Null controllability of parabolic coupled system with control under constraints

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Abstract. We prove the null controllability of a parabolic system. The single control is common to both PDEs, distributed and subject to constraints. The studied model can be applied in dynamics of biological systems or in physics. First we study the problem associated to a similar linearized system. Then appropriate Carleman inequalities and a fixed-point argument are used to prove the null controllability results.

M.S.C. 2010: 35K40, 35K59, 93B05.
Key words: Null controllability; non-linear system; parabolic system; Carleman inequality.

1 Introduction

Let $n \in \mathbb{N}$ and Ω be a bounded domain of $\mathbb{R}^n$ with boundary $\Gamma = \partial\Omega$ of class C^2. Consider a non-empty open set $\omega \subset \Omega$ and ν the outward unit normal to Γ. For a time $T > 0$, consider $\Sigma = \Gamma \times (0,T)$ the lateral boundary of the cylinder $Q = \Omega \times (0,T)$, and G the small cylinder $\omega \times (0,T)$. We consider the following nonlinear parabolic coupled system:

$$(1.1) \qquad \left\{ \begin{array}{r} \partial_t y_1 - A(t)y_1 + a_1 y_1 + b_1 y_2 = f_1 + k\chi_\omega \text{ in } Q, \\ \partial_t y_2 - A(t)y_2 + a_2 y_1 + b_2 y_2 = f_2 + k\chi_\omega \text{ in } Q, \\ y_1 = y_2 = 0 \text{ on } \Sigma, \\ y_1(0) = y_1^0, y_2(0) = y_2^0 \text{ in } \Omega, \end{array} \right.$$

where $f_i \in L^2(Q)$, $y_i^0 \in L^2(\Omega)$, $a_i, b_i \in L^\infty(Q)$, $i = 1,2$, k is the control acting on the system through G and

$$(1.2) \qquad A(t)w = \sum_{\kappa,l=1}^{n} B_{\kappa l}(w(.,t),t)\frac{\partial^2 w}{\partial x_\kappa \partial x_l},$$

the functions $B_{\kappa l} : L^1(\Omega) \times [0,T] \to \mathbb{R}$ are given $\forall \kappa, l \in \{1, \dots n\}$. We will make some hypotheses on the $B_{\kappa l}$ in the remainder.
Such models can be applied in the context of dynamics of biological systems to describe

Applied Sciences, Vol. 21, 2019, pp. 146-158.

the migration of population. They can also describe in physics the distribution of heat in a conductor and the behavior in systems of interacting components in chemistry. In the case of migration of populations, $y_i, i = 1, 2$, can be the density of two bacterial species, $\partial_t y_i, i = 1, 2$, stands for the population variation, the coefficients $a_i, b_i, i = 1, 2$, characterize the interactions of the two species and we can have:

$$B_{\kappa l}(y_i(.,t), t) = a_{\kappa l}\left(\int_\Omega y_i(x,t)dx\right)$$

where $a_{\kappa l}$ is a positive and continuous real function depending on the population itself and indicating the speed at which the movement is executed (see [2]). In the context of biochemical reaction processes between two mobile species in Ω, y_i, $i = 1, 2$ denotes the concentration of the species. The operator $B_{\kappa l}$ can be written as:

$$B_{\kappa l}(y_i(.,t), t) = a_{\kappa l}(\langle l_0, y_i(t)\rangle_{L^2(\Omega), L^2(\Omega)})$$

where l_0 is a linear form on $L^2(\Omega)$ and $a_{\kappa l}$ is a real positive continuous function.

It is said that (1.1) is null controllable at time T if for any given $y_i^0 \in L^2(\Omega)$, $i = 1, 2$, there exists a control $k \in L^2(G)$ such that the associated solution satisfied

$$y_i(T) = 0 \text{ in } \Omega,\ i = 1, 2,$$

with an estimate of the form

$$\|k\|_{L^2(G)} \leqslant C(\|y_1^0\|_{L^2(\Omega)} + \|y_2^0\|_{L^2(\Omega)} + \|f_1\|_{L^2(Q)} + \|f_2\|_{L^2(Q)}),\ C > 0.$$

In this work, we study another type of controllability problem introduced by Nakoulima in [11]. In addition to reach the null trajectory at time T, the control must satisfy an additional condition that we will clarify. Let $\mathcal{H}$ be a finite dimensional vector subspace of $L^2(G)$ and $\mathcal{H}^\perp$ the orthogonal of $\mathcal{H}$ in $L^2(G)$. We focus on the following null controllability problem: *for any given $f_i \in L^2(Q)$ and $y_i^0 \in L^2(\Omega)$, $i = 1, 2$, find*

$$k \in \mathcal{H}^\perp \tag{1.3}$$

such that the associated solution (y_1, y_2) of (1.1) satisfies

$$y_i(T) = 0 \text{ in } \Omega,\ i = 1, 2. \tag{1.4}$$

From this work, we deduce existence results of optimal control satisfying a null controllability problem with constraints on the state.
In the linear case, we showed that the null controllability of two coupled diffusion equations in the presence of constraints on the control holds (see [8]). We applied in [9] this result to prove the existence of a control solving the null controllability of a nonlinear system, the state being submitted to constraints.

The rest of the paper is organized as follows: In Section 2 we state the main result of the paper. Theorem 2.2 reads the existence of a control under constraints solving the null controllability problem. In Section 3 we give some intermediate estimates arising from Carleman estimates. Then we prove an observability inequality which will be useful to obtain the null controllability of the linearized system. In Section 4 we give the proof of the main result stated in Section 2. Section 5 is devoted to an application of our work to a null controllability problem with integral constraints. Finally we end with a conclusion.

2 Preliminaries and main result

First let us introduce the following notations (see for instance [5, 10]).

- Let P be the orthogonal projection operator from $L^2(G)$ into $\mathcal{H}$,
- $C^0(\Omega)$ is the set of continuous functions defined on Ω,
- $C^l(\Omega) = \{u : \Omega \to \mathbb{R}; \forall\alpha \in \mathbb{N}^n, |\alpha| \leqslant l, D^\alpha u \in C^0(\Omega)\}$, with $D^\alpha u = \dfrac{\partial^{\alpha_1+\alpha_2+\ldots+\alpha_n} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}, |\alpha| = \sum_{i=1}^{n} \alpha_i$,
- for $\delta \in (0,1)$, $u \in C^0(\overline{Q})$: $[u]_{\delta,\frac{\delta}{2}} = \sup_{\substack{\overline{Q}\\ x\neq x'}} \dfrac{|u(x,t)-u(x',t)|}{|x-x'|^\delta} + \sup_{\substack{\overline{Q}\\ t\neq t'}} \dfrac{|u(x,t)-u(x,t')|}{|t-t'|^{\delta/2}}$
- $C^{\delta,\delta/2}(\overline{Q}) = \{u \in C^0(\overline{Q}) : [u]_{\delta,\delta/2} < \infty\}$ is a Banach space with the norm $|u|_{\delta,\delta/2;\overline{Q}} = \|u\|_{L^\infty(Q)} + [u]_{\delta,\delta/2}$,
- $C^{1+\delta,\frac{1+\delta}{2}}(\overline{Q}) = \left\{u \in C^0(\overline{Q}) : \dfrac{\partial u}{\partial x_i} \in C^{\delta,\frac{\delta}{2}}(\overline{Q}) \forall i, \sup_{\substack{\overline{Q}\\ t\neq t'}} \dfrac{|u(x,t)-u(x,t')|}{|t-t'|^{\frac{1+\delta}{2}}} < \infty\right\}$ is a Banach space which the norm is denoted by $|.|_{1+\delta,\frac{1+\delta}{2};\overline{Q}}$,
- $Z = \{z \in L^1(0,T;L^2(\Omega)) : z_t \in L^\infty(0,T;L^2(\Omega))\}$,
- $X = \{(k,(y_1,y_2)) : k \in C^{\delta,\delta/2}(\overline{\omega}\times[0,T]), (y_1,y_2) \in (C^{1+\delta,\frac{1+\delta}{2}}(\overline{Q}))^2\}$.

We will need the result below which is due to Fursikov and Imanuvilov.

Lemma 2.1. *([4]) There exists a function $\beta \in C^2(\overline{\Omega})$ satisfying*

$$\begin{cases} \beta(x) > 0 & \forall x \in \Omega, \\ \beta(x) = 0 & \forall x \in \Gamma, \\ |\nabla\beta(x)| \neq 0 & \forall x \in \overline{\Omega}\setminus\omega', \end{cases}$$

where ω' is a non-empty open set with $\omega' \Subset \omega$. By $\omega' \Subset \omega$ we mean that ω' is compactly embedded in ω i.e. $\omega' \subseteq \overline{\omega'} \subseteq \omega$ and $\overline{\omega'}$ is compact.

In addition for every $\lambda > 0$ and for $(x,t) \in Q$, let us introduce the functions:

$$\rho(x,t) = \frac{e^{\lambda\beta(x)}}{t(T-t)} \text{ and } \alpha(x,t) = \frac{e^{2\lambda\|\beta\|_{L^\infty(\Omega)}} - e^{\lambda\beta(x)}}{t(T-t)}.$$

Note that $\rho(.,t)$ and $\alpha(.,t) \to +\infty$ when $t \to 0$ or $t \to T$.
We introduce the function $\xi \in C^\infty(\mathbb{R}^n)$ satisfying:

$$\begin{cases} \xi(x) = 1, & \forall x \in \omega', \\ 0 < \xi(x) \leqslant 1, & \forall x \in \omega'', \\ \xi(x) = 0, & \forall x \in \mathbb{R}^n \setminus \omega'', \end{cases} \tag{2.1}$$

where $\omega' \Subset \omega'' \Subset \omega \Subset \Omega$. We will assume that for every $\kappa, l \in \{1,\ldots,n\}$,

$$B_{\kappa l} = B_{l\kappa}, \tag{2.2}$$

$$(2.3) \qquad -\infty < \gamma_0 \leqslant B_{\kappa l} \leqslant \gamma_1 < +\infty.$$

Besides we suppose that for every $\kappa, l \in \{1, \dots, n\}$, $B_{\kappa l}$ is continuous and globally Lipschitz in $L^2(\Omega) \times [0, T]$. By this, we mean that there is $L > 0$ such that for any $(z, t), (y, s) \in L^2(\Omega) \times [0, T]$, we have:

$$(2.4) \qquad |B_{\kappa l}(z, t) - B_{\kappa l}(y, s)| \leqslant L(\|z - y\|_{L^2(\Omega)} + |t - s|).$$

We also suppose that there exists $\alpha_0 > 0$ such that for all $z \in L^2(\Omega)$, almost every (a.e.) in $t \in (0, T)$ and for all $\phi \in \mathbb{R}^n$,

$$(2.5) \qquad e^{-\gamma\alpha}\xi^2 \sum_{\kappa,l=1}^{n} B_{\kappa l}(z, t)\phi_\kappa \phi_l \geqslant \alpha_0 |\phi|^2, \ \forall \gamma > 0.$$

We use (2.5) particularly in Theorem 3.2 in Section 3.1 to obtain an observability inequality from which we will deduce null controllability results for (1.1). We set

$$(2.6) \qquad \begin{aligned} a &= -\frac{1}{2}(a_1 + a_2 + b_1 + b_2), \quad & b &= -\frac{1}{2}(a_1 + a_2 - b_1 - b_2), \\ c &= -\frac{1}{2}(a_1 - a_2 + b_1 - b_2), \quad & d &= -\frac{1}{2}(a_1 - a_2 - b_1 + b_2), \end{aligned}$$

and define $\mathcal{L}_\xi = \sum_{\kappa,l=1}^{n} \xi_{\kappa l} \dfrac{\partial^2}{\partial x_\kappa \partial x_l}$, with $\xi_{\kappa l} \in \mathbb{R}$ for each $\kappa, l \in \{1, \dots, n\}$.
We assume that for $p_{\kappa l}, q_{\kappa l} \in \mathbb{R}$, any function $\varphi \in \mathcal{H}$ such that (φ, σ) satisfies

$$(2.7) \qquad \begin{aligned} & -\partial_t \varphi - \frac{1}{2}(\mathcal{L}_p \varphi + \mathcal{L}_q \sigma) - a\varphi - c\sigma \\ &= -\partial_t \sigma - \frac{1}{2}(\mathcal{L}_q \varphi + \mathcal{L}_p \sigma) - b\varphi - d\sigma = 0 \end{aligned}$$

in G for some σ, is null in G.

Such an assumption was used by Lions in [6] (p.33) to solve a problem of discriminating sentinel. We are now able to state the main result of this work.

Theorem 2.2. *Let $A(t)(.)$ be the operator defined by (1.2) with each function $B_{\kappa l}$ satisfying (2.2)-(2.5). Assume (2.7) and that there exist a constant $c_0 > 0$ and a set ω_c such that*

$$\omega_c \Subset \omega \text{ and } |c| \geqslant c_0 \text{ in } \omega_c \times (0, T_0) \text{ for } T_0 > 0.$$

Then there exists a positive continuous function θ in Q, such that for every $f_1, f_2 \in L^2(Q)$ with $\theta f_1, \theta f_2 \in L^2(Q)$, the null controllability problem (1.1),(1.3),(1.4) admits a unique solution. (The definition of θ is given later by (3.11)).

3 Controllability of the linearized system

This section is devoted to the proof of the null controllability problem of the linearized system. For given $z = (z_1, z_2) \in Z \times Z$, we will consider the linearized system

$$(3.1) \qquad \left\{ \begin{aligned} \partial_t y_1 - B(t; z_1)y_1 + a_1 y_1 + b_1 y_2 &= f_1 + k\chi_\omega \text{ in } Q, \\ \partial_t y_2 - B(t; z_2)y_2 + a_2 y_1 + b_2 y_2 &= f_2 + k\chi_\omega \text{ in } Q, \\ y_1 = y_2 &= 0 \text{ on } \Sigma, \\ y_1(0) = y_1^0, y_2(0) &= y_2^0 \text{ in } \Omega, \end{aligned} \right.$$

where $B(t;y)w = \sum_{\kappa,l=1}^n B_{\kappa l}(y(t),t)\dfrac{\partial^2 w}{\partial x_\kappa \partial x_l}$. Note that for z fixed in Z, the function $t \mapsto B_{\kappa l}(z(t),t)$ is a.e. differentiable and because of (2.4), we have for $1 \leqslant \kappa, l \leqslant n$,

$$\pi(z) = \max_{1\leqslant \kappa,l\leqslant n} \|(B_{\kappa,l})_t(z(t),t)\|_{L^\infty(0,T)} \leqslant L(1+\|z_t\|_{L^\infty(0,T;L^2(\Omega))}).$$

Under (2.2)-(2.5) and the above assumptions, for each $y_1^0, y_2^0 \in L^2(\Omega)$, $f_1, f_2 \in L^2(\Omega)$ and each $k \in L^2(G)$, the system (3.1) possesses exactly one solution (y_1,y_2) in $C([0,T];L^2(\Omega)^2) \cap L^2(0,T;H_0^1(\Omega)^2)$ (see [2]). Let (y_1,y_2) be a solution of (3.1). Setting

$$u = y_1+y_2, \quad v = y_1-y_2, \quad f = f_1+f_2, \quad g = f_1-f_2, \quad h = 2k, \tag{3.2}$$

one gets that (u,v) is solution of

$$\left\{\begin{array}{rcll} u_t-(B(t;z_1)y_1+B(t;z_2)y_2)-au-bv &=& f+h\chi_\omega & \text{in } Q,\\ v_t-(B(t;z_1)y_1-B(t;z_2)y_2)-cu-dv &=& g & \text{in } Q,\\ u=v &=& 0 & \text{on } \Sigma,\\ u(0)=u^0, v(0) &=& v^0 & \text{in } \Omega. \end{array}\right. \tag{3.3}$$

Note that: $B(t;z_1)y_1+B(t;z_2)y_2 = \dfrac{1}{2}\Big((B(t;z_1)+B(t;z_2))u+(B(t;z_1)-B(t;z_2))v\Big)$ and $B(t;z_1)y_1-B(t;z_2)y_2 = \dfrac{1}{2}\Big((B(t;z_1)-B(t;z_2))u+(B(t;z_1)+B(t;z_2))v\Big)$.

3.1 Carleman estimates

An observability inequality is established in this part, the obtention of such an estimate being useful for the study of exact controllability problems.
For $z \in Z$, $f \in L^2(Q)$ and $y_T \in L^2(\Omega)$, consider the parabolic system

$$\left\{\begin{array}{lcll} \partial_t y + B(t;z)y &=& f & \text{in } Q,\\ y &=& 0 & \text{on } \Sigma,\\ y(T) &=& y_T & \text{in } \Omega. \end{array}\right. \tag{3.4}$$

Then the following Carleman inequality holds.

Theorem 3.1 ([3], Theorem 2.1)**.** *There are positive constants s_0, λ_0 and C_0 such that for any $s \geqslant s_0$, $\lambda \geqslant \lambda_0$, $f \in L^2(Q)$ and $y_T \in L^2(\Omega)$, the associated solution to (3.4) satisfies*

$$\begin{aligned} &\int_Q e^{-2s\alpha}\Big[(s\rho)^{-1}\Big(|\partial_t y|^2 + \sum_{i,j=1}^n \Big|\frac{\partial^2 y}{\partial x_i \partial x_j}\Big|^2\Big) + s\lambda^2\rho|\nabla y|^2 + \lambda^4 (s\rho)^3|y|^2\Big]dxdt \\ &\qquad \leqslant C_0\Big(\int_Q e^{-2s\alpha}|f|^2dxdt + \int_0^T\int_{y'} e^{-2s\alpha}\lambda^4(s\rho)^3|y|^2dxdt\Big). \end{aligned} \tag{3.5}$$

Furthermore ,C_0 and λ_0 only depend on Ω, ω, γ_0, γ_1 and α_0; s_0 can be chosen on the form $s_0 = \sigma_0(T+T^2)+\sigma_1\pi(z)T^2$, where σ_0 and σ_1 depend on Ω, ω, γ_0, γ_1 and α_0.

Now let us introduce the following notations for $z=(z_1,z_2)\in Z\times Z$,

(3.6)
$$\begin{cases} \mathcal{V} &= \{\varphi\in C^\infty(\overline{Q});\varphi|_\Sigma=0\},\\ \mathcal{W} &= \mathcal{V}\times\mathcal{V},\\ M(\varphi,\sigma) &= -\partial_t\varphi-\dfrac{1}{2}\Big((B(t;z_1)+B(t;z_2))\varphi+(B(t;z_1)-B(t;z_2))\sigma\Big)-a\varphi-c\sigma,\\ N(\varphi,\sigma) &= -\partial_t\sigma-\dfrac{1}{2}\Big((B(t;z_1)-B(t;z_2))\varphi+(B(t;z_1)+B(t;z_2))\sigma\Big)-b\varphi-d\sigma,\end{cases}$$

and set $\|a,b,c,d\|_\infty^2=\|a\|^2_{L^\infty(Q)}+\|b\|^2_{L^\infty(Q)}+\|c\|^2_{L^\infty(Q)}+\|d\|^2_{L^\infty(Q)}$.
Moreover, the following observability inequality holds:

Theorem 3.2. *Assume that there exist a constant $c_0>0$ and a domain ω_c such that*

$$\omega_c\Subset\omega \text{ and } |c|\geqslant c_0 \text{ in } \omega_c\times(0,T_0) \text{ for some } T_0>0. \tag{3.7}$$

Then for $r\in[0,2)$, $s\geqslant s_0$, $\lambda\geqslant\lambda_1=\left(\frac{T^6C\|a,b,c,d\|_\infty^2}{(2s)^3}\right)^{\frac14}$, $\varphi=(\varphi_1,\varphi_2)\in\mathcal{W}$, there exists a positive constant depending on γ_0, γ_1, n, c_0, α_0, $\|a,b,c,d\|_\infty$, $\|\beta\|_{L^\infty(\Omega)}$ and T such that for any $\varphi=(\varphi_1,\varphi_2)\in\mathcal{W}$, we have

$$\begin{aligned}\int_0^T\int_{\omega'}(|\varphi_1|^2+|\varphi_2|^2)e^{-2\alpha}dxdt\leqslant C\Big(&\int_G|\varphi_1|^2e^{-r\alpha}dxdt\\ &+\int_Q(|M(\varphi)|^2+|N(\varphi)|^2)e^{-2\alpha}dxdt\Big)\end{aligned} \tag{3.8}$$

for any ω' such that $\omega'\Subset\omega_c\Subset\omega$.

Proof. The proof of this result is technical. We followed the approach of Annex E in [7]. The following is the main tools that we used to prove (3.8).
We assume for instance that $c\geqslant c_0>0$ in $\omega_c\times(0,T)$. We let $\xi\in C^\infty(\mathbb{R}^n)$ satisfy (2.1). For $\beta_0,\beta_1,m>0$, we define

$$\Lambda(t)=\int_\Omega(e^{-2m\alpha}\eta^{\frac76}|\varphi_2|^2-\beta_0e^{-2\alpha}\eta\varphi_2\varphi_1+\beta_1e^{-2\alpha}\eta|\varphi_1|^2)dx.$$

Then we derive Λ with respect to t and replace $(\varphi_1)_t$ and $(\varphi_2)_t$ with their expressions given by (3.6). Integrating by parts over $(0,T)$ and using $\Lambda(0)=\Lambda(T)=0$, we get

$$\begin{aligned}\beta_0\int_Q e^{-2\alpha}\eta c|\varphi_2|^2dxdt=&\int_Q\{(2m\alpha_t+2d)e^{-2m\alpha}\eta^{\frac76}|\varphi_2|^2\\ &+[2\beta_1(\alpha_t+a)e^{-2\alpha}\eta-\beta_0e^{-2\alpha}\eta b]|\varphi_1|^2\\ &-[\beta_0(2\alpha_t+a+d)e^{-2\alpha}\eta-2\beta_1e^{-2\alpha}\eta c-2e^{-2m\alpha}\eta^{7/6}b]\varphi_1\varphi_2\}dxdt\\ &+\int_Q e^{-2m\alpha}\eta^{7/6}\varphi_2(B(t;z_1)+B(t;z_2))\varphi_2dxdt\\ &-\frac{\beta_0}{2}\int_Q e^{-2\alpha}\eta\varphi_2(B(t;z_1)-B(t;z_2))\varphi_2dxdt\\ &+\int_Q e^{-2m\alpha}\eta^{7/6}\varphi_2(B(t;z_1)-B(t;z_2))\varphi_1dxdt\\ &+\beta_1\int_Q e^{-2\alpha}\eta\varphi_1(B(t;z_1)-B(t;z_2))\varphi_2dxdt\end{aligned} \tag{3.9}$$

$$-\frac{\beta_0}{2}\int_Q e^{-2\alpha}\eta\Big(\varphi_2(B(t;z_1)+B(t;z_2))\varphi_1+\varphi_1(B(t;z_1)+B(t;z_2))\varphi_2\Big)dxdt$$

$$-\frac{\beta_0}{2}\int_Q e^{-2\alpha}\eta\varphi_1(B(t;z_1)-B(t;z_2))\varphi_1 dxdt$$

$$+\beta_1\int_Q e^{-2\alpha}\eta\varphi_1(B(t;z_1)+B(t;z_2))\varphi_1 dxdt$$

$$+2\int_Q e^{-2m\alpha}\eta^{7/6}\varphi_2 N(\varphi)dxdt-\beta_0\int_Q e^{-2\alpha}\eta\varphi_2 M(\varphi)dxdt$$

$$(3.10)\quad -\beta_0\int_Q e^{-2\alpha}\eta\varphi_1 N(\varphi)dxdt+2\beta_1\int_Q e^{-2\alpha}\eta\varphi_1 M(\varphi)dxdt=J_1+\cdots+J_{12}.$$

Then we estimate each of the terms $J_1,\ldots,J_{12}$, in particular, each time that the integral $\int_G e^{-2s\alpha}\rho^3(|\varphi_1|^2+|\varphi_2|^2)dxdt$ appears, we estimate it by $\int_G|\varphi_1|^2e^{-r\alpha}dxdt$. □

We set

$$(3.11)\qquad \frac{1}{\theta^2}=\rho^3e^{-2s\alpha}.$$

We recall that P is the orthogonal projection operator from $L^2(G)$ into $\mathcal{H}$.

The following observability inequality follows from (3.8).

Lemma 3.3. *Assume (2.7). Then with the hypotheses of Theorem 3.2, there exists a constant C depending on C_0, λ_1, s_0, γ_0, γ_1, n, c_0, α_0, $\|a,b,c,d\|_\infty$, T, $\|\beta\|_{L^\infty(\Omega)}$ and on the Poincare constant K, such that for every $\varphi=(\varphi_1,\varphi_2)\in\mathcal{W}$, we have*

$$(3.12)\quad \int_\Omega(|\varphi_1(0)|^2+|\varphi_2(0)|^2)dx+\int_Q\frac{1}{\theta^2}(|\varphi_1|^2+|\varphi_2|^2)dxdt$$
$$\leqslant C\Big(\int_Q(|M(\varphi)|^2+|N(\varphi)|^2)dxdt+\int_G|\varphi_1-P\varphi_1|^2dxdt\Big).$$

Proof. First we state the result for the norm of $\varphi(x,0)$ in Ω, where $\varphi\in\mathcal{W}$. The second part of the inequality (3.12) is a consequence of (3.8). □

3.2 Null controllability of the linearized system

In this part, we prove the controllability problem associated to (3.3). Consider the bilinear form defined on $\mathcal{W}\times\mathcal{W}$ by

$$\mathcal{B}(\varphi,\sigma)=\int_Q M(\varphi)M(\sigma)dxdt+\int_Q N(\varphi)N(\sigma)dxdt+\int_G(\varphi_1-P\varphi_1)(\sigma_1-P\sigma_1)dxdt,$$

$\forall\varphi=(\varphi_1,\varphi_2),\sigma=(\sigma_1,\sigma_2)\in\mathcal{W}$. The bilinear form $\mathcal{B}(.,.)$ is a scalar product on $\mathcal{W}$. Let W be the completion of the pre-Hilbert space $\mathcal{W}$ with respect to the norm $\mathcal{B}(\varphi,\varphi)$. We deduce from the observability estimate (3.12) null controllability results for (3.3). Proceeding as in the proof of Theorem 3.4.4. in [7], we show the

Theorem 3.4. *Recall the notations (3.2) and (3.11). Assume (2.7) and (3.7). Assume also that $f_1,f_2\in L^2(Q)$ are such that $\theta f_1,\theta f_2\in L^2(Q)$. For all $z\in Z\times Z$,* *there exists a unique control $\tilde{h}$ of minimal norm in $L^2(G)$ such that $\tilde{h} \in \mathcal{H}^\perp$ and the associated solution $(\tilde{u}, \tilde{v})$ of (3.3) satisfies $\tilde{u}(T) = \tilde{v}(T) = 0$ in Ω. The control $\tilde{h}$ is given by*

$$\tilde{h} = \tilde{\eta}_1 \chi_\omega - P\tilde{\eta}_1, \tag{3.13}$$

where $\tilde{\eta} = (\tilde{\eta}_1, \tilde{\eta}_2)$ satisfies

$$\begin{cases} M(\tilde{\eta}_1, \tilde{\eta}_2) = N(\tilde{\eta}_1, \tilde{\eta}_2) = 0 \ \text{in } Q, \\ \tilde{\eta}_1 = \tilde{\eta}_2 = 0 \ \text{on } \Sigma. \end{cases} \tag{3.14}$$

Furthermore there exists a constant $C > 0$ depending on Ω, ω, c_0, r, T, $\|\beta\|_{L^\infty(\Omega)}$ and $\|a_1, a_2, b_1, b_2\|_\infty$ such that

$$\|\tilde{h}\|_{L^2(G)} \leqslant C(\|\theta f\|_{L^2(Q)} + \|\theta g\|_{L^2(Q)} + \|u^0\|_{L^2(\Omega)} + \|v^0\|_{L^2(\Omega)}), \tag{3.15}$$

$$\|\tilde{\eta}\|_W \leqslant C(\|\theta f\|_{L^2(Q)} + \|\theta g\|_{L^2(Q)} + \|u^0\|_{L^2(\Omega)} + \|v^0\|_{L^2(\Omega)}), \tag{3.16}$$

$$\|\tilde{\eta}_1\|_{L^2(G)} \leqslant C(\|\theta f\|_{L^2(Q)} + \|\theta g\|_{L^2(Q)} + \|u^0\|_{L^2(\Omega)} + \|v^0\|_{L^2(\Omega)}). \tag{3.17}$$

The following result gives another estimate indicating that the control can be chosen depending continuously on the initial data.

Lemma 3.5. *For all $z = (z_1, z_2) \in Z \times Z$ there is $k \in L^2(G)$ satisfying $k \in \mathcal{H}^\perp$ and such that the associated solution to (3.1) satisfies (1.4). Moreover*

$$\|(k, (y_1, y_2))\|_X \leqslant C(\|z\|_{Z\times Z})(\|\theta f_1\|_{L^2(Q)} + \|\theta f_2\|_{L^2(Q)} + \|y_1^0\|_{L^2(\Omega)} + \|y_2^0\|_{L^2(\Omega)}). \tag{3.18}$$

Proof. Let τ be such that $0 < \tau < T$ and let us take $k(x, t) = 0$ for $0 < t < \tau$. From the regularizing effect of the parabolic equations in (3.1), the associated state satisfies $(y_1(., \tau), y_2(., \tau)) \in (C^{2+\delta}(\overline{\Omega}))^2$, with

$$\|(y_1(., \tau), y_2(., \tau))\|_{C^{2+\delta}(\overline{\Omega})\times C^{2+\delta}(\overline{\Omega})} \leqslant C(\|z\|_{Z\times Z})(\|\theta f_1\|_{L^2(Q)} + \|\theta f_2\|_{L^2(Q)} + \|y_1^0\|_{L^2(\Omega)} + \|y_2^0\|_{L^2(\Omega)}).$$

Therefore, it is not restrictive to assume that $y_1^0, y_2^0 \in C^{2+\delta}(\overline{\Omega})$. We conclude, using Lemma 3.4.5 of [7] (see also Theorem 2.3, [3]), that the control $k(z)$ can be chosen such that $(k, (y_1, y_2))$ is an element of X satisfying (3.18). □

4 Proof of the main result

We are now able to prove Theorem 2.2. Let $R > 0$ and B_R the closed ball in Z of radius R and center 0. According to Theorem 3.4, for each $z = (z_1, z_2) \in Z \times Z$, there is a unique control $\tilde{k}(z) = \frac{1}{2}\tilde{h}(z)$ which solves (1.3),(1.4),(3.1), $\tilde{h} \in \mathcal{H}^\perp$ being defined by (3.13). In view of Lemma 3.5, the control $\tilde{k}(z)$ can be chosen such that $(\tilde{k}, (\tilde{y}_1, \tilde{y}_2))$ is an element of X satisfying

$$\|(\tilde{k}, (\tilde{y}_1, \tilde{y}_2))\|_X \leqslant C(R)(\|\theta f_1\|_{L^2(Q)} + \|\theta f_2\|_{L^2(Q)} + \|y_1^0\|_{L^2(\Omega)} + \|y_2^0\|_{L^2(\Omega)}). \tag{4.1}$$

Let us set $\mathcal{S}(z) = \{(y_1(z), y_2(z)) \in Z \times Z : (k,(y_1,y_2))$is a control-state, $(k,(y_1,y_2)) \in X$, (1.3), (1.4) and (4.1) hold$\}$. The multi-valued mapping $\mathcal{S} : Z \times Z \to 2^{Z\times Z}$ satisfies the hypotheses of the Kakutani fixed-point theorem.

First for each $z \in B_R \times B_R$ the mapping $\mathcal{S}$ is non-empty and convex, it is a consequence of Theorem 3.4.

Then for all $z \in B_R \times B_R$, $\mathcal{S}(z)$ is uniformly bounded in the Hölder space $C^{1+\delta,\frac{1+\delta}{2}}(\overline{Q})^2$, $\delta \in (0,1)$, since $(\tilde{y}_1(z),\ \tilde{y}_2(z))$ satisfies (4.1). The injection from $C^{1+\delta,\frac{1+\delta}{2}}(\overline{Q})$ into Z being compact (see for instance [3]), there is $K \subset Z \times Z$ compact such that $\mathcal{S}(z) \in K$. Moreover if $\|\theta f_1\|_{L^2(Q)}$, $\|\theta f_2\|_{L^2(Q)}$, $\|y_1^0\|_{L^2(\Omega)}$ and $\|y_2^0\|_{L^2(\Omega)}$ are small enough, there is $R > 0$ such that $\mathcal{S}(B_R \times B_R) \in B_R \times B_R$.

The mapping $\mathcal{S}$ has a closed graph. Indeed, let $(z_n)_n = ((z_{1_n}, z_{2_n}))_n \in Z \times Z$ and $(y_{1_n}, y_{2_n})_n \in \mathcal{S}(z_n)$. Assume that $z_n \to z = (z_1, z_2)$ strongly in $Z \times Z$ and that $(y_{1_n}, y_{2_n}) \to (y_1, y_2)$ strongly in $Z \times Z$. We prove that $(y_1, y_2) \in \mathcal{S}(z)$. This concludes the proof of Theorem 2.2.

5 Applications

This section is devoted to show that our work is used to solve a null controllability problem with constraints on the state.
Let $(e_j)_{j=1,\ldots,m}$ be a family of m vectors of $L^2(Q)$ such that:

(5.1) $$\text{the } (e_j\chi_\omega)_{j=1,\ldots,m} \text{ are linearly independent.}$$

Let us consider the following null controllability problem: *Given* $e_j \in L^2(Q)$ $j = 1,\ldots,m$, *find* $k \in L^2(G)$ *such that if* (y_1, y_2) *solves*

(5.2) $$\begin{cases} \partial_t y_1 - B(t;z_1)y_1 + a_1 y_1 + b_1 y_2 = f_1 + k\chi_\omega \text{ in } Q, \\ \partial_t y_2 - B(t;z_2)y_2 + a_2 y_1 + b_2 y_2 = f_2 + k\chi_\omega \text{ in } Q, \\ y_1 = y_2 = 0 \text{ on } \Sigma, \\ y_1(0) = y_1^0, y_2(0) = y_2^0 \text{ in } \Omega, \end{cases}$$

then

(5.3) $$\int_Q y_1 e_j\, dxdt = \int_Q y_2 e_j\, dxdt = 0;\ j = 1,\ldots,m,$$

and

(5.4) $$y_1(T) = y_2(T) = 0 \text{ in } \Omega.$$

We introduce a family of adjoint systems of (5.2)

(5.5) $$\begin{cases} -\partial_t p_j - B(t;z_1)p_j + a_1 p_j + a_2 q_j = e_j \text{ in } Q, \\ -\partial_t q_j - B(t;z_2)q_j + b_1 p_j + b_2 q_j = e_j \text{ in } Q, \\ p_j = q_j = 0 \text{ on } \Sigma, \\ p_j(T) = q_j(T) = 0 \text{ in } \Omega, \end{cases}$$

and define for each $j = 1,\ldots,m$: $\mu_j = p_j + q_j$ and $\nu_j = p_j - q_j$. We assume that for any $z = (z_1, z_2) \in Z \times Z$,

(5.6) $$(a_1 + b_1 - a_2 - b_2)I = B(t;z_1) - B(t;z_2) \text{ in } G \text{ with } a_2 \neq b_1,$$

I denoting the identity operator.

Remark 5.1. Due to the notations (2.6), the assumption (5.6) can also be written as $cI = -\frac{1}{2}(B(t;z_1) - B(t;z_2))$ in G for any $z = (z_1, z_2) \in Z \times Z$.

Then the following lemma holds:

Lemma 5.1. *Under hypotheses (5.1) and (5.6), let θ be the positive function given by the formula (3.11). Then the functions μ_j and ν_j, $j = 1, \dots, m$, are linearly independent in G for any $z \in Z \times Z$. Moreover, the functions $\frac{1}{\theta}\mu_j$ and $\frac{1}{\theta}\nu_j$, $j = 1, \dots, m$, are also linearly independent in G for any $z \in Z \times Z$.*

Proof. Let $\xi_j \in \mathbb{R}$, $j = 1, \dots, m$ be such that $\sum_{j=1}^m \xi_j \mu_j = 0$ in G, and let $z \in Z \times Z$. Since

$$-\partial_t \mu_j - \frac{1}{2}[(B(t;z_1) + B(t;z_2))\mu_j + (B(t;z_1) - B(t;z_2))\nu_j] - a\mu_j - c\nu_j = 2e_j$$

holds in Q for each $j \in \{1, \dots, m\}$, we obtain in G:

$$-\Big(\frac{1}{2}(B(t;z_1) - B(t;z_2)) + cI\Big)\sum_{j=1}^m \xi_j \nu_j = 2\sum_{j=1}^m \xi_j e_j.$$

In view of (5.6), $\sum_{j=1}^m \xi_j e_j = 0$ in G, and (5.1) implies that $\xi_j = 0$ for all j. So the functions μ_j, $j = 1, \dots, m$ are linearly independent in G.

Now, let $\xi_j \in \mathbb{R}$, $j = 1, \dots, m$ such that $\sum_{j=1}^m \xi_j \nu_j = 0$ in G. Since

$$-\partial_t \nu_j - \frac{1}{2}[(B(t;z_1) - B(t;z_2))\mu_j + (B(t;z_1) + B(t;z_2))\nu_j] - b\mu_j - d\nu_j = 0$$

holds in Q, then

$$\Big(\frac{1}{2}(B(t;z_1) - B(t;z_2)) + bI\Big)\sum_{j=1}^m \xi_j \mu_j = 0$$

in G. This implies that $(b-c)I\sum_{j=1}^m \xi_j \mu_j = 0$ in G following (5.6). Thus $\sum_{j=1}^m \xi_j \mu_j = 0$ in G, since $b \neq c$ in G. Finally, $\xi_j = 0$ for each $j \in \{1, \dots, m\}$. The second assertion of Lemma 5.1 follows. □

Now we prove the announced result in the following:

Proposition 5.2. *With the hypotheses of Lemma 5.1, consider the vector subspace $\frac{1}{\theta}\mathcal{H}$ of $L^2(G)$ generated by the functions $\frac{1}{\theta}\mu_j\chi_\omega$, $j = 1, \dots, m$. Then for any $z \in Z \times Z$, there exists a unique $h_0 \in \frac{1}{\theta}\mathcal{H}$ such that the problem (5.2)-(5.4) is equivalent to the following problem: Given $a_i, b_i \in L^\infty(Q)$ and $y_i^0 \in L^2(\Omega)$ $i = 1, 2$, find a control*

$$h_1 \in \mathcal{H}^\perp \tag{5.7}$$

such that if (y_1, y_2) solves

$$\begin{cases} \partial_t y_1 - B(t;z_1)y_1 + a_1 y_1 + b_1 y_2 = (\frac{1}{\theta}h_0 + h_1)\chi_\omega \text{ in } Q, \\ \partial_t y_2 - B(t;z_2)y_2 + a_2 y_1 + b_2 y_2 = (\frac{1}{\theta}h_0 + h_1)\chi_\omega \text{ in } Q, \\ y_1 = y_2 = 0 \text{ on } \Sigma, \\ y_1(0) = y_1^0, y_2(0) = y_2^0 \text{ in } \Omega, \end{cases} \tag{5.8}$$

then

$$y_1(T) = y_2(T) = 0 \text{ in } \Omega. \tag{5.9}$$

Proof. Suppose that (5.2)-(5.4) holds. First we multiply (5.2) by the solution (p_j, q_j) to (5.5), and we integrate by parts over Q. Then we add and substract the results. In view of (5.3), it follows that

$$-\int_\Omega y_1^0 p_j(0)dx - \int_\Omega y_2^0 q_j(0)dx = \int_G k\mu_j dxdt, \tag{5.10}$$

$$-\int_\Omega y_1^0 p_j(0)dx + \int_\Omega y_2^0 q_j(0)dx - 2\int_Q a_2 y_1 q_j dxdt + 2\int_Q b_1 y_2 p_j dxdt = \int_G k\nu_j dxdt. \tag{5.11}$$

Let $\frac{1}{\theta}\mathcal{H}$ and $\frac{1}{\theta}\mathcal{K}$ be the vector subspaces of $L^2(G)$ respectively generated by the functions $\frac{1}{\theta}\mu_j\chi_\omega$ and $\frac{1}{\theta}\nu_j\chi_\omega$, $j = 1, \dots, m$. Then there is one and only one $(h_0, l_0) \in \frac{1}{\theta}\mathcal{H} \times \frac{1}{\theta}\mathcal{K}$ such that

$$\int_G \frac{1}{\theta} h_0 \mu_j dxdt = -\int_\Omega (y_1^0 p_j(0) + y_2^0 q_j(0))dx, \tag{5.12}$$

$$\int_G \frac{1}{\theta} l_0 \nu_j dxdt = -\int_\Omega (y_1^0 p_j(0) - y_2^0 q_j(0))dx - 2\int_Q a_2 y_1 q_j dxdt + 2\int_Q b_1 y_2 p_j dxdt. \tag{5.13}$$

Thus, according to (5.10) and (5.11), we have for any $j \in \{1, \dots, m\}$,

$$\int_G \frac{1}{\theta} h_0 \mu_j dxdt = \int_G k\mu_j dxdt, \qquad \int_G \frac{1}{\theta} l_0 \nu_j dxdt = \int_G k\nu_j dxdt.$$

Then $(k - \frac{1}{\theta}h_0, k - \frac{1}{\theta}l_0) \in \mathcal{H}^\perp \times \mathcal{K}^\perp$. There are $(h_1, l_1) \in \mathcal{H}^\perp \times \mathcal{K}^\perp$ such that $k = \frac{1}{\theta}h_0 + h_1 = \frac{1}{\theta}l_0 + l_1$. Now, replacing k by $\frac{1}{\theta}h_0 + h_1$ in (5.2), we obtain (5.8).

Conversely, assume that (5.7)-(5.9) holds. Let $k \in \left(\frac{1}{\theta^2}\mathcal{H} + \mathcal{H}^\perp\right) \cap \left(\frac{1}{\theta^2}\mathcal{K} + \mathcal{K}^\perp\right)$ be such that $k = \frac{1}{\theta}h_0 + h_1$, where $h_0 \in \frac{1}{\theta}\mathcal{H}$ is defined by relation (5.12). Multiplying (5.8) by (p_j, q_j), then integrating by parts over Q and adding the results, we have

$$\int_Q (y_1 + y_2)e_j dxdt = \int_G h_1 \mu_j dxdt, \text{ for } j \in \{1, \dots, m\},$$

and since $h_1 \in \mathcal{H}^\perp$, we get

$$\int_Q (y_1 + y_2)e_j dxdt = 0, \text{ for any } j \in \{1, \dots, m\}.$$

Now let $l_0 \in \frac{1}{\theta}\mathcal{K}$ be defined by (5.13) and let l_1 satisfying $l_1 = \frac{1}{\theta}h_0 + h_1 - \frac{1}{\theta}l_0$. Multiplying (5.8) by (p_j, q_j), then integrating by parts over Q and subtracting the results, we obtain

$$\int_Q (y_1 - y_2)e_j dxdt = \int_G l_1 \nu_j dxdt, \text{ for } j \in \{1, \dots, m\},$$

which ends the proof of Proposition 5.2, since $l_1 \in \mathcal{K}^\perp$ by construction. □

Remark 5.2. We will show that assuming the independence of the functions e_j, $j = 1, \dots, m$, the hypothese (2.7) will be useless.

Conclusion

In this paper we proved a null controllability problem associated to a nonlinear parabolic system with a nonlocal operator. As a consequence of this work, a null controllability problem with integral constraints will be the purpose of the next work. Thus we generalized the results established in the linear case for the Laplacian. The next step would be to focus on the equations governed by more general operators, as fractional operators.

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