Omega| \geq kx.\nu \text{ on } \partial\Omega.$$

Suppose that Ω is starshaped w.r.t. the origin, then

1. *either, there exists $\Omega^* \supsetneq \bar{\Omega}$ such that Ω^* is solution to $\mathcal{QS}(u_\Omega, kN\frac{|\Omega|}{|\partial\Omega|})$,*
2. *or, Ω is an N-ball.*

Proof. Since Ω is starshaped w.r.t. the origin,

$$kN\frac{|\Omega|}{|\partial\Omega|}|\partial\Omega| = k\int_{\partial\Omega} x.\nu \leq \int_{\partial\Omega} |\nabla v_\Omega| = \int_\Omega u_\Omega.$$

Then

1. either, there exists $\Omega^* \supsetneq \bar{\Omega}$ such that Ω^* is solution to $\mathcal{QS}(u_\Omega, kN\frac{|\Omega|}{|\partial\Omega|})$,
2. or, $|\nabla v_\Omega| = kx.\nu$ on $\partial\Omega$ and so Ω is an N-ball.

According to item 2. of Theorem 2.8, Ω is an N-ball. □

4 Symmetry results for some overdetermined problems

In this section, we consider new overdetermined boundary value problems. We use essentially the domain derivative to get the symmetry result.

Proposition 4.1. *Let $\Omega \subset D$, and let u_Ω and v_Ω be the solutions of the Dirichlet problems $P(\Omega, 1)$ and $P(\Omega, u_\Omega)$. Suppose that*

$$(OC)\begin{cases} |\nabla v_\Omega| = \frac{N}{N+2}|\nabla u_\Omega|^3 \text{ on } \partial\Omega, \text{ and} \\ \int_{\partial\Omega} |\nabla u_\Omega|^2 \frac{\partial u'_\Omega}{\partial\nu} d\sigma = 0. \end{cases}$$

Then Ω is an N-ball.

Before proving Proposition 4.1, we check that a ball B_R is a solution to problem $\mathcal{P}(1, c)$ which satisfies (OC).
Let u_R be the solution of $P(B_R, 1)$. Using polar coordinates, u_R verifies

$$\begin{aligned} -u''_R - \frac{N-1}{r}u'_R &= 1 \quad \text{for } r \in]0, R[, \\ u_R(R) &= 0. \end{aligned}$$

By the first equation, $\left(r^{N-1}u'_R\right)' = -r^{N-1}$. Since $u_R(R) = 0$, we get

$$r^{N-1}u'_R(r) = R^{N-1}u'_R(R) + \int_r^R s^{N-1}ds.$$

As $r \to 0$, $r^{N-1}u'_R(r) \to 0$ (otherwise we get a distributional contribution to Δu_R at the origin). Thus

$$-u'_R(R) = \frac{1}{R^{N-1}} \int_0^R s^{N-1} ds = \frac{R}{N}.$$

Now let v_R) be the solution of $P(B_R, u_R)$). On one hand, we have $-u'_R(R) = \frac{R}{N}$. Then a simple calculation shows that

$$u_R(r) = \frac{1}{2N}\left(R^2 - r^2\right) \quad \text{for} \quad r \in]0, R[\,.$$

On the other hand, the radial function v_R satisfies

$$\begin{cases} -v''_R - \frac{N-1}{r} v'_R = u_R \quad \text{for } r \in]0, R[\,, \\ v_R(R) = 0 \\ -v'_R(R) = \frac{1}{R^{N-1}} \int_0^R s^{N-1} u_R(s) ds = \frac{N}{N+2}\left(\frac{R}{N}\right)^3. \end{cases}$$

Therefore B_R is a solution of $\mathcal{P}(1, c)$ and so

$$|\nabla v_R| = \frac{N}{N+2}\left(\frac{R}{N}\right)^3 = \frac{N}{N+2} |\nabla u_R|^3, \text{ and}$$

$$\int_{\partial\Omega} |\nabla u_\omega|^2 \frac{\partial u'_\Omega}{\partial \nu} d\sigma = \left(\frac{R}{N}\right)^2 \int_{\partial\Omega} \frac{\partial u'_\Omega}{\partial \nu} d\sigma = 0.$$

Proof. Let $\mathcal{O}$ be the set of all bounded open, connected domains of class C^2 in $\mathbb{R}^N$. Consider the following functional of a domain, introduced in [14]:

$$J(\omega) = N \int_{\partial\omega} |\nabla u_\omega|^3 d\sigma - (N+2) \int_\omega |\nabla u_\omega|^2 dx.$$

u_ω being the solution of $P(\omega, 1)$. As in [14], $J(\omega) \geq 0$ for any $\omega \in \mathcal{O}$. Now, by Green's formula

$$\frac{N}{N+2} \int_{\partial\Omega} |\nabla u_\Omega|^3 = \int_{\partial\Omega} |\nabla v_\Omega| = \int_\Omega u_\Omega = \int_\Omega |\nabla u_\Omega|^2,$$

then $J(\Omega) = 0$.
The domain derivative of the functional J at $\omega \in \mathcal{O}$ in the direction V is given by

$$dJ(\omega, V) = \int_{\partial\omega} ([2(N-1)|\nabla u_\omega|^2 - 2N(N-1)H_\omega |\nabla u_\omega|^3] V.\nu - 3N |\nabla u_\omega|^2 \frac{\partial u'_\omega}{\partial \nu}) d\sigma.$$

H_ω is the mean curvature of $\partial\omega$ and u'_ω is the domain derivative of u_ω. Since Ω minimizes the functional J, then for every vector field $V \in C^2(\mathbb{R}^N, \mathbb{R}^N)$ we have $dJ(\Omega, V) = 0$. Now, according to (OC), we obtain

$$|\nabla u_\Omega| = \frac{1}{NH_\Omega} \text{ on } \partial\Omega.$$

This means that Ω is an N-ball, according to Lemma 2.7. $\square$

Remark 4.1. Proposition 4.1 corresponds to the following biharmonic problem

$$\begin{cases} \Delta^2 v_\Omega = 1 \quad \text{in } \Omega\,, v_\Omega = \Delta v_\Omega = 0 \quad (N+2)|\nabla v_\Omega| = N|\nabla u_\Omega|^3 \text{ on } \partial\Omega\,, \\ \int_{\partial\Omega} |\nabla u_\Omega|^2 \frac{\partial u'_\Omega}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases}$$

The work bellows is motivated by the following result obtained by Didenko and Emamizadeh [16]

Theorem 4.2. *The domain Ω is a ball if and only if there exists a constant c that the following integral equation is valid*

$$\int_\Omega u'_\Omega dx = c \int_{\partial\Omega} u'_\Omega d\sigma,$$

for every vector field $V \in C^2(\mathbb{R}^N, \mathbb{R}^N)$, where

$$\triangle u'_\Omega = 0 \text{ in } \Omega \qquad u'_\Omega = -\frac{\partial u_\Omega}{\partial \nu} V.\nu\ .$$

Let u_Ω, v_Ω, and w_Ω be respectively the solution of $P(\Omega, 1)$, $P(\Omega, u_\Omega)$, $P(\Omega, v_\Omega)$.

Proposition 4.3. *Ω is an N-ball if one of the following conditions is satisfied*

(i) $-\frac{\partial v'_\Omega}{\partial \nu} = c^2 V.\nu$ *on* $\partial\Omega$

(ii) $-\frac{\partial v'_\Omega}{\partial \nu} = c|\nabla u_\Omega| V.\nu$ *on* $\partial\Omega$

(iii) $-\frac{\partial v'_\Omega}{\partial \nu} = c^2|\nabla u_\Omega|^2 V.\nu$ *on* $\partial\Omega$.

Proof. By Green's formula

$$\begin{aligned} \int_{\partial\Omega} -\frac{\partial v'_\Omega}{\partial \nu} = \int_\Omega u'_\Omega &= \int_\Omega -\Delta u_\Omega u'_\Omega \\ &= \int_\Omega -\Delta u'_\Omega u_\Omega + \int_{\partial\Omega} u_\Omega \frac{\partial u'_\Omega}{\partial \nu} - \int_{\partial\Omega} u'_\Omega \frac{\partial u_\Omega}{\partial \nu} \\ &= \int_{\partial\Omega} |\nabla u_\Omega|^2 V.\nu. \end{aligned}$$

Then, we get

(i) $|\nabla u_\Omega| = c$ on $\partial\Omega$;

(ii) $|\nabla u_\Omega| = c$ on $\partial\Omega$;

(iii) $|\nabla v_\Omega| = \frac{1}{c}|\nabla u_\Omega|$ on $\partial\Omega$.

Each of the above items tells us that Ω is an N-ball. □

Remark 4.2. Proposition 4.3 corresponds to the following biharmonic problem

$$\begin{cases} \Delta^2 v'_\Omega = 0 \quad \text{in } \Omega\,, \\ v'_\Omega = |\nabla v_\Omega| V.\nu, \quad \Delta v'_\Omega = |\nabla u_\Omega| V.\nu \text{ on } \partial\Omega\,, \\ -\frac{\partial w'_\Omega}{\partial \nu} = \Lambda \text{ on } \partial\Omega. \end{cases}$$

where $\Lambda = c^2 V.\nu$, $\Lambda = c|\nabla u_\Omega| V.\nu$, or $\Lambda = c^2|\nabla u_\Omega|^2 V.\nu$.

Proposition 4.4. *Ω is an N-ball if one of the following conditions is satisfied:*

(i) $-\frac{\partial w'_\Omega}{\partial \nu} = cV.\nu$ *on* $\partial\Omega$

(ii) $-\frac{\partial w'_\Omega}{\partial \nu} = c|\nabla u_\Omega|V.\nu$ *on* $\partial\Omega$

(iii) $-\frac{\partial w'_\Omega}{\partial \nu} = c|\nabla u_\Omega|^2 V.\nu$ *on* $\partial\Omega$

(iv) $-\frac{\partial w'_\Omega}{\partial \nu} = c|\nabla v_\Omega|V.\nu$ *on* $\partial\Omega$.

Proof. By Green's formula

$$\begin{aligned}
\int_{\partial\Omega} -\frac{\partial w'_\Omega}{\partial \nu} = \int_\Omega v'_\Omega &= \int_\Omega -\Delta u_\Omega u'_\Omega \\
&= \int_\Omega -\Delta v'_\Omega u_\Omega + \int_{\partial\Omega} u_\Omega \frac{\partial v'_\Omega}{\partial \nu} - \int_{\partial\Omega} v'_\Omega \frac{\partial u_\Omega}{\partial \nu} \\
&= \int_\Omega u u'_\Omega + \int_{\partial\Omega} |\nabla u_\Omega||\nabla v_\Omega| V.\nu. \\
&= 2\int_{\partial\Omega} |\nabla u_\Omega||\nabla v_\Omega| V.\nu.
\end{aligned}$$

Then, we get

(i) $|\nabla u_\Omega||\nabla v_\Omega| = \frac{c}{2}$ on $\partial\Omega$

(ii) $|\nabla v_\Omega| = \frac{c}{2}$ on $\partial\Omega$

(iii) $|\nabla v_\Omega| = \frac{c}{2}|\nabla u_\Omega|$ on $\partial\Omega$

(iv) $|\nabla u_\Omega| = \frac{c}{2}$ on $\partial\Omega$.

Each of the above items tells us that Ω is an N-ball. □

Remark 4.3. Proposition 4.4 corresponds to the following triharmonic problem

$$\begin{cases}
\Delta^3 w'_\Omega = 0 \quad \text{in } \Omega\,, \\
w'_\Omega = |\nabla w_\Omega| V.\nu, \quad \Delta w'_\Omega = |\nabla v_\Omega| V.\nu, \quad \Delta^2 w'_\Omega = |\nabla u_\Omega| V.\nu \text{ on } \partial\Omega\,, \\
-\frac{\partial w'_\Omega}{\partial \nu} = \Lambda \text{ on } \partial\Omega.
\end{cases}$$

where $\Lambda = cV.\nu$, $\Lambda = c|\nabla u_\Omega|V.\nu$, $\Lambda = c|\nabla u_\Omega|^2 V.\nu$ or $\Lambda = c|\nabla v_\Omega|V.\nu$.

5 Concluding remarks

Remark 5.1. Let z_Ω be the solution of $P(\Omega, \frac{1}{2}u_\Omega^2)$.

Proposition 5.1. *Ω is an N-ball if one of the following conditions is satisfied*

(i) $-\frac{\partial z'_\Omega}{\partial \nu} = cV.\nu$ *on* $\partial\Omega$

(ii) $-\frac{\partial z'_\Omega}{\partial \nu} = c|\nabla u_\Omega|V.\nu$ *on* $\partial\Omega$

(iii) $-\frac{\partial z'_\Omega}{\partial \nu} = c|\nabla u_\Omega|^2 V.\nu$ *on* $\partial\Omega$

(iv) $-\frac{\partial z'_\Omega}{\partial \nu} = c|\nabla v_\Omega| V.\nu$ *on* $\partial\Omega$.

Proof. By Green's formula

$$\begin{aligned}\int_{\partial\Omega} -\frac{\partial z'_\Omega}{\partial \nu} = \int_\Omega u_\Omega u'_\Omega &= \int_\Omega -\Delta v_\Omega u'_\Omega \\ &= \int_\Omega -\Delta u'_\Omega v_\Omega + \int_{\partial\Omega} v_\Omega \frac{\partial u'_\Omega}{\partial \nu} - \int_{\partial\Omega} u'_\Omega \frac{\partial v_\Omega}{\partial \nu} \\ &= \int_{\partial\Omega} |\nabla u_\Omega||\nabla v_\Omega| V.\nu.\end{aligned}$$

Then, we get

(i) $|\nabla u_\Omega||\nabla v_\Omega| = c$ on $\partial\Omega$

(ii) $|\nabla v_\Omega| = c$ on $\partial\Omega$

(iii) $|\nabla v_\Omega| = c|\nabla u_\Omega|$ on $\partial\Omega$

(iv) $|\nabla u_\Omega| = c$ on $\partial\Omega$

Each of the above items tells us that Ω is an N-ball. □

Remark 5.2.

Proposition 5.2. *Balls are stationary for the functional*

$$J(\Omega) = \frac{F(\omega)}{G(\omega)},$$

where

1. $F(\omega) = |\omega|$ *and* $G(\omega) = \int_\omega u_\omega$*;*
2. $F(\omega) = \int_\omega u_\omega$ *and* $G(\omega) = \int_\omega u_\omega^2$*;*
3. $F(\omega) = |\omega|$ *and* $G(\omega) = \int_\omega u_\omega^2$*;*
4. $F(\omega) = \int_\omega u_\omega v_\omega$ *and* $G(\omega) = \int_\omega u_\omega^2$*;*
5. $F(\omega) = \int_\omega u_\omega v_\omega$ *and* $G(\omega) = \int_\omega u_\omega$*.*

u_ω *and* v_ω *being respectively the solution of* $P(\omega, 1)$ *and* $P(\omega, u_\omega)$*.*

Proof. If Ω is of class C^2, then for every vector field $V \in C^2(\mathbb{R}^N, \mathbb{R}^N)$

$$dJ(\Omega, V) = \frac{dF(\Omega, V)G(\Omega) - F(\Omega)dG(\Omega, V)}{[G(\Omega)]^2}.$$

Then $dJ(\Omega, V) = 0$ implies

1. $|\nabla u_\Omega| = \frac{1}{J(\Omega)}$ on $\partial\Omega$;

2. $|\nabla v_\Omega| = \frac{1}{2J(\Omega)}|\nabla u_\Omega|$ on $\partial\Omega$;
3. $|\nabla v_\Omega||\nabla u_\Omega| = \frac{1}{2J(\Omega)}$ on $\partial\Omega$;
4. $|\nabla v_\Omega| = 2J(\Omega)|\nabla u_\Omega|$ on $\partial\Omega$;
5. $|\nabla v_\Omega| = J(\Omega)|\nabla u_\Omega|$ on $\partial\Omega$.

Each of the overdetermined conditions obtained above says that Ω is an N-ball. □

Remark 5.3. One can have the same existence result for the functional used in Section 4. if we replace $\mathcal{O}_C$ by the following class of open sets: For $\varepsilon > 0$

$$\mathcal{O}_\varepsilon = \{\omega \subset D,\ \forall x \in \partial\omega,\ \partial\omega \cap B(x,\varepsilon) \text{ has } \bar{B}'_\varepsilon - PGN\}$$

B'_ε is the $(N-1)$-ball with center x.

Remark 5.4. For the problem $\mathcal{QS}(f,g)$ (g is positive ant it attains its maximum on ∂C), one can have

$$\int_C f dx > \int_{\partial C} g d\sigma$$

as a sufficient condition of existence.
Now, let $a > 0$ and put $C = [-1,1] \times \{0\} \subset \mathbb{R}^2$. Consider the problem

$$\mathcal{QS}(a,g) \begin{cases} -\Delta u_\Omega = a\delta_C \quad \text{in } \Omega \\ u_\Omega = 0 \text{ on } \partial\Omega \\ -\frac{\partial u_\Omega}{\partial \nu} = g \text{ on } \partial\Omega. \end{cases}$$

The problem $Q_S(a,g)$ has a solution if $a > \frac{1}{\pi}\int_{\partial B} g(\sigma)d\sigma$. B being the unit ball in $\mathbb{R}^2$.
Notice that in the special case where $g \equiv k = const.$,the condition above becomes $a > 2k$ and it is a necessary and sufficient condition of existence for $\mathcal{QS}(a,k)$ [3].

Remark 5.5. Let Ω be a bounded and convex set which contains the interior of C the convex hull of the support of f. The existence of a domain C_Ω which minimizes the ratio

$$R(\omega) = \frac{\int_\omega f}{|\partial\omega|}$$

is obtained in the class of convex subset of Ω. One calls the minimum above the f-Cheeger set of Ω and $\lambda(\Omega)$ the f-Cheeger constant. Using the same arguments as in Section 3., one can prove that

1. either there exists $\Omega^* \supsetneq \bar{\Omega}$ which is a solution to $\mathcal{QS}(f,\lambda(\Omega))$,
2. or $C_\Omega = \Omega$, i.e Ω is f-Cheeger in itself.

Remark 5.6. Let Ω be an open subset of D. Consider the Cheeger constant

$$h(\Omega) = \min_{\omega\subset\Omega} \frac{|\partial\omega|}{|\omega|}.$$

Using the domain derivative of h [28], we prove the following

Proposition 5.3. *If* $\int_C f(x)dx > \sqrt{h(C)}|\partial C|$*, then the problem* $(Q_S(f, \sqrt{g(\Omega)}))$ *admits a solution where*

$$g(\Omega) = \frac{|\Omega|}{|\partial C_\Omega|}((N-1)H_{\partial C} - h(\Omega))\chi_{\partial\Omega\cap\partial C_\Omega} + h(\Omega),$$

C_Ω *being the unique Cheeger set of* Ω.

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